Program Analysis (70020)
Overview

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Autumn 2021
Lectures 12 October and 25 November 2021

Hybrid Lectures in Lecture Theatre 145 and on Teams on Tuesday (11am-1pm) and Thursday (2-4pm).
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Tutorials typically Tuesdays, second hour.

Materials and Notes on https://www.doc.ic.ac.uk/~herbert/teaching.html

Assessment

Coursework Test I: Thu 28 October at 15:00
Coursework Test II: Tue 23 November at 12:00
Examination: Week 11, 14 December at 10:00
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**Static Analysis vs Dynamic Testing**
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Static Analysis vs Dynamic Testing

- Compiler Optimisation
Program analysis is an **automated** technique for finding out properties of programs **without** having to execute them.

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Unfortunately, the achieving the aims of (static) program analysis tend to be computationally extremely hard.
Program Properties

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**Decidable Problems** There exists an algorithm or computational process which computes a solution (in finite time) for all instances of the problem.

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**Rice Theorem** Any non-trivial program property is undecidable.
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**Decidable Problems**  There exists an algorithm or computational process which computes a solution (in finite time) for all instances of the problem.

**Halting Problem**  There is no general computational process or machine which can decide whether or not any given program terminates.

**Rice Theorem**  Any non-trivial program property is undecidable.

The approach is to find terminating algorithms for program analysis while not always finding a “meaningful” solution.
Fermat’s Program – Terminates

1: \texttt{try} \leftarrow \texttt{true};
2: \texttt{x} \leftarrow 1;
3: \texttt{while } \texttt{try} \texttt{ do}
4: \quad \texttt{y} \leftarrow 1;
5: \quad \texttt{while } \texttt{y} \leq \texttt{x} \texttt{ && } \texttt{try} \texttt{ do}
6: \quad \quad \texttt{z} \leftarrow 1;
7: \quad \quad \texttt{while } \texttt{z} \leq \texttt{y} \texttt{ && } \texttt{try} \texttt{ do}
8: \quad \quad \quad \texttt{try} \leftarrow \texttt{x}^3 + \texttt{y}^3 \neq \texttt{z}^3
9: \quad \quad \quad \texttt{z} \leftarrow \texttt{z} + 1;
10: \quad \quad \texttt{end while}
11: \quad \texttt{y} \leftarrow \texttt{y} + 1;
12: \quad \texttt{end while}
13: \texttt{x} \leftarrow \texttt{x} + 1;
14: \texttt{end while}
Collatz Problem – Unknown

Take an integer \( x \) and compute a sequence of updates:

1: \textbf{while} \( x \neq 1 \) \textbf{do}
2: \hspace{1em} \textbf{if} \ x \mod 2 = 0 \ \textbf{then}
3: \hspace{2em} x \leftarrow x / 2;
4: \hspace{1em} \textbf{else}
5: \hspace{2em} x \leftarrow 3 \times x + 1
6: \hspace{1em} \textbf{end if}
7: \hspace{1em} \textbf{end while}
Collatz Problem – Unknown

Take an integer $x$ and compute a sequence of updates:

1: while $x \neq 1$ do
2: if $x \mod 2 = 0$ then
3: $x \leftarrow x/2$;
4: else
5: $x \leftarrow 3 \times x + 1$
6: end if
7: end while

Currently it is unknown whether this terminates for all $x$. 
Some techniques used in program analysis include:

- Data Flow Analysis
- Control Flow Analysis
- Types and Effects Systems
- Abstract Interpretation

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Consider the following fragment in *some* procedural language.

1: \( m \leftarrow 2; \)
2: \textbf{while} \( n > 1 \) \textbf{do}
3: \( m \leftarrow m \times n; \)
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We annotate a program such that it becomes clear about what program point we are talking about.
A Parity Analysis

**Claim:** This program fragment always returns an even $m$, independently of the initial values of $m$ and $n$. 
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A Parity Analysis

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We can statically determine that in any circumstances the value of \( m \) at the last statement will be even for any input \( n \).

A program analysis, so-called parity analysis, can determine this by propagating the even/odd or parity information forwards form the start of the program.
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- **unknown** — the parity of the value is unknown
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- **even** — the value is known to be even
- **odd** — the value is known to be odd
- **unknown** — the parity of the value is unknown

For both variables $m$ and $n$ we record its parity at each stage of the computation (beginning of each statement).
A First Example

Executing the program with *abstract* values, parity, for $m$ and $n$.

1:  $m \leftarrow 2$
2:  **while** $n > 1$ **do**
3:    $m \leftarrow m \times n$
4:    $n \leftarrow n - 1$
5:  **end while**
6:  **stop**
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\( \triangleright \) unknown(m) – unknown(n)
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5: \( \textbf{end} \ \textbf{while} \)
6: \( \textbf{stop} \)

\[\begin{align*}
\triangleright \ & \text{unknown}(m) - \text{unknown}(n) \\
\triangleright \ & \text{even}(m) - \text{unknown}(n)
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▷ unknown($m$) – unknown($n$)
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▷ **unknown(\( m \)) – unknown(\( n \))**
▷ **even(\( m \)) – unknown(\( n \))**

Important: We can restart the loop!
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A First Example

Executing the program with *abstract* values, parity, for \( m \) and \( n \).

1: \( m \leftarrow 2; \)  \( \triangleright \) unknown\((m) – unknown\((n)\)  
2: \textbf{while} \( n > 1 \) \textbf{do}  \( \triangleright \) even\((m) – unknown\((n)\)  
3: \( m \leftarrow m \times n; \)  \( \triangleright \) even\((m) – unknown\((n)\)  
4: \( n \leftarrow n – 1 \)  \( \triangleright \) even\((m) – unknown\((n)\)  
5: \textbf{end while}  \( \triangleright \) even\((m) – unknown\((n)\)  
6: \textbf{stop}
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Executing the program with *abstract* values, parity, for \( m \) and \( n \).

1: \( m \leftarrow 2; \) \( \triangleright \) unknown\( (m) \) – unknown\( (n) \)
2: **while** \( n > 1 \) **do** \( \triangleright \) even\( (m) \) – unknown\( (n) \)
3: \( m \leftarrow m \times n; \) \( \triangleright \) even\( (m) \) – unknown\( (n) \)
4: \( n \leftarrow n – 1 \) \( \triangleright \) even\( (m) \) – unknown\( (n) \)
5: **end while** \( \triangleright \) even\( (m) \) – unknown\( (n) \)
6: **stop** \( \triangleright \) even\( (m) \) – unknown\( (n) \)

Important: We can restart the loop!
A First Example

The first program computes 2 times the factorial for any positive value of \( n \). Replacing ‘2’ by ‘1’ in the first statement gives:

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i.e. the factorial – but then the program analysis is unable to tell us anything about the parity of \( m \) at the end of the execution.
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3: \hspace{0.5cm} m \leftarrow m \times n$
4: \hspace{0.5cm} $n \leftarrow n - 1$
5: \textbf{end while}$
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\[\text{\triangleright unknown}(m) - \text{unknown}(n)\]
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\end{align*}
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i.e. the factorial – but then the program analysis is unable to tell us anything about the parity of \( m \) at the end of the execution.
Loss of Precision

The analysis of the new program does not give a satisfying result because:

▶ \( m \) could be even — if the input \( n > 1 \), or
▶ \( m \) could be odd — if the input \( n \leq 1 \).

However, even if we fix/require the input to be positive and even — e.g. by some suitable conditional assignment — the program analysis still might not be able to accurately predict that \( m \) will be even at statement 5.
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However, even if we fix/require the input to be positive and **even** — e.g. by some suitable conditional assignment — the program analysis still might not be able to accurately predict that $m$ will be **even** at statement 5.
Safe Approximations

Such a loss of precession is a common feature of program analysis: many properties that we are interested in are essentially **undecidable** and therefore we cannot hope to detect (all of) them accurately.
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We only aim to ensure that the answers/results we obtain by program analysis are at least **safe**, i.e.

- **yes** means *definitely* yes,
- **no** means *maybe* no.
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The data flow analysis is usually specified as a set of equations which associate analysis information with program points which correspond to the nodes in the control flow graph. This information may be propagated forwards through the program (e.g. parity analysis) or backwards.
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When the control flow graph is not explicitly given, we need a preliminary control flow analysis
Control Flow Information

This allows us to determine the predecessors \(\text{pred}\) and successors \(\text{succ}\) of each statement, e.g. \(\text{pred}(2) = \{1, 4\}\).
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RD_{entry}(p) = \begin{cases} 
RD_{init} & \text{if } p \text{ is initial} \\
\bigcup_{p' \in \text{pred}(p)} RD_{exit}(p') & \text{otherwise}
\end{cases}
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\]

\[
RD_{\text{exit}}(p) = (RD_{\text{entry}}(p) \setminus \text{kill}_{RD}(p)) \cup \text{gen}_{RD}(p)
\]
At each program point some definitions get “killed” (those which define the same variable as at the program point) while others are “generated”.

A suitable representation for properties are sets of pairs, where each pair contains a variable \( x \) and a program point \( p \): the meaning of the pair \((x, p)\) is that the assignment to \( x \) at point \( p \) is the current one.

The initial value in this case is:

\[
RD_{init} = \{ (x, ?) \mid x \text{ is a variable in the program} \}
\]
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Analysis Information

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Reaching Definitions is a forward analysis and we require the least (most precise) solutions to the set of equations.
Equations & Solutions

For our initial program fragment

\[ m \leftarrow 2 \]

\textbf{while} \ [n > 1] \do

\[ m \leftarrow m \times n \]
\[ n \leftarrow n - 1 \]

\textbf{end while}

\[ \text{stop} \]
Equations & Solutions

For our initial program fragment

\[
[m \leftarrow 2]^1;
\]

\textbf{while} \ [n > 1]^2 \textbf{ do}

\[
[m \leftarrow m \times n]^3;
\]

\[
[n \leftarrow n - 1]^4
\]

\textbf{end while}

\[
\text{stop}^5
\]

some of the \textit{RD} equations we get are:

\[
\text{RD}_{\text{entry}}(1) = \{(m,?), (n,?)\}
\]

\[
\text{RD}_{\text{entry}}(2) = \text{RD}_{\text{exit}}(1) \cup \text{RD}_{\text{exit}}(4)
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Equations & Solutions

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\end{align*}
\]
Equations & Solutions

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<table>
<thead>
<tr>
<th></th>
<th>\text{RD}_{\text{entry}}</th>
<th>\text{RD}_{\text{exit}}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{(m,?), (n,?)}</td>
<td>{(m,1), (n,?)}</td>
</tr>
<tr>
<td>2</td>
<td>{(m,1), (m,3), (n,?), (n,4)}</td>
<td>{(m,1), (m,3), (n,?), (n,4)}</td>
</tr>
<tr>
<td>3</td>
<td>{(m,1), (m,3), (n,?), (n,4)}</td>
<td>{(m,3), (n,?), (n,4)}</td>
</tr>
<tr>
<td>4</td>
<td>{(m,3), (n,?), (n,4)}</td>
<td>{(m,3), (n,4)}</td>
</tr>
<tr>
<td>5</td>
<td>{(m,1), (m,3), (n,?), (n,4)}</td>
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How can we construct solution to the data flow equations?
Answer: Iteratively, by improving approximations/guesses.
Solving Equations

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INPUT: Control Flow Graph i.e. initial(p), pred(p).
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OUTPUT: Reaching Definitions \( RD \).
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INPUT: Control Flow Graph
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OUTPUT: Reaching Definitions $RD$.

METHOD: Step 1: Initialisation
Step 2: Iteration
Some Examples

Some examples of data flow analyses — and the possible applications and optimisations they allow for — are:

- Reaching Definitions — Constant Folding
- Available Expressions — Avoid Re-computations
- Very Busy Expressions — Hoisting
- Live Variables — Dead Code Elimination
- Information Flow — Computer Security
- (Probabilistic) Program Slicing
- Shape Analysis — Pointer Analysis — etc.
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To illustrate the ideas we shall show how Reaching Definitions can be used to perform Constant Folding.
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- Replace the use of a variable in some expression by a constant if it is known that the value of that variable will always be a constant.
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- Replace the use of a variable in some expression by a constant if it is known that the value of that variable will always be a constant.
- Simplify an expression by partially evaluating it: subexpressions that contain no variables can be evaluated.
\[
RD \vdash [ x := a ]^\ell \triangleright [ x := a[y \mapsto n] ]^\ell
\]

\[
\text{if } \left\{ \begin{array}{l}
y \in FV(a) \land (y, ?) \notin RD_{\text{entry}}(\ell) \land \\
\forall (y', \ell') \in RD_{\text{entry}}(\ell) : \\
y' = y \Rightarrow [\ldots]^{\ell'} = [ y := n ]^{\ell'}
\end{array} \right.
\]
Constant Folding I

\[ RD \vdash [ x := a ]^{\ell} \triangleright [ x := a[y \mapsto n] ]^{\ell} \]

if \( \begin{cases} y \in \text{FV}(a) \land (y, ?) \not\in \text{RD}_{\text{entry}}(\ell) \land \\
\forall (y', \ell') \in \text{RD}_{\text{entry}}(\ell) : \\
y' = y \Rightarrow [\ldots]^{\ell'} = [ y := n ]^{\ell'} \end{cases} \)

\[ RD \vdash [ x := a ]^{\ell} \triangleright [ x := n ]^{\ell} \]

if \( \begin{cases} \text{FV}(a) = \emptyset \land a \text{ is not constant} \land \\
a \text{ evaluates to } n \end{cases} \)
Constant Folding II

\[
\begin{align*}
RD \vdash S_1 & \triangleright S'_1 \\
RD \vdash S_1; S_2 & \triangleright S'_1; S_2
\end{align*}
\]
Constant Folding II

\[
\begin{align*}
RD \vdash S_1 & \triangleright S'_1 \\
RD \vdash S_1 ; S_2 & \triangleright S'_1 ; S_2 \\
RD \vdash S_2 & \triangleright S'_2 \\
RD \vdash S_1 ; S_2 & \triangleright S_1 ; S'_2
\end{align*}
\]
Constant Folding II

\[
\begin{align*}
RD ⊢ S_1 & \succ S'_1 \\
RD ⊢ S_1 ; S_2 & \succ S'_1 ; S_2 \\
RD ⊢ S_2 & \succ S'_2 \\
RD ⊢ S_1 ; S_2 & \succ S_1 ; S'_2 \\
RD ⊢ S_1 & \succ S'_1 \\
RD ⊢ \text{if } [b]^\ell \text{ then } S_1 \text{ else } S_2 & \succ \text{if } [b]^\ell \text{ then } S'_1 \text{ else } S_2
\end{align*}
\]
Constant Folding II

\[
\begin{align*}
RD \vdash S_1 & \triangleright S'_1 \\
RD \vdash S_1; S_2 & \triangleright S'_1; S_2 \\
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RD \vdash S_1; S_2 & \triangleright S_1; S'_2 \\
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RD \vdash \text{if } [b]^{\ell} \text{ then } S_1 \text{ else } S_2 & \triangleright \text{if } [b]^{\ell} \text{ then } S'_1 \text{ else } S_2 \\
RD \vdash S_2 & \triangleright S'_2 \\
RD \vdash \text{if } [b]^{\ell} \text{ then } S_1 \text{ else } S_2 & \triangleright \text{if } [b]^{\ell} \text{ then } S_1 \text{ else } S'_2
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\]
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\begin{align*}
RD \vdash S_1 & \triangleright S'_1 \\
RD \vdash S_1; S_2 & \triangleright S'_1; S_2 \\
RD \vdash S_2 & \triangleright S'_2 \\
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RD \vdash S & \triangleright S' \\
RD \vdash \text{while } [b]^\ell \text{ do } S & \triangleright \text{while } [b]^\ell \text{ do } S'
\end{align*}
\]
An Example

To illustrate the use of the transformation consider:

\[
\begin{align*}
x &:= 10 \\
y &:= x + 10 \\
z &:= y + 10
\end{align*}
\]
An Example

To illustrate the use of the transformation consider:

\[
\begin{align*}
[\ x := 10 ]^1; & \ [\ y := x + 10 ]^2; \ [\ z := y + 10 ]^3 \\
\end{align*}
\]

The (least) solution to the Reaching Definition analysis is:

\[
\begin{align*}
\text{RD}_{\text{entry}}(1) &= \{(x,?), (y,?), (z,?)\} \\
\text{RD}_{\text{exit}}(1) &= \{(x,1), (y,?), (z,?)\} \\
\text{RD}_{\text{entry}}(2) &= \{(x,1), (y,?), (z,?)\} \\
\text{RD}_{\text{exit}}(2) &= \{(x,1), (y,2), (z,?)\} \\
\text{RD}_{\text{entry}}(3) &= \{(x,1), (y,2), (z,?)\} \\
\text{RD}_{\text{exit}}(3) &= \{(x,1), (y,2), (z,3)\}
\end{align*}
\]
We have for example the following:

\[
RD \vdash [ y := x + 10 ]^2 \triangleright [ y := 10 + 10 ]^2
\]
We have for example the following:

$$RD \vdash [y := x + 10]^2 \triangleright [y := 10 + 10]^2$$

and therefore the rules for sequential composition allow us to do the following transformation:

$$RD \vdash [x := 10]^1; [y := x + 10]^2; [z := y + 10]^3 \triangleright [x := 10]^1; [y := 10 + 10]^2; [z := y + 10]^3$$
We can continue this kind of transformation and obtain:
Transformation

We can continue this kind of transformation and obtain:

\[
RD \vdash [\ x := 10 \ ]^1; [\ y := x + 10 \ ]^2; [\ z := y + 10 \ ]^3
\]
We can continue this kind of transformation and obtain:

\[
RD \vdash [x := 10]^1; [y := x + 10]^2; [z := y + 10]^3
\]
\[
\Rightarrow [x := 10]^1; [y := 10 + 10]^2; [z := y + 10]^3
\]
We can continue this kind of transformation and obtain:

\[ RD \vdash [ x := 10 ]^1; [ y := x + 10 ]^2; [ z := y + 10 ]^3 \]
\[ \triangleright [ x := 10 ]^1; [ y := 10 + 10 ]^2; [ z := y + 10 ]^3 \]
\[ \triangleright [ x := 10 ]^1; [ y := 20 ]^2; [ z := y + 10 ]^3 \]
We can continue this kind of transformation and obtain:

\[
\begin{align*}
RD & \vdash [x := 10]^1; [y := x + 10]^2; [z := y + 10]^3 \\
& \triangleright [x := 10]^1; [y := 10 + 10]^2; [z := y + 10]^3 \\
& \triangleright [x := 10]^1; [y := 20]^2; [z := y + 10]^3 \\
& \triangleright [x := 10]^1; [y := 20]^2; [z := 20 + 10]^3
\end{align*}
\]
We can continue this kind of transformation and obtain:

\[
\begin{align*}
RD \vdash [ \ x := 10 \ ]^1; & \ [ \ y := x + 10 \ ]^2; \ [ \ z := y + 10 \ ]^3 \\
\triangleright [ \ x := 10 \ ]^1; & \ [ \ y := 10 + 10 \ ]^2; \ [ \ z := y + 10 \ ]^3 \\
\triangleright [ \ x := 10 \ ]^1; & \ [ \ y := 20 \ ]^2; \ [ \ z := y + 10 \ ]^3 \\
\triangleright [ \ x := 10 \ ]^1; & \ [ \ y := 20 \ ]^2; \ [ \ z := 20 + 10 \ ]^3 \\
\triangleright [ \ x := 10 \ ]^1; & \ [ \ y := 20 \ ]^2; \ [ \ z := 30 \ ]^3
\end{align*}
\]
Transformation

We can continue this kind of transformation and obtain:

\[
\begin{align*}
RD & \quad \vdash [ \, x := 10 \, ]^1; [ \, y := x + 10 \, ]^2; [ \, z := y + 10 \, ]^3 \\
& \quad \quad \triangleright [ \, x := 10 \, ]^1; [ \, y := 10 + 10 \, ]^2; [ \, z := y + 10 \, ]^3 \\
& \quad \quad \triangleright [ \, x := 10 \, ]^1; [ \, y := 20 \, ]^2; [ \, z := y + 10 \, ]^3 \\
& \quad \quad \triangleright [ \, x := 10 \, ]^1; [ \, y := 20 \, ]^2; [ \, z := 20 + 10 \, ]^3 \\
& \quad \quad \triangleright [ \, x := 10 \, ]^1; [ \, y := 20 \, ]^2; [ \, z := 30 \, ]^3 \\
\end{align*}
\]

after which no more steps are possible.
The above example shows that optimisation is in general the result of a number of successive transformations.

\[ RD \vdash S_1 \triangleright S_2 \triangleright \ldots \triangleright S_n. \]
Additional Issues

The above example shows that optimisation is in general the result of a number of successive transformations.

\[ RD \vdash S_1 \quad \triangleright \quad S_2 \quad \triangleright \quad \ldots \quad \triangleright \quad S_n. \]

This could be costly because one \( S_1 \) has been transformed into \( S_2 \) we might have to *re-compute* the Reaching Definition analysis before the next transformation step can be done.
The above example shows that optimisation is in general the result of a number of successive transformations.

\[ RD \vdash S_1 \triangleright S_2 \triangleright \ldots \triangleright S_n. \]

This could be costly because one \( S_1 \) has been transformed into \( S_2 \) we might have to \emph{re-compute} the Reaching Definition analysis before the next transformation step can be done.

It could also be the case that different sequences of transformations either lead to different end results or are of very different length.
Correctness

Any Program Analysis should be:

- unambiguously specified,
- efficiently computable,
- most importantly: correct.
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For example, why not consider in $RD$ before:

$$RD_{entry}(2) = RD_{exit}(1) \cap RD_{exit}(4)$$

instead of $RD_{entry}(2) = RD_{exit}(1) \cup RD_{exit}(4)$.
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For example, why not consider in $RD$ before:

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instead of $RD_{\text{entry}}(2) = RD_{\text{exit}}(1) \cup RD_{\text{exit}}(4)$. 

It requires formal (mathematical) proof whether an \textit{analysis} (or \textit{program transformation}) is \textit{correct} with respect to some model of execution or semantics.
A program is foremost a text but it has intended meaning or semantics describing its execution.
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A simple example: Why is $0.9 = 0.99999 \ldots = 1$?
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Obviously, these are different strings! However, they have a meaning or semantics as specification of a real number in $\mathbb{R}$. 
A program is foremost a text but it has intended meaning or semantics describing its execution.

A simple example: Why is 0.\dot{9} = 0.99999 \ldots = 1? Obviously, these are different strings! However, they have a meaning or semantics as specification of a real number in \( \mathbb{R} \). More concretely, infinite strings refer to the limit of their expansion, so \([0.\dot{9}] = \lim(0.9, 0.99, 0.999, \ldots) = 1 = [1]\).
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A simple example: Why is $0.\overline{9} = 0.99999\ldots = 1$?

Obviously, these are different strings! However, they have a meaning or semantics as specification of a real number in $\mathbb{R}$. More concretely, infinite strings refer to the limit of their expansion, so $[0.\overline{9}] = \lim(0.9, 0.99, 0.999, \ldots) = 1 = [1]$.

This course will mostly be concerned with intuitive or light-weight semantics when it comes to the “meaning” of a program and the correctness of a program analysis.
Topics Covered – Executive Summary

- Data Flow Analysis
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- Data Flow Analysis
- Monotone Frameworks
Topics Covered – Executive Summary

- Data Flow Analysis
- Monotone Frameworks
- Control Flow Analysis
Topics Covered – Executive Summary

- Data Flow Analysis
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- Abstract Interpretation
Topics Covered – Executive Summary

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