Program Analysis (70020)

Overview

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Spring 2026

Lectures: 13 January until 26 February 2026

Lecture Theatre 144 on Tuesday (4pm-6pm)
and Thursday (2pm-4pm).

Tutorials typically Tuesdays, first hour.

Coursework Tests 27 January and 17 February

Material and Notes on

https://www.doc.ic.ac.uk/∼herbert/teaching.html

Scientia, Panopto, etc.

Assessment

Coursework Test I: Tue 27 January, 16:00

Coursework Test II: Tue 17 February, 16:00

Examination: Week 11, 16–20 March 2026

Program Analysis

Program analysis is an automated technique for finding out properties of programs without having to execute them.

Static Analysis vs **Dynamic Testing**

- Compiler Optimisation
- Program Verification
- Security Analysis

Unfortunately, the achieving the aims of (static) program analysis tend to be computationally extremely hard.

Program Properties

In some sense Program Analysis is an impossible task.

Decidable Problems There exists an algorithm or computational process which computes a solution (in finite time) for all instances of the problem.

Halting Problem There is no general computational process or machine which can decide whether or not any given program terminates.

Rice Theorem Any non-trivial program property is undecidible.

The approach is to find terminating algorithms for program analysis while not always finding a “meaningful” solution.

Fermat's Program – Terminates?

```
try ← true;
x ← 1;
while try do
    y ← 1;
    while $y \leq x$ && try do
        z ← 1;
        while $z \leq y$ && try do
            try ← $x^3 + y^3 \neq z^3$
            $z \leftarrow z + 1$;
        end while
        $y \leftarrow y + 1$;
    end while
    $x \leftarrow x + 1$;
end while
```

Collatz Problem – Unknown

Take an integer x and compute a sequence of updates:

```
while $x \neq 1$ do
    if $x \bmod 2 = 0$ then
        $x \leftarrow x/2$;
    else
        $x \leftarrow 3 \times x + 1$
    end if
end while
```

Currently it is unknown whether this terminates for all x.

Techniques

Some techniques used in program analysis include:

- Data Flow Analysis
- Control Flow Analysis
- Types and Effects Systems
- Abstract Interpretation

Flemming Nielson, Hanne Riis Nielson and Chris Hankin: *Principles of Program Analysis*. Springer Verlag, 1999/2005.

Xavier Rival and Kwangkeun Yi: *Introduction to Static Analysis – An Abstract Interpretation Perspective*. MIT Press, 2020.

Patrick Cousot: *Principles of Abstract Interpretation*. 2021.

A First Example

Consider the following fragment in *some* procedural language.

```
1: $m \leftarrow 2$;
2: while $n > 1$ do
3:     $m \leftarrow m \times n$;
4:     $n \leftarrow n - 1$
5: end while
6: stop
```

```
$[m \leftarrow 2]^1$;
while $[n > 1]^2$ do
    $[m \leftarrow m \times n]^3$;
    $[n \leftarrow n - 1]^4$
end while
$[\textbf{stop}]^5$
```

We annotate a program such that it becomes clear about what program point we are talking about.

A Parity Analysis

Claim: This program fragment always returns an **even** m, idependently of the initial values of m and n.

We can statically determine that in any circumstances the value of m at the last statement will be **even** for any input n.

A program analysis, so-called parity analysis, can determine this by propagating the even/odd or *parity* information *forwards* form the start of the program.

Properties

We will assign to each variable one of three properties:

- **even** — the value is known to be even
- **odd** — the value is known to be odd
- **unknown** — the parity of the value is unknown

For both variables m and n we record its parity at each stage of the computation (beginning of each statement).

A First Example

Executing the program with *abstract* values, parity, for m and n.

```
1: $m \leftarrow 2$;                    ▷ unknown(m) – unknown(n)
2: while $n > 1$ do                         ▷ even(m) – unknown(n)
3:     $m \leftarrow m \times n$;                 ▷ even(m) – unknown(n)
4:     $n \leftarrow n - 1$                     ▷ even(m) – unknown(n)
5: end while                                ▷ even(m) – unknown(n)
6: stop                                     ▷ even(m) – unknown(n)
```

Important: We can restart the loop!

A First Example

The first program computes 2 times the factorial for any positive value of n. Replacing ‘2’ by ‘1’ in the first statement gives:

```
1: $m \leftarrow 1$;                    ▷ unknown(m) – unknown(n)
2: while $n > 1$ do                     ▷ unknown(m) – unknown(n)
3:     $m \leftarrow m \times n$;             ▷ unknown(m) – unknown(n)
4:     $n \leftarrow n - 1$                 ▷ unknown(m) – unknown(n)
5: end while                            ▷ unknown(m) – unknown(n)
6: stop                                 ▷ unknown(m) – unknown(n)
```

i.e. the factorial – but then the program analysis is unable to tell us anything about the parity of m at the end of the execution.

Loss of Precision

The analysis of the new program does not give a satisfying result because:

- m could be **even** — if the input $n > 1$, or
- m could be **odd** — if the input $n \leq 1$.

However, even if we fix/require the input to be positive and **even** — e.g. by some suitable conditional assignment — the program analysis still might not be able to accurately predict that m will be **even** at statement **5**.

Safe Approximations

Such a loss of precession is a common feature of program analysis: many properties that we are interested in are essentially undecidable and therefore we cannot hope to detect (all of) them accurately.

We only aim to ensure that the answers/results we obtain by program analysis are at least safe, i.e.

- yes means *definitely* yes,
- no means *maybe* no.

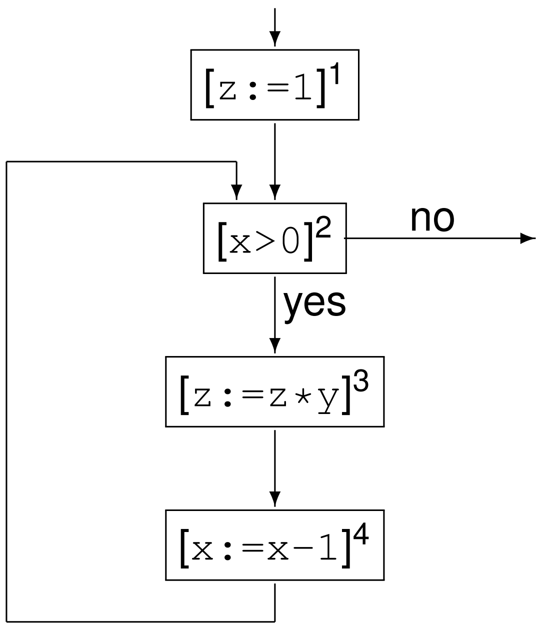
Data Flow Analysis

The starting point for data flow analysis is a representation of the control flow graph of the program: the nodes of such a graph may represent individual statements – as in a flowchart – or sequences of statements; arcs specify how control may be passed during program execution.

The data flow analysis is usually specified as a set of equations which associate analysis information with program points which correspond to the nodes in the control flow graph. This information may be propagated *forwards* through the program (e.g. parity analysis) or *backwards*.

When the control flow graph is not explicitly given, we need a preliminary control flow analysis

Control Flow Information



This allows us to determine the predecessors *pred* and successors *succ* of each statement, e.g. $\textit{pred}(2) = \{1, 4\}$.

Reaching Definition

Reaching Definition (*RD*) analysis determines which set of definitions (i.e. assignments) are current when control reaches a certain program point p.

The analysis can be specified by equations of the form:

$$\mathrm{RD}_{entry}(p) = \begin{cases} \mathrm{RD}_{init} & \text{if } p \text{ is initial} \\ \bigcup_{p' \in pred(p)} \mathrm{RD}_{exit}(p') & \text{otherwise} \end{cases}$$

$$\mathrm{RD}_{exit}(p) = (\mathrm{RD}_{entry}(p) \backslash kill_{\mathrm{RD}}(p)) \cup gen_{\mathrm{RD}}(p)$$

Analysis Information

At each program point some definitions get "killed" (those which define the same variable as at the program point) while others are "generated".

A suitable representation for properties are sets of pairs, where each pair contains a variable x and a program point p: the meaning of the pair (x, p) is that the assignment to x at point p is the current one. The initial value in this case is:

$$\mathrm{RD}_{init} = \{(x, ?) \mid x \text{ is a variable in the program}\}$$

Reaching Definitions is a forward analysis and we require the least (most precise) solutions to the set of equations.

Equations & Solutions

For our initial program fragment

```
$[m \leftarrow 2]^1$;
**while** $[n > 1]^2$ **do**
    $[m \leftarrow m \times n]^3$;
    $[n \leftarrow n - 1]^4$
**end while**
$[\textbf{stop}]^5$
```

some of the *RD* equations we get are:

$$\begin{array}{lcl} \mathrm{RD}_{entry}(1) & = & \{(m,?),(n,?)\} \\ \mathrm{RD}_{entry}(2) & = & \mathrm{RD}_{exit}(1) \cup \mathrm{RD}_{exit}(4) \end{array}$$

Equations & Solutions

$$\begin{array}{lcl} \mathrm{RD}_{entry}(1) & = & \{(m,?),(n,?)\} \\ \mathrm{RD}_{entry}(2) & = & \mathrm{RD}_{exit}(1) \cup \mathrm{RD}_{exit}(4) \end{array}$$

	RD_{entry}	RD_{exit}
1	$\{(m,?),(n,?)\}$	$\{(m,1),(n,?)\}$
2	$\{(m,1),(m,3),(n,?),(n,4)\}$	$\{(m,1),(m,3),(n,?),(n,4)\}$
3	$\{(m,1),(m,3),(n,?),(n,4)\}$	$\{(m,3),(n,?),(n,4)\}$
4	$\{(m,3),(n,?),(n,4)\}$	$\{(m,3),(n,4)\}$
5	$\{(m,1),(m,3),(n,?),(n,4)\}$	$\{(m,1),(m,3),(n,?),(n,4)\}$

Solving Equations

How can we construct solution to the data flow equations?
Answer: Iteratively, by improving approximations/guesses.

INPUT: Control Flow Graph
i.e. initial(p), pred(p).

OUTPUT: Reaching Definitions *RD*.

METHOD: Step 1: Initialisation
Step 2: Iteration

Some Examples

Some examples of data flow analyses — and the possible applications and optimisations they allow for — are:

- Reaching Definitions — Constant Folding
- Available Expressions — Avoid Re-computations
- Very Busy Expressions — Hoisting
- Live Variables — Dead Code Elimination

- *Information Flow — Computer Security*
- *(Probabilistic) Program Slicing*
- *Shape Analyis — Pointer Analysis — etc.*

Code Optimisation

To illustrate the ideas we shall show how Reaching Definitions can be used to perform Constant Folding.

There are two ingredients to this:

- Replace the use of a variable in some expression by a constant if it is known that the value of that variable will always be a constant.
- Simplify an expression by partially evaluating it: subexpressions that contain no variables can be evaluated.

Constant Folding I

$$RD \vdash [\, x := a \,]^{\ell} \;\triangleright\; [\, x := a[y \mapsto n] \,]^{\ell}$$

$$\text{if} \begin{cases} y \in FV(a) \;\wedge\; (y, ?) \notin \mathrm{RD}_{entry}(\ell) \;\wedge \\ \forall (y', \ell') \in \mathrm{RD}_{entry}(\ell) : \\ y' = y \;\Rightarrow\; [\ldots]^{\ell'} = [\, y := n \,]^{\ell'} \end{cases}$$

$$RD \vdash [\, x := a \,]^{\ell} \;\triangleright\; [\, x := n \,]^{\ell}$$

$$\text{if} \begin{cases} FV(a) = \emptyset \;\wedge\; a \text{ is not constant} \;\wedge \\ a \text{ evaluates to } n \end{cases}$$

Constant Folding II

$$\frac{RD \vdash S_1 \ \triangleright\ S_1'}{RD \vdash S_1; S_2 \ \triangleright\ S_1'; S_2}$$

$$\frac{RD \vdash S_2 \ \triangleright\ S_2'}{RD \vdash S_1; S_2 \ \triangleright\ S_1; S_2'}$$

$$\frac{RD \vdash S_1 \ \triangleright\ S_1'}{RD \vdash \textbf{if } [b]^\ell \textbf{ then } S_1 \textbf{ else } S_2 \ \triangleright\ \textbf{if } [b]^\ell \textbf{ then } S_1' \textbf{ else } S_2}$$

$$\frac{RD \vdash S_2 \ \triangleright\ S_2'}{RD \vdash \textbf{if } [b]^\ell \textbf{ then } S_1 \textbf{ else } S_2 \ \triangleright\ \textbf{if } [b]^\ell \textbf{ then } S_1 \textbf{ else } S_2'}$$

$$\frac{RD \vdash S \ \triangleright\ S'}{RD \vdash \textbf{while } [b]^\ell \textbf{ do } S \ \triangleright\ \textbf{while } [b]^\ell \textbf{ do } S'}$$

An Example

To illustrate the use of the transformation consider:

$$[\ x := 10\ ]^1;\ [\ y := x + 10\ ]^2;\ [\ z := y + 10\ ]^3$$

The (least) solution to the Reaching Definition analysis is:

$$\begin{aligned}
\mathrm{RD}_{entry}(1) &= \{(x,?),(y,?),(z,?)\}\\
\mathrm{RD}_{exit}(1) &= \{(x,1),(y,?),(z,?)\}\\
\mathrm{RD}_{entry}(2) &= \{(x,1),(y,?),(z,?)\}\\
\mathrm{RD}_{exit}(2) &= \{(x,1),(y,2),(z,?)\}\\
\mathrm{RD}_{entry}(3) &= \{(x,1),(y,2),(z,?)\}\\
\mathrm{RD}_{exit}(3) &= \{(x,1),(y,2),(z,3)\}
\end{aligned}$$

Constant Folding

We have for example the following:

$$RD \vdash [\, y := x + 10 \,]^2 \;\triangleright\; [\, y := 10 + 10 \,]^2$$

and therfore the rules for sequential composition allow us to do the following transformation:

$$\begin{array}{ll} RD \vdash & [\, x := 10 \,]^1; [\, y := x + 10 \,]^2; [\, z := y + 10 \,]^3 \triangleright \\ & [\, x := 10 \,]^1; [\, y := 10 + 10 \,]^2; [\, z := y + 10 \,]^3 \end{array}$$

Transformation

We can continue this kind of transformation and obtain:

$$\begin{array}{lll} RD & \vdash & [\, x := 10 \,]^1; [\, y := x + 10 \,]^2; [\, z := y + 10 \,]^3 \\ & \triangleright & [\, x := 10 \,]^1; [\, y := 10 + 10 \,]^2; [\, z := y + 10 \,]^3 \\ & \triangleright & [\, x := 10 \,]^1; [\, y := 20 \,]^2; [\, z := y + 10 \,]^3 \\ & \triangleright & [\, x := 10 \,]^1; [\, y := 20 \,]^2; [\, z := 20 + 10 \,]^3 \\ & \triangleright & [\, x := 10 \,]^1; [\, y := 20 \,]^2; [\, z := 30 \,]^3 \end{array}$$

after which no more steps are possible.

Additional Issues

The above example shows that optimisation is in general the result of a number of successive transformations.

$$RD \vdash S_1 \ \triangleright \ S_2 \ \triangleright \ \ldots \ \triangleright \ S_n.$$

This could be costly because one S_1 has been transformed into S_2 we might have to *re-compute* the Reaching Definition analysis before the next transformation step can be done.

It could also be the case that different sequences of transformations either lead to different end results or are of very different length.

Correctness

Any Program Analysis should be:

- unambigously **specified**,
- efficiently **computable**,
- most importantly: **correct**.

For example, why not consider in *RD* before:

$$\mathrm{RD}_{entry}(2) \quad = \quad \mathrm{RD}_{exit}(1) \ \cap \ \mathrm{RD}_{exit}(4)$$

instead of $\mathrm{RD}_{entry}(2) = \mathrm{RD}_{exit}(1) \ \cup \ \mathrm{RD}_{exit}(4)$.

It requires formal (mathematical) proof whether an **analysis** (or **program transformation**) is correct with respect to some model of execution or semantics.

Formal Semantics

A **program** is foremost a text but it has intended **meaning** or semantics describing its execution.

A simple example: Why is $0.\dot{9} = 0.99999\ldots = 1$?

Obviously, these are different strings! However, they have a meaning or semantics as specification of a real number in $\mathbb{R}$. More concretely, infinite strings refer to the limit of their expansion, so $[\![0.\dot{9}]\!] = \lim(0.9, 0.99, 0.999, \ldots) = 1 = [\![1]\!]$.

This course will mostely be concerned with intutive or light-weight semantics when it comes to the “meaning” of a program and the correctness of a program analysis.

Modelling and Specification

Architecture and Structural Engineering

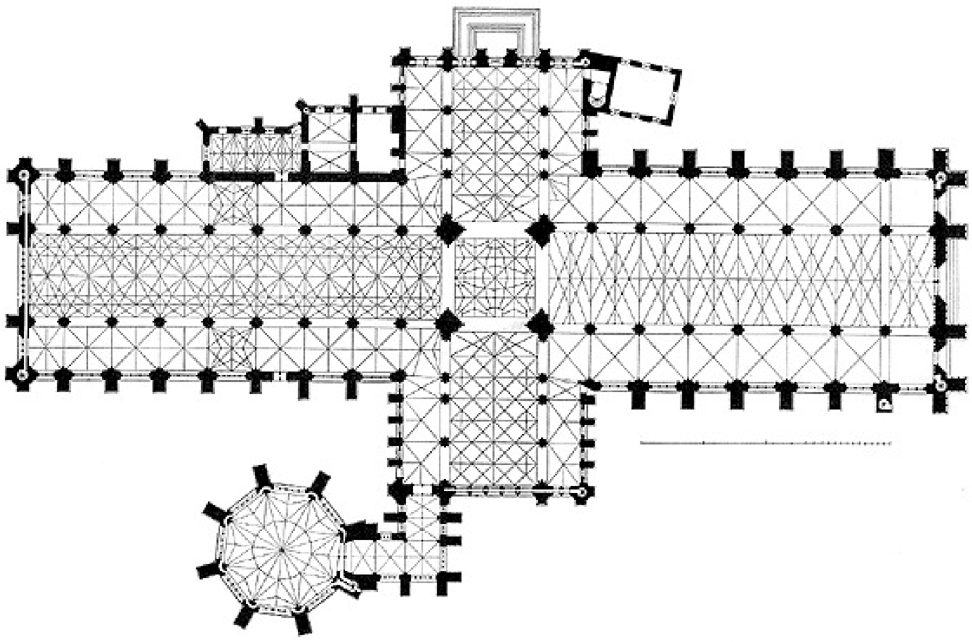


Figure: York Minster

Topics Covered – Executive Summary

- Data Flow Analyis
- Monotone Frameworks
- Control Flow Analysis
- Abstract Interpretation
- Probabilistic Programs
- Probabilistic Abstract Interpretation
- *Further Topics*