

A note on integrating products of linear forms over the unit simplex

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Abstract

Integrating a product of linear forms over the unit simplex can be done in polynomial time if the number of variables n is fixed [1]. In this note, we highlight that this problem is equivalent to obtaining the normalizing constant of state probabilities for a popular class of Markov processes used in queueing network theory. In light of this equivalence, we survey existing computational algorithms developed in queueing theory that can be used for exact integration. For example, under some regularity conditions, queueing theory algorithms can exactly integrate a product of linear forms of total degree N by solving N systems of linear equations.

1 Introduction

Let $\Delta = \{\mathbf{x} \in \mathbb{R}^n | x_i \geq 0, \sum_{i=1}^n x_i = 1\}$ be the unit simplex and denote by dm the integral Lebesgue measure on Δ . Denote by $\theta_1, \theta_2, \dots, \theta_d$ a collection of linear forms on $\mathbb{R}^n$, where θ_{ij} is the i th element of θ_j and define the coefficient matrix $\boldsymbol{\theta} = (\theta_{ij}) \in \mathbb{R}^{nd}$. Also let $N_1, \dots, N_d$ be a set of nonnegative integers, with $N = \sum_{i=1}^d N_i$ and $\mathbf{N} = (N_1, \dots, N_d)$.

Recently, [1] considers the problem of integrating a homogeneous polynomial function f of degree N over Δ , observing that this can be reduced to computing a finite collection of integrals of the type

$$J(\boldsymbol{\theta}, \mathbf{N}) = \int_{\Delta} \prod_{j=1}^d \left(\sum_{i=1}^n \theta_{ij} x_i \right)^{N_j} dm \quad (1)$$

A polynomial-time algorithm to compute (1) is then proposed in [1], which obtains this integral by determining the coefficient of $z_1^{N_1} \dots z_d^{N_d}$ in the Taylor expansion of

$$T(z_1, \dots, z_d) = \frac{1}{\prod_{i=1}^n (1 - \sum_{j=1}^d z_j \theta_{ij})} \quad (2)$$

in $O(N^d)$ time. Our goal is to examine alternative methods to compute (1) that arise from queueing network theory [4].

We first observe that (2) corresponds to the generating function of the *normalizing constant* $G(\boldsymbol{\theta}, \mathbf{N})$ of state probabilities for a class of finite Markov processes known as *product-form closed queueing networks*, which have been extensively used in performance evaluation of computer and

communication systems [2,20]. We provide in Appendix A a brief introduction to this class of Markov models; additional background may be found in [4,27].

The connection between normalizing constants and (1) is explicitly noted in [11], which shows that

$$J(\boldsymbol{\theta}, \mathbf{N}) = \frac{N_1!N_2!\cdots N_d!}{(N+n-1)!}G(\boldsymbol{\theta}, \mathbf{N}) \quad (3)$$

Therefore, algorithms developed in queueing theory to compute $G(\boldsymbol{\theta}, \mathbf{N})$ may be readily used to integrate products of linear forms over the unit simplex.

The rest of this note reviews exact computational methods for $G(\boldsymbol{\theta}, \mathbf{N})$. Although in queueing theory it is assumed that $\theta_{ij} \geq 0$, computational algorithms developed therein for $G(\boldsymbol{\theta}, \mathbf{N})$ do not rely on this assumption and can therefore be applied also to linear forms with arbitrary values of θ_{ij} .

In the following sections, we denote by $\mathbf{N} - \mathbf{1}_j$ the vector obtained from $\mathbf{N}$ by decreasing its j th element N_j by one unit. We also indicate with $\boldsymbol{\theta} - \theta_j$ the matrix obtained from $\boldsymbol{\theta}$ by removing a row with elements $\theta_{j1}, \dots, \theta_{jd}$, and similarly denote by $\boldsymbol{\theta} + \theta_j$ the matrix obtained by appending to $\boldsymbol{\theta}$ a new row with elements $\theta_{j1}, \dots, \theta_{jd}$.

2 Exact computational algorithms

2.1 Recurrence relations

Recurrence relations are the standard method used in queueing theory to compute $G(\boldsymbol{\theta}, \mathbf{N})$. Existing methods differ for computational requirements and ease of implementation. We here review two classic methods, convolution and RECAL, and the method of moments, a class of algorithms that scales efficiently under large degree N . We point to [12, 19, 20, 25, 26] for other recursive algorithms.

2.1.1 Convolution algorithm

The convolution algorithm is a recurrence relation for normalizing constants presented in its most general form in [28], extending the case $d = 1$ first developed in [5]. The method relies on the recurrence relation

$$G(\boldsymbol{\theta}, \mathbf{N}) = G(\boldsymbol{\theta} - \theta_n, \mathbf{N}) + \sum_{j=1}^d \theta_{nj} G(\boldsymbol{\theta}, \mathbf{N} - \mathbf{1}_j) \quad (4)$$

with termination conditions $G(\emptyset, \cdot) = 0$, $G(\cdot, \mathbf{0}) = 1$, where $\mathbf{0} = (0, \dots, 0)$, and $G(\cdot, \mathbf{N} - \mathbf{1}_j) = 0$ if $N_j = 0$. It can be verified that the algorithm requires $\mathcal{O}(N^d)$ time and space for fixed n and d . Variants of this algorithm exist [12, 26], for example a specialized version has been proposed for problems with sparse $\boldsymbol{\theta}$ [25].

2.1.2 Recurrence by chain algorithm (RECAL)

Compared to convolution, the RECAL algorithm is more efficient on models with large d [14]. The method is based on the recurrence relation

$$G(\boldsymbol{\theta}, \mathbf{N}) = N_d^{-1} \sum_{i=1}^n \theta_{id} G(\boldsymbol{\theta} + \theta_i, \mathbf{N} - \mathbf{1}_d) \quad (5)$$

with similar termination conditions as the convolution algorithm. The computational complexity is $\mathcal{O}(N^n)$ time and space for fixed n and d . The method is well-suited for parallel implementation and can be optimized for sparse θ_{ij} coefficients [17].

2.1.3 Method of moments

The method of moments simultaneously applies (4) and (5) to a set of normalizing constants that differ for the elements and composition of θ and N . Different rules exist to define this set of normalizing constants, called the *basis*, leading to multiple variants of the method [8–10]. Assume that the normalizing constants in the basis are grouped in a vector $g(N)$, then the method of moments defines a matrix recurrence relation

$$A(\theta, N)g(N) = B(\theta, N)g(N - 1_j) \quad (6)$$

for any $1 \leq j \leq d$ and where $g(0)$ can be computed explicitly from the termination conditions of (4)-(5). Here $A(\theta, N)$ and $B(\theta, N)$ are square matrices, with constant or decreasing sizes as the recurrence progresses, depending on the implementation. The coefficients within these matrices depend on j . Thus, to avoid to recompute these matrices at each step, the method of moments first performs a recursion along dimension $j = d$, then along $j = d - 1$, and so forth up to reaching $N = 0$. An explicit example of the basic method may be found in [7]. Provided that $A(\theta, N)$ is invertible at all steps of the recursion, it is possible to solve (6) and obtain $G(\theta, N)$ from the elements of $g(N)$. A necessary condition for invertibility is that $\theta_{ir} \neq \theta_{jr}, \forall i, j, r$.

Compared to existing algorithms, (6) finds $G(\theta, N)$ after solving N systems of linear equations. Since the order of $A(\theta, N)$ does not depend on N , the theoretical complexity of the method is $\mathcal{O}(N)$ time and $\mathcal{O}(1)$ space for fixed n and d . However, implementation complexity is larger due to the cost of exact algebra, which is normally required to solve (6) without error propagation [6].

2.2 Explicit solutions

2.2.1 Case $d = 1$

In this section we consider the case $d = 1$, where we can simplify notation by indicating N with N and θ_{i1} with θ_i . It has been known since long time that in the case $d = 1$, and provided that all θ_i coefficients are distinct, the normalizing constant can be written as [22]

$$G(\theta, N) = \sum_{i=1}^n \frac{\theta_i^{N+n-1}}{\prod_{k \neq i} (\theta_i - \theta_k)} \quad (7)$$

that is simply the divided difference $[\theta_1, \dots, \theta_n]x^{N+n-1}$. It is useful to note that in the case where some coefficients θ_{ij} coincide, (7) generalizes to [3]

$$\begin{aligned} G(\theta, N) &= [\theta_1, \dots, \theta_n]x^{N+n-1} \\ &= \sum_{j=1}^{n'} (-1)^{m_j-1} \theta_j^{N+n-m_j} \sum_{\substack{r \geq 0 \\ r=m_j-1}} (-1)^{r_j} \binom{N+r_j}{r_j} \prod_{\substack{k=1 \\ k \neq j}}^{n'} \binom{m_k+r_k-1}{r_k} \frac{\theta_k^{r_k}}{(\theta_j - \theta_k)^{m_k+r_k}} \end{aligned} \quad (8)$$

where $n' \leq n$ is the number of distinct coefficients θ_i , and m_j denotes the number of coefficients identical to θ_j .

2.2.2 Case $d > 1$

For arbitrary values of d , [11] derives some generalizations of (7). The first one is

$$G(\boldsymbol{\theta}, \mathbf{N}) = \sum_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{N}} \frac{(-1)^{N-t}}{N_1! \cdots N_d!} \prod_{j=1}^d \binom{N_j}{t_j} \sum_{i=1}^n \frac{(\sum_{j=1}^d t_j \theta_{ij})^{N+n-1}}{\prod_{k \neq i} (\sum_{j=1}^d t_j (\theta_{ij} - \theta_{kj}))} \quad (9)$$

where $\mathbf{t} = (t_1, \dots, t_d)$, $t = \sum_{j=1}^d t_j$, which has the same $\mathcal{O}(N^d)$ time complexity of the convolution algorithm, but $\mathcal{O}(1)$ space requirements. This expression holds assuming that θ_{ij} values for given j are all distinct, otherwise the following generalized expression based on (8) should be used [11]

$$G(\boldsymbol{\theta}, \mathbf{N}) = \sum_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{N}} \frac{(-1)^{N-t}}{N_1! \cdots N_d!} \prod_{r=1}^d \binom{N_r}{t_r} \sum_{j=1}^{n'} (-1)^{m_j-1} (\sum_{s=1}^d t_s \theta_{js})^{N+n-m_j} \\ \times \sum_{\substack{\mathbf{r} \geq \mathbf{0} \\ r = m_j - 1}} (-1)^{r_j} \binom{N+r_j}{r_j} \prod_{\substack{k=1 \\ k \neq j}}^{n'} \binom{m_k + r_k - 1}{r_k} \frac{(\sum_{s=1}^d t_s \theta_{ks})^{r_k}}{(\sum_{s=1}^d t_s (\theta_{js} - \theta_{ks}))^{m_k + r_k}} \quad (10)$$

Another explicit expression is given by

$$G(\boldsymbol{\theta}, \mathbf{N}) = \sum_{\substack{\mathbf{h} \geq \mathbf{0} \\ h \leq N}} \frac{(-1)^{N-h}}{N_1! \cdots N_d!} \binom{N+n-1}{N-h} \prod_{j=1}^d \left(\sum_{i=1}^n h_i \theta_{ij} \right)^{N_j} \quad (11)$$

where $\mathbf{h} = (h_1, \dots, h_n)$, $h = \sum_{i=1}^n h_i$. This expression has $\mathcal{O}(N^{n+1})$ time and $\mathcal{O}(1)$ space requirements.

It is also worth noting that since the integrand of $J(\boldsymbol{\theta}, \mathbf{N})$ is a polynomial, the cubature rules proposed in [18] provide alternative explicit expressions for $G(\boldsymbol{\theta}, \mathbf{N})$, which become exact for a sufficiently high interpolation degree, corresponding to a $\mathcal{O}((N/2)^d)$ time complexity. We point to [3, 15, 16] for other works on explicit expressions for the normalizing constant.

3 Conclusion

In this note, we have highlighted a connection between queueing network theory and the integration of products of linear forms over the unit simplex as in (1). In order to solve queueing network models, a normalizing constant is required, which can be computed with the recurrence relations and the explicit expressions that we have reviewed. This normalizing constant readily provides the value of the integral (1).

While the scope of the present note is restricted to exact methods, numerical approximations and asymptotic expansions are also available for $G(\boldsymbol{\theta}, \mathbf{N})$, such as [13, 21, 23, 24, 29].

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References

- [1] V. Baldoni, N. Berline, J. A. d. Loera, M. Köppe, and M. Vergne. How to integrate a polynomial over a simplex. *Mathematics of Computation*, 80:297–325, 2011.
- [2] F. Baskett, K. M. Chandy, R. R. Muntz, and F. G. Palacios. Open, closed, and mixed networks of queues with different classes of customers. *JACM*, 22:248–260, 1975.
- [3] A. Bertozzi and J. McKenna. Multidimensional residues, generating functions, and their application to queueing networks. *SIAM Review*, 35(2):239–268, 1993.
- [4] G. Bolch, S. Greiner, H. de Meer, and K. S. Trivedi. *Queueing Networks and Markov Chains*. Wiley, 2006.
- [5] J. P. Buzen. Computational algorithms for closed queueing networks with exponential servers. *Comm. of the ACM*, 16(9):527–531, 1973.
- [6] G. Casale. An efficient algorithm for the exact analysis of multiclass queueing networks with large population sizes. *Proc. of ACM SIGMETRICS*, 169–180, 2006.
- [7] G. Casale. An application of exact linear algebra to capacity planning models. *ACM Communications in Computer Algebra*, 42(4):202–205, 2008.
- [8] G. Casale. CoMoM: Efficient class-oriented evaluation of multiclass performance models. *IEEE Trans. on Software Engineering*, 35(2):162–177, 2009.
- [9] G. Casale. A generalized method of moments for closed queueing networks. *Performance Evaluation*, 68(2):180–200, 2011.
- [10] G. Casale. Exact analysis of performance models by the method of moments. *Performance Evaluation*, 68(6):487–506, 2011.
- [11] G. Casale. Accelerating performance inference over closed systems by asymptotic methods. *Proc. ACM Meas. Anal. Comput. Syst (POMACS)*, 1(1), 2017. To be presented at ACM SIGMETRICS 2017, preprint available at <https://spiral.imperial.ac.uk/handle/10044/1/43431>.
- [12] K. M. Chandy and C. H. Sauer. Computational algorithms for product-form queueing networks models of computing systems. *Comm. of the ACM*, 23(10):573–583, 1980.
- [13] G L. Choudhury, K. K. Leung, and W. Whitt. Calculating normalization constants of closed queueing networks by numerically inverting their generating functions. *JACM*, 42(5):935–970, 1995.
- [14] A. E. Conway and N. d. Georganas. RECAL - A new efficient algorithm for the exact analysis of multiple-chain closed queueing networks. *Journal of the ACM*, 33(4):768–791, 1986.
- [15] A. I. Gerasimov. On normalizing constants in multiclass queueing networks. *Oper. Res.*, 43(4):704–711, 1995.
- [16] J. J. Gordon. The evaluation of normalizing constants in closed queueing networks. *Oper. Res.*, 38(5):863–869, 1990.

- [17] A. G. Greenberg and J. McKenna. Solution of closed, product form, queueing networks via the RECAL and TREE-RECAL methods on a shared memory multiprocessor. *ACM SIGMETRICS Performance Evaluation Review (PER)*, 17(1):127–135, 1989.
- [18] A. Grundmann, H.M. Möller. Invariant integration formulas for the n-simplex by combinatorial methods. *SIAM J. on Numerical Analysis*, 15(2):282–290, 1978.
- [19] P. G. Harrison and Ting Ting Lee. A New Recursive Algorithm for Computing Generating Functions in Closed Multi-Class Queueing Networks. *Proc. of IEEE MASCOTS*, 231–238, 2004.
- [20] P. G. Harrison and S. Coury. On the asymptotic behaviour of closed multiclass queueing networks. *Performance Evaluation*, 47(2):131–138, 2002.
- [21] C. Knessl, C. Tier. Asymptotic expansions for large closed queueing networks with multiple job classes. *IEEE Trans. Computers*, 41(4):480–488, 1992.
- [22] E. Koenigsberg. Cyclic queues. *Operational Research Quarterly*, 9, 1:22–35, 1958.
- [23] Y. Kogan. Asymptotic expansions for probability distributions in large loss and closed queueing networks. *Perform. Eval. Rev.*, 29(3):25–27, Dec. 2001.
- [24] Y. Kogan, A. Yakovlev. Asymptotic analysis for closed multichain queueing networks with bottlenecks. *QUESTA*, 23:235–258, 1996.
- [25] S. Lam. Dynamic scaling and growth behavior of queueing network normalization constants. *Journal of the ACM*, 29(2):492–513, 1982.
- [26] S. Lam. A simple derivation of the MVA and LBANC algorithms from the convolution algorithm. *IEEE Trans. on Computers*, 32:1062–1064, 1983.
- [27] S. S. Lavenberg. A perspective on queueing models of computer performance. *Performance Evaluation*, 10(1):53–76, 1989.
- [28] M. Reiser and H. Kobayashi. Queueing networks with multiple closed chains: Theory and computational algorithms. *IBM J. Res. Dev.*, 19(3):283–294, 1975.
- [29] W. Wang, G. Casale, C. Sutton. A Bayesian Approach to Parameter Inference in Queueing Networks. *ACM TOMACS*, 27(1), 2016.

A Product-form closed queueing networks

A foundational result in queueing theory is the BCMP theorem [2], which establishes that networks of n queueing stations with d job classes admit a product-form equilibrium distribution when each station operates under one of four scheduling disciplines: First-Come First-Served (FCFS) with class-independent exponential service times, Processor Sharing (PS), Last-Come First-Served with Pre-emptive Resume (LCFS-PR), or Infinite Server (IS). Under these conditions, the equilibrium state probability factorizes across stations as

$$\mathbb{P}(\mathbf{k}) = \frac{1}{G(\boldsymbol{\theta}, \mathbf{N})} \prod_{i=1}^n f_i(\mathbf{k}_i) \quad (12)$$

where $\mathbf{k}_i = (k_{i1}, \dots, k_{id})$ is the vector of jobs at station i , $f_i(\mathbf{k}_i)$ is a function determined by the scheduling discipline and service rates at station i , and $G(\boldsymbol{\theta}, \mathbf{N})$ is a normalizing constant ensuring that probabilities sum to one. The BCMP theorem applies to open, closed, and mixed networks; in what follows, we focus on the closed case.

In closed queueing networks, a finite set of N jobs circulate among a network of n nodes, called queueing stations, according to a set of transition probabilities. Jobs are partitioned into d types, called classes, and the network topology is closed, meaning that the number of jobs inside the network remains constant over time and equal to N_j in class j . Upon visiting a station i , a job of class j is processed at rate $\mu_{ij} = \theta_{ij}^{-1} k_i^{-1}$ when a total of k_i jobs are simultaneously present at the station. Therefore, the larger the number of jobs at a station, the smaller the rate.

A common question that arises in these models is to determine the joint stationary distribution of the number k_{ij} of class- j jobs residing at station i when the Markov process reaches equilibrium. The analytic form of this distribution is known to be [2]

$$\mathbb{P}(\mathbf{k}) = \frac{1}{G(\boldsymbol{\theta}, \mathbf{N})} \prod_{i=1}^n \frac{k_i!}{\prod_{j=1}^d k_{ij}!} \prod_{l=1}^d \theta_{il}^{k_{il}} \quad (13)$$

where $\mathbf{k} \in \mathcal{S}$ is a state vector, $\mathcal{S}$ is a state space given by

$$\mathcal{S} = \left\{ \mathbf{k} \in \mathbb{R}^{nd} \mid k_{ij} \geq 0, \sum_{i=1}^n k_{ij} = N_j \right\}$$

and the normalizing constant is obtained by requiring that $\sum_{\mathbf{k} \in \mathcal{S}} \mathbb{P}(\mathbf{k}) = 1$, *i.e.*,

$$G(\boldsymbol{\theta}, \mathbf{N}) = \sum_{\mathbf{k} \in \mathcal{S}} \prod_{i=1}^n \frac{k_i!}{\prod_{j=1}^d k_{ij}!} \prod_{l=1}^d \theta_{il}^{k_{il}} \quad (14)$$