Program Analysis (CO470/97128/97146)
Monotone Frameworks

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$$\sqsubseteq : L \times L \rightarrow \{\tt, \ff\} \quad \text{or} \quad \sqsubseteq \subseteq L \times L$$

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- **reflexive** $\forall l : l \sqsubseteq l$, 

A partially ordered set $(L, \sqsubseteq)$ is a set $L$ equipped with a partial ordering $\sqsubseteq$ (sometimes written $\sqsubseteq_L$). We shall write $l_2 \sqsubseteq l_1$ for $l_1 \sqsubseteq l_2$ and $l_1 \sqsubseteq l_2$ for $l_1 \sqsubseteq l_2$ and $l_1 \neq l_2$. 

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Partially Ordered Set

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- **reflexive** $\forall l : l \sqsubseteq l$,
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- **anti-symmetric** $\forall l_1, l_2 : l_1 \subseteq l_2 \land l_2 \subseteq l_1 \Rightarrow l_1 = l_2$. 


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- **anti-symmetric** \( \forall l_1, l_2 : l_1 \sqsubseteq l_2 \land l_2 \sqsubseteq l_1 \Rightarrow l_1 = l_2 \).

A partially ordered set \((L, \sqsubseteq)\) is a set \( L \) equipped with a partial ordering \( \sqsubseteq \) (sometimes written \( \sqsubseteq_L \)). We shall write \( l_2 \sqsubseteq l_1 \) for \( l_1 \sqsubseteq l_2 \) and \( l_1 \sqsubset l_2 \) for \( l_1 \sqsubseteq l_2 \land l_1 \neq l_2 \).
Examples of POS’s

Example: Integers
The integers $\mathbb{Z}$ ordered in the usual way, i.e. for $i_1, i_2 \in \mathbb{Z}$:

$$i_1 \sqsubseteq i_2 \text{ iff } i_1 \leq i_2$$
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Example: Power-Set
Take a (finite) set $X$ and consider at the set of all sub-sets of $X$, i.e. its power set $\mathcal{P}(X)$. A partial ordering on $\mathcal{P}(X)$ is given by inclusion, i.e. for two sub-sets $S_1, S_2 \in \mathcal{P}(X)$:

\[ S_1 \equiv S_2 \text{ iff } S_1 \subseteq S_2 \]
Upper/Lower Bounds

Given a partially ordered set \((L, \sqsubseteq)\).

A subset \(Y\) of \(L\) has \(l \in L\) as an upper bound if

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\forall l' \in Y : l' \sqsubseteq l
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and as a lower bound if

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\forall l' \in Y : l' \sqsupseteq l.
\]
Least Upper/Greatest Lower Bounds

Given a partially ordered set \((L, \sqsubseteq)\) and \(Y \subseteq L\).

A least upper bound \(l\) of \(Y\) is an upper bound of \(Y\) that satisfies \(l \sqsubseteq l_0\) whenever \(l_0\) is another upper bound of \(Y\);
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Note that subsets \(Y\) of a partially ordered set \(L\) need not have least upper bounds nor greatest lower bounds but when they exist they are unique (since \(\sqsubseteq\) is anti-symmetric) and they are denoted \(\sqcup Y\) and \(\sqcap Y\), respectively.
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Note that subsets \(Y\) of a partially ordered set \(L\) need not have least upper bounds nor greatest lower bounds but when they exist they are unique (since \(\sqsubseteq\) is anti-symmetric) and they are denoted \(\sqcup Y\) and \(\sqcap Y\), respectively.

Sometimes \(\sqcup\) is called the **join operator** and \(\sqcap\) the **meet operator** and we shall write \(l_1 \sqcup l_2\) for \(\sqcup\{l_1, l_2\}\) and similarly \(l_1 \sqcap l_2\) for \(\sqcap\{l_1, l_2\}\).
A complete lattice

\[ L = (L, \sqsubseteq) = (L, \sqsubseteq, \sqcup, \sqcap, \bot, \top) \]

is a partially ordered set \((L, \sqsubseteq)\) such that all subsets have least upper bounds as well as greatest lower bounds.
A complete lattice

\[ L = (L, \sqsubseteq) = (L, \sqsubseteq, \sqcup, \sqcap, \perp, \top) \]

is a partially ordered set \((L, \sqsubseteq)\) such that all subsets have least upper bounds as well as greatest lower bounds.

Furthermore, define \( \perp = \bigcup \emptyset = \bigcap L \) is the least element and \( \top = \bigcap \emptyset = \bigcup L \) is the greatest element.
Take a (finite) set $X$ and look again at its power set $\mathcal{P}(X)$. A partial ordering ‘$\subseteq$’ on $\mathcal{P}(X)$ is given as above by inclusion ‘$\subseteq$’.
Power-Set Lattice

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The *meet* and *join* operators are given by (set) *intersection*

$$S_1 \cap S_2 = S_1 \cap S_2$$

and (set) *union*

$$S_1 \cup S_2 = S_1 \cup S_2.$$
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The least and greatest elements in $\mathcal{P}(X)$ are given by $\bot = \emptyset$ and $\top = X$. 
Power-Set: Hasse Diagrams

\{1, 2, 3\}

\{1, 2\} \rightarrow \{1, 3\} \rightarrow \{2, 3\}

\{1\} \rightarrow \{2\} \rightarrow \{3\}

\emptyset
Power-Set: Hasse Diagrams
A function $f : L_1 \to L_2$ between two partially ordered sets $L_1 = (L_1, \sqsubseteq_1)$ and $L_2 = (L_2, \sqsubseteq_2)$ is **monotone** (or **isotone** or **order-preserving**) if

$$\forall l, l' \in L_1 : l \sqsubseteq_1 l' \Rightarrow f(l) \sqsubseteq_2 f(l')$$
Properties of Functions I

A function \( f : L_1 \to L_2 \) between two partially ordered sets \( L_1 = (L_1, \sqsubseteq_1) \) and \( L_2 = (L_2, \sqsubseteq_2) \) is monotone (or isotone or order-preserving) if

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\forall l, l' \in L_1 : l \sqsubseteq_1 l' \Rightarrow f(l) \sqsubseteq_2 f(l')
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A function \( f : L_1 \to L_2 \) is an additive function (or a join morphism, sometimes called a distributive function) if

\[
\forall l_1, l_2 \in L_1 : f(l_1 \sqcup l_2) = f(l_1) \sqcup f(l_2)
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A function $f : L_1 \rightarrow L_2$ is an additive function (or a join morphism, sometimes called a distributive function) if

$$\forall l_1, l_2 \in L_1 : f(l_1 \sqcup l_2) = f(l_1) \sqcup f(l_2)$$

and it is called a multiplicative function (or a meet morphism) if

$$\forall l_1, l_2 \in L_1 : f(l_1 \sqcap l_2) = f(l_1) \sqcap f(l_2)$$
The function $f : L_1 \rightarrow L_2$ is a *completely additive* function (or a *complete join morphism*) if for all $Y \subseteq L_1$:

$$f \left( \bigsqcup_1 Y \right) = \bigsqcup_2 \{ f(l') \mid l' \in Y \}$$

whenever $\bigsqcup_1 Y$ exists.
The function \( f : L_1 \to L_2 \) is a completely additive function (or a complete join morphism) if for all \( Y \subseteq L_1 \):

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f \left( \bigsqcup_1 Y \right) = \bigsqcup_2 \{ f(l') \mid l' \in Y \} \text{ whenever } \bigsqcup_1 Y \text{ exists}
\]

and it is completely multiplicative (or a complete meet morphism) if for all \( Y \subseteq L_1 \):

\[
f \left( \bigcap_1 Y \right) = \bigcap_2 \{ f(l') \mid l' \in Y \} \text{ whenever } \bigcap_1 Y \text{ exists}
\]
Cartesian Product $L_1 \times L_2$

Let $L_1 = (L_1, \sqsubseteq_1)$ and $L_2 = (L_2, \sqsubseteq_2)$ be partially ordered sets. Define $L = (L, \sqsubseteq)$ by

$L = L_1 \times L_2 = \{(l_1, l_2) \mid l_1 \in L_1 \land l_2 \in L_2\}$

$(l_{11}, l_{21}) \sqsubseteq (l_{12}, l_{22})$ iff $l_{11} \sqsubseteq_1 l_{12} \land l_{21} \sqsubseteq_2 l_{22}$
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$$L = L_1 \times L_2 = \{ (l_1, l_2) | l_1 \in L_1 \land l_2 \in L_2 \}$$

$$(l_{11}, l_{21}) \sqsubseteq (l_{12}, l_{22}) \text{ iff } l_{11} \sqsubseteq_1 l_{12} \land l_{21} \sqsubseteq_2 l_{22}$$

If additionally each $L_i = (L_i, \sqsubseteq_i, \sqcup_i, \sqcap_i, \bot_i, \top_i)$ is a complete lattice then so is $L = (L, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ and furthermore

$$\bigcup Y = ( \bigcup_1 \{ l_1 | \exists l_2 : (l_1, l_2) \in Y \} , \bigcup_2 \{ l_2 | \exists l_1 : (l_1, l_2) \in Y \} )$$

and $\bot = (\bot_1, \bot_2)$ and similarly for $\bigcap Y$ and $\top$. 

Cartesian Product $L_1 \times L_2$
Let $L_1 = (L_1, \sqsubseteq_1)$ be a partially ordered set and let $S$ be a set. Define $L = (L, \sqsubseteq)$ by

$$L = \{ f : S \to L_1 \mid f \text{ is a total function} \}$$

$$f \sqsubseteq f' \iff \forall s \in S : f(s) \sqsubseteq_1 f'(s)$$
Total Function Space $S \to L_1$

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If additionally $L_1 = (L_1, \sqsubseteq_1, \bigcup_1, \bigcap_1, \bot_1, \top_1)$ is a complete lattice then so is $L = (L, \sqsubseteq, \bigcup, \bigcap, \bot, \top)$ and furthermore

$$\bigcup Y = \lambda s. \bigcup_1 \{ f(s) \mid f \in Y \}$$

and $\bot = \lambda s. \bot_1$ and similarly for $\bigcap Y$ and $\top$. 
Chains

A subset $Y \subseteq L$ of a partially ordered set $L = (L, \sqsubseteq)$ is a chain if

$$\forall l_1, l_2 \in Y : (l_1 \sqsubseteq l_2) \lor (l_2 \sqsubseteq l_1)$$

Thus a chain is a (possibly empty) subset of $L$ that is totally ordered.

We shall say that it is a finite chain if it is a finite subset of $L$. 
Ascending and Descending Chains

A sequence \((l_n)_n = (l_n)_{n \in \mathbb{N}}\) of elements in \(L\) is an \textit{ascending chain} if

\[ n \leq m \Rightarrow l_n \subseteq l_m \]

Writing \((l_n)_n\) also for \(\{l_n \mid n \in \mathbb{N}\}\) it is clear that an ascending chain also is a chain.
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Similarly, a sequence \((l_n)_n\) is a \textit{descending chain} if

\[ n \leq m \Rightarrow l_n \sqsupseteq l_m \]
We shall say that a sequence \((l_n)_n\) *eventually stabilises* if and only if

\[ \exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N} : n \geq n_0 \Rightarrow l_n = l_{n_0} \]
Stabilising Chains

We shall say that a sequence \((l_n)_n\) \textit{eventually stabilises} if and only if

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For the sequence \((l_n)_n\) we write \(\bigcup_n l_n\) for \(\bigcup \{l_n \mid n \in \mathbb{N}\}\) and similarly we write \(\bigcap_n l_n\) for \(\bigcap \{l_n \mid n \in \mathbb{N}\}\).
We shall say that a partially ordered set $L = (L, \sqsubseteq)$ has \textit{finite height} if and only if all chains are finite.
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It has finite height \textit{at most} \( h \) if all chains contain at most \( h + 1 \) elements; it has finite height \( h \) if additionally there is a chain with \( h + 1 \) elements.
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A partially ordered set $L$ satisfies the \textit{Ascending Chain Condition} (ACC) if and only if all ascending chains eventually stabilise.
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A partially ordered set $L$ satisfies the \textit{Ascending Chain Condition} (ACC) if and only if all ascending chains eventually stabilise.

A partially ordered set $L$ satisfies the \textit{Descending Chain Condition} (DCC) if and only if all descending chains eventually stabilise.
Chain Examples

\[ \bullet \quad 0 \quad \bullet \quad -1 \quad \bullet \quad -2 \quad \bullet \quad -3 \quad \bullet \quad -\infty \]
Chain Examples

\[ \begin{align*}
0 & \quad -1 & \quad -2 & \quad -3 & \quad -\infty \\
\infty & \quad 3 & \quad 2 & \quad 1 & \quad 0
\end{align*} \]
Reductive and Extensive Functions

Consider a **monotone** function $f : L \rightarrow L$ on a **complete** lattice $L$. 

A fixed point of $f$ is an element $l \in L$ such that $f(l) = l$, we write $\text{Fix}(f) = \{l | f(l) = l\}$ for the set of fixed points.

The function $f$ is **reductive** at $l$ if and only if $f(l) \sqsubseteq l$ and we write $\text{Red}(f) = \{l | f(l) \sqsubseteq l\}$ for the set of elements upon which $f$ is reductive; we shall say that $f$ itself is **reductive** if $\text{Red}(f) = L$.

Similarly, the function $f$ is **extensive** at $l$ if and only if $f(l) \sqsupseteq l$, we write $\text{Ext}(f) = \{l | f(l) \sqsupseteq l\}$.
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$$\text{Ext}(f) = \{ l \mid f(l) \sqsupseteq l \}$$
Fixed Points

Since $L$ is a complete lattice it is always the case that the set $\text{Fix}(f)$ will have a greatest lower bound in $L$ and we denote it by $\text{lfp}(f)$:

$$\text{lfp}(f) = \bigsqcap \text{Fix}(f) = \bigsqcap \text{Red}(f) \in \text{Fix}(f) \subseteq \text{Red}(f)$$
Fixed Points

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\]

Similarly, the set $\text{Fix}(f)$ will have a least upper bound in $L$ and we denote it by $\text{gfp}(f)$:

\[
\text{gfp}(f) = \bigcup \text{Fix}(f) = \bigcup \text{Ext}(f) \in \text{Fix}(f) \subseteq \text{Ext}(f)
\]
Existence of Fixed Points

If $L$ satisfies the Ascending Chain Condition then there exists $n$ such that $f^n(\bot) = f^{n+1}(\bot)$ and hence

$$\text{lfp}(f) = f^n(\bot).$$
Existence of Fixed Points

If $L$ satisfies the Ascending Chain Condition then there exists $n$ such that $f^n(\bot) = f^{n+1}(\bot)$ and hence

$$lfp(f) = f^n(\bot).$$

If $L$ satisfies the Descending Chain Condition then there exists $n$ such that $f^n(\top) = f^{n+1}(\top)$ and hence

$$gfp(f) = f^n(\top).$$
Existence of Fixed Points

If $L$ satisfies the Ascending Chain Condition then there exists $n$ such that $f^n(\bot) = f^{n+1}(\bot)$ and hence

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If $L$ satisfies the Descending Chain Condition then there exists $n$ such that $f^n(\top) = f^{n+1}(\top)$ and hence

$$\text{gfp}(f) = f^n(\top).$$

Indeed any monotone function $f$ over a partially ordered set satisfying the Ascending Chain Condition is continuous.
Fix-points etc.

\[ \text{Red}(f) \rightarrow \text{Fix}(f) \rightarrow \text{Ext}(f) \]

\[ \top \]
\[ f^n(\top) \]
\[ \bigcap_n f^n(\top) \]
\[ \text{gfp}(f) \]
\[ \text{lfp}(f) \]
\[ \bigcup_n f^n(\bot) \]
\[ f^n(\bot) \]
\[ \bot \]
Fixed Points and Solutions

Given equations over some domain, e.g. integers

\[ 6x^3 - 3x^2 - x = 7 \]

We look at it as a “recursive” equation:

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or simply:

\[ f(x) = x. \]
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We look at it as a “recursive” equation:

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or simply:

\[ f(x) = x. \]

If \( x \) is a fixed point of \( f \) then it is a solution to the equation.
Lattice Equations

Given a system of equations with unknowns $x_1, \ldots, x_n$ over a complete lattice $L$ (fulfilling ACC/DCC).

\[
\begin{align*}
x_1 &= f_1(x_1, \ldots, x_n) \\
\ldots &= \ldots \\
x_n &= f_m(x_1, \ldots, x_n)
\end{align*}
\]
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\vdots &= \vdots \\
x_n &= f_m(x_1, \ldots, x_n)
\end{align*}
\]

Consider the equations as defining a function $F : L^n \rightarrow L^n$

\[
F(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))
\]
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Given a system of equations with unknowns $x_1, \ldots, x_n$ over a complete lattice $L$ (fulfilling ACC/DCC).

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Consider the equations as defining a function $F : L^n \to L^n$

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\]

In our case we start with a recursive set of equations:

\[
\text{Analysis}(i) = f_i(\text{Analysis}(1), \ldots, \text{Analysis}(n)).
\]
**Chaotic Iteration**

**Iteration:** Construct iteratively the smallest or largest solution/fixed point, i.e. $\text{lfp}(F)$ or $\text{gfp}(F)$, by starting with

$$x_i = x_i^0 = \bot \quad \text{or} \quad x_i = x_i^0 = \top$$

and construct a sequence of approximations like:
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and construct a sequence of approximations like:

$$x_i^0 = \bot$$
$$x_i^1 = f(x_1^0, \ldots, x_n^0)$$
$$\ldots \quad \ldots$$
$$x_i^k = f(x_1^{k-1}, \ldots, x_n^{k-1})$$

until we converge, i.e. the sequence stabilises.
An Example

Look at the complete lattice $\mathcal{P}(X) = \mathcal{P}([a, b, c, d])$. Construct solutions to the following set equations:

\[
\begin{align*}
S_1 &= \{a\} \cup S_4 \\
S_2 &= S_1 \cup S_3 \\
S_3 &= S_4 \cap \{b\} \\
S_4 &= S_2 \cup \{b, c\}
\end{align*}
\]
Two Solutions

Starting from $\bot$ gives:

<table>
<thead>
<tr>
<th>$S_1$</th>
<th>$\emptyset$</th>
<th>${a}$</th>
<th>${a, b, c}$</th>
<th>${a, b, c}$</th>
<th>${a, b, c}$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_2$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>${a}$</td>
<td>${a, b, c}$</td>
<td>${a, b, c}$</td>
<td>${a, b, c}$</td>
</tr>
<tr>
<td>$S_3$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>${b}$</td>
<td>${b}$</td>
<td>${b}$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$S_4$</td>
<td>$\emptyset$</td>
<td>${b, c}$</td>
<td>${b, c}$</td>
<td>${a, b, c}$</td>
<td>${a, b, c}$</td>
<td>${a, b, c}$</td>
</tr>
</tbody>
</table>
Two Solutions

Starting from $\bot$ gives:

\[
\begin{align*}
S_1 &= \emptyset & \{a\} & \{a, b, c\} & \{a, b, c\} & \{a, b, c\} & \ldots \\
S_2 &= \emptyset & \emptyset & \{a\} & \{a, b, c\} & \{a, b, c\} & \ldots \\
S_3 &= \emptyset & \emptyset & \{b\} & \{b\} & \{b\} & \ldots \\
S_4 &= \emptyset & \{b, c\} & \{b, c\} & \{a, b, c\} & \{a, b, c\} & \ldots
\end{align*}
\]

Starting from $\top$ gives:

\[
\begin{align*}
S_1 &= \{a, b, c, d\} & \{a, b, c, d\} & \{a, b, c, d\} & \ldots \\
S_2 &= \{a, b, c, d\} & \{a, b, c, d\} & \{a, b, c, d\} & \ldots \\
S_3 &= \{a, b, c, d\} & \{b\} & \{b\} & \ldots \\
S_4 &= \{a, b, c, d\} & \{a, b, c, d\} & \{a, b, c, d\} & \ldots
\end{align*}
\]
Mathematics literature is full of Fixed Point Theorems, e.g.
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Theorem (Knaster-Tarski)

Let $L$ be a complete lattice and assume that $f : L \rightarrow L$ is an order-preserving map. Then

$$\bigsqcup \{x \in L \mid x \sqsubseteq f(x)\} \in \text{Fix}(f).$$
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Classical Analyses

Each of the four classical analyses considers equations for a label consistent program $S_\star$ and they take the form:

$$Analysis_\circ(\ell) = \begin{cases} \iota, & \text{if } \ell \in E \\ \bigsqcup\{Analysis_\bullet(\ell') \mid (\ell', \ell) \in F\}, & \text{otherwise} \end{cases}$$

$$Analysis_\bullet(\ell) = f_\ell(Analysis_\circ(\ell))$$
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$\bigcup$ is $\cap$ or $\bigcup$ (and $\sqcup$ is $\cup$ or $\cap$).
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$E$ is $\{init(S_\star)\}$ or $final(S_\star)$,
Classical Analyses

Each of the four classical analyses considers equations for a label consistent program $S_\star$ and they take the form:

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$$\text{Analysis}_\bullet(\ell) = f_\ell(\text{Analysis}_\circ(\ell))$$

$\iota$ specifies the initial or final analysis information, and
Each of the four classical analyses considers equations for a label consistent program $S_*$ and they take the form:

$$Analysis_\circ(\ell) = \begin{cases} \iota, & \text{if } \ell \in E \\ \bigsqcup \{Analysis_\bullet(\ell') \mid (\ell', \ell) \in F\}, & \text{otherwise} \end{cases}$$

$$Analysis_\bullet(\ell) = f_\ell(Analysis_\circ(\ell))$$

$f_\ell$ is the transfer function associated with $B_\ell \in \text{blocks}(S_*)$. 
Forward vs Backward Analysis

The forward analyses have $F$ to be $flow(S_\star)$ and then $\text{Analysis}_\circ$ concerns entry conditions and $\text{Analysis}_\bullet$ concerns exit conditions; also the equation system presupposes that $S_\star$ has isolated entries.
The **forward analyses** have $F$ to be $\text{flow}(S_*)$ and then $\text{Analysis}_\circ$ concerns entry conditions and $\text{Analysis}_\bullet$ concerns exit conditions; also the equation system presupposes that $S_*$ has isolated entries.

The **backward analyses** have $F$ to be $\text{flow}^R(S_*)$ and then $\text{Analysis}_\circ$ concerns exit conditions and $\text{Analysis}_\bullet$ concerns entry conditions; also the equation system presupposes that $S_*$ has isolated exits.
Must vs May Analysis

When $\sqcap$ is $\cap$ we require the *greatest* sets that solve the equations and we are able to detect properties satisfied by *all* paths of execution reaching (or leaving) the entry (or exit) of a label; these analyses are often called **must analyses**.
Must vs May Analysis

When $\sqsubseteq$ is $\cap$ we require the *greatest* sets that solve the equations and we are able to detect properties satisfied by *all* paths of execution reaching (or leaving) the entry (or exit) of a label; these analyses are often called must analyses.

When $\sqsubseteq$ is $\cup$ we require the *least* sets that solve the equations and we are able to detect properties satisfied by *at least one* execution path to (or from) the entry (or exit) of a label; these analyses are often called may analyses.
Alternative Formulation

It is occasionally awkward to have to assume that forward analyses have isolated entries and that backward analyses have isolated exits. This motivates reformulating the above equations to be of the form:

\[
\text{Analysis}_\circ(\ell) = \bigsqcup \{ \text{Analysis}_\bullet(\ell') \mid (\ell', \ell) \in F \} \sqcup \iota^\ell_E
\]

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\text{Analysis}_\bullet(\ell) &= f_\ell(\text{Analysis}_\circ(\ell))
\end{align*}
\]

where

\[
\iota^\ell_E = \begin{cases} 
\iota & \text{if } \ell \in E \\
\bot & \text{if } \ell \notin E
\end{cases}
\]

and \(\bot\) satisfies \(l \sqcup \bot = l\) (hence \(\bot\) is not really there).
The view that we take here is that a program is a *transition system*; the nodes represent blocks and each block has a *transfer function* associated with it that specifies how the block acts on the “input” state.
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Note that for forward analyses, the input state is the entry state, and for backward analyses, it is the exit state.
Monotone & Distributive Frameworks

A Monotone Framework consists of:

- A complete lattice, \( L \), that satisfies the Ascending Chain Condition, and we write \( \bigvee \) for the least upper bound operator;
- A set \( F \) of monotone functions from \( L \) to \( L \) that contains the identity function and that is closed under function composition.

A Distributive Framework is a Monotone Framework where additionally all functions \( f \) in \( F \) are required to be distributive:

\[ f(l_1 \bigvee l_2) = f(l_1) \bigvee f(l_2) \]
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Instance of a Framework

An *instance*, Analysis, of a Monotone or Distributive Framework to consists of:

1. the complete lattice, $L$,
2. the space of transfer functions, $F$,
3. a finite flow, $\text{flow} (\star)$ or $\text{flow} R (\star)$;
4. a finite set of so-called extremal labels, $E$, that typically is $\{\text{init} (\star)\}$ or $\{\text{final} (\star)\}$;
5. an extremal value, $\iota \in L$, for the extremal labels; and
6. a mapping, $f \cdot$, from the labels $\text{Lab} \star$ of $F$ to transfer functions in $F$. 


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- an *extremal value*, $\iota \in L$, for the extremal labels; and
- a mapping, $f$, from the labels $\text{Lab}_\star$ of $F$ to transfer functions in $\mathcal{F}$. 
An instance gives rise to a set of equations, $\text{Analysis}^=\text{Analysis}$, of the form considered earlier:

\[
\text{Analysis}_\circ(\ell) = \bigsqcup \{ \text{Analysis}_\bullet(\ell') \mid (\ell', \ell) \in F \} \sqcup \iota^\ell_E
\]

where $\iota^\ell_E = \begin{cases} 
\iota & \text{if } \ell \in E \\
\bot & \text{if } \ell \notin E
\end{cases}$

\[
\text{Analysis}_\bullet(\ell) = f_\ell(\text{Analysis}_\circ(\ell))
\]
### Classical Instances

<table>
<thead>
<tr>
<th>Available Expressions</th>
<th>Reaching Definitions</th>
<th>Very Busy Expressions</th>
<th>Live Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L )</td>
<td>( \mathcal{P}(A\text{Exp}_\star) )</td>
<td>( \mathcal{P}(\text{Var}<em>\star \times \text{Lab}</em>\star) )</td>
<td>( \mathcal{P}(\text{Var}_\star) )</td>
</tr>
<tr>
<td>( \sqsupseteq )</td>
<td>( \sqsupseteq )</td>
<td>( \subseteq )</td>
<td>( \subseteq )</td>
</tr>
<tr>
<td>( \sqcap )</td>
<td>( \sqcap )</td>
<td>( \sqcup )</td>
<td>( \sqcup )</td>
</tr>
<tr>
<td>( \bot )</td>
<td>( A\text{Exp}_\star )</td>
<td>( \bot )</td>
<td>( \bot )</td>
</tr>
<tr>
<td>( L \sqsubseteq E )</td>
<td>( { \text{init}(S_\star) } )</td>
<td>( {(x, ?)</td>
<td>x \in \text{FV}(S_\star)} )</td>
</tr>
<tr>
<td>( L \sqsubseteq F )</td>
<td>( \text{flow}(S_\star) )</td>
<td>( \text{flow}(S_\star) )</td>
<td>( \text{flow}^R(S_\star) )</td>
</tr>
<tr>
<td>( \mathcal{F} )</td>
<td>( { f : L \to L</td>
<td>\exists l_k, l_g : f(l) = (l \setminus l_k) \cup l_g } )</td>
<td>( { f \in \mathcal{F}</td>
</tr>
<tr>
<td>( f_\ell )</td>
<td>( f_\ell(l) = (l \setminus \text{kill}([B]<em>\ell)) \cup \text{gen}([B]</em>\ell) ) where ([B]<em>\ell \in \text{blocks}(S</em>\star))</td>
<td>( \emptyset )</td>
<td>( \emptyset )</td>
</tr>
</tbody>
</table>
Lemma: Each of the four classical data flow analyses is a Monotone Framework as well as a Distributive Framework.
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It is worth pointing out that in order to get this result we have made the frameworks dependent upon the actual program – this is needed to enforce that the Ascending Chain Condition is fulfilled.
A Non-Distributive Example

The Constant Propagation Analysis (CP) will determine:

For each program point, whether or not a variable has a constant value whenever execution reaches that point.
A Non-Distributive Example

The Constant Propagation Analysis (CP) will determine:

*For each program point, whether or not a variable has a constant value whenever execution reaches that point.*

Such information can be used as the basis for an optimisation known as *Constant Folding*: all uses of the variable may be replaced by the constant value.
CP State: $\mathbb{Z}^\top$

The (abstract) states for the CP Analysis are given by:

$$\widehat{\text{State}}_{\text{CP}} = ((\text{Var}_* \rightarrow \mathbb{Z}^\top)_{\perp}, \sqsubseteq, \sqcup, \sqcap, \bot, \lambda x. \top)$$

where $\text{Var}_*$ is the set of variables appearing in the program.
The (abstract) states for the CP Analysis are given by:

\[
\text{State}_{\text{CP}} = ((\text{Var}_* \to \mathbb{Z}^\top) \perp, \sqsubseteq, \cup, \cap, \bot, \lambda x. \top)
\]

where \text{Var}_* is the set of variables appearing in the program.

\[\mathbb{Z}^\top = \mathbb{Z} \cup \{\top\}\] is partially ordered as follows:

\[
\forall z \in \mathbb{Z}^\top : z \sqsubseteq \top
\]

\[
\forall z_1, z_2 \in \mathbb{Z} : (z_1 \sqsubseteq z_2) \iff (z_1 = z_2)
\]
We construct a non-standard partial order on $\mathbb{Z}$:

\[ \cdots -2 -1 0 +1 +2 \cdots \]
CP State: Partial Order $\mathbb{Z}^\top$ vs Lattice $\mathbb{Z}^\perp$

We construct a non-standard partial order on $\mathbb{Z}$:

\[ \cdots -2 -1 0 +1 +2 \cdots \]
We construct a non-standard partial order on $\mathbb{Z}$:

\[
\begin{array}{c}
\vdots \\
-2 & -1 & 0 & +1 & +2 & \ldots \\
\end{array}
\]
We construct a non-standard partial order on $\mathbb{Z}$:

```
          ⊤
         /   \
       /     \n     ⊤       ⊤
    / |     / |  \\
   0 -2 -1  +1 +2 ...
  / |     / |     / |
 ∙ ∙ ∙ ∙ ∙ ∙ ∙ ∙ ∙
```

We could also consider a complete lattice $\mathbb{Z}^\perp$ (but don’t):
We construct a non-standard partial order on $\mathbb{Z}$:

```
\cdots -2 -1 0 +1 +2 \cdots
```

We could also consider a complete lattice $\mathbb{Z}^\top$ (but don’t):

```
\cdots -2 -1 0 +1 +2 \cdots
```
We construct a non-standard partial order on $\mathbb{Z}$:

\[
\begin{array}{cccccc}
\vdots & -2 & -1 & 0 & +1 & +2 & \vdots \\
\end{array}
\]

We could also consider a complete lattice $\mathbb{Z}^\top$ (but don’t):

\[
\begin{array}{cccccc}
\vdots & -2 & -1 & 0 & +1 & +2 & \vdots \\
\end{array}
\]
We construct a non-standard partial order on \( \mathbb{Z} \):

We could also consider a complete lattice \( \mathbb{Z} \uparrow \) (but don’t):
CP State: Partial Order $\mathbb{Z}^\top$ vs Lattice $\mathbb{Z}^\top$

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We construct a non-standard partial order on \( \mathbb{Z} \):

\[
\cdots -2 \quad -1 \quad 0 \quad +1 \quad +2 \quad \cdots
\]

We could also consider a complete lattice \( \mathbb{Z} \) (but don’t):

\[
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To capture the case where no information is available we extend $\text{Var}_\star \rightarrow \mathbb{Z}^\top$ with a least element $\bot$, written $(\text{Var}_\star \rightarrow \mathbb{Z}^\top)\bot$. 
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The partial ordering $\sqsubseteq$ on $\widehat{\text{State}}_{\text{CP}} = (\text{Var}_\ast \rightarrow \mathbb{Z}^\top)_\bot$ is:

\[
\forall \hat{\sigma} \in (\text{Var}_\ast \rightarrow \mathbb{Z}^\top)_\bot : \quad \bot \sqsubseteq \hat{\sigma}
\]

\[
\forall \hat{\sigma}_1, \hat{\sigma}_2 \in \text{Var}_\ast \rightarrow \mathbb{Z}^\top : \quad \hat{\sigma}_1 \sqsubseteq \hat{\sigma}_2 \text{ iff } \forall x : \hat{\sigma}_1(x) \sqsubseteq \hat{\sigma}_2(x)
\]
To capture the case where no information is available we extend \( \text{Var}_\star \to \mathbb{Z}^\top \) with a least element \( \bot \), written \( (\text{Var}_\star \to \mathbb{Z}^\top)\bot \).

The partial ordering \( \sqsubseteq \) on \( \hat{\text{State}}_{\text{CP}} = (\text{Var}_\star \to \mathbb{Z}^\top)\bot \) is:

\[
\forall \hat{\sigma} \in (\text{Var}_\star \to \mathbb{Z}^\top)\bot : \quad \bot \sqsubseteq \hat{\sigma}
\]

\[
\forall \hat{\sigma}_1, \hat{\sigma}_2 \in \text{Var}_\star \to \mathbb{Z}^\top : \quad \hat{\sigma}_1 \sqsubseteq \hat{\sigma}_2 \iff \forall x : \hat{\sigma}_1(x) \sqsubseteq \hat{\sigma}_2(x)
\]

and the binary least upper bound operation is then:

\[
\forall \hat{\sigma} \in (\text{Var}_\star \to \mathbb{Z}^\top)\bot : \quad \hat{\sigma} \sqcup \bot = \hat{\sigma} = \bot \sqcup \hat{\sigma}
\]

\[
\forall \hat{\sigma}_1, \hat{\sigma}_2 \in \text{Var}_\star \to \mathbb{Z}^\top : \quad \forall x : (\hat{\sigma}_1 \sqcup \hat{\sigma}_2)(x) = \hat{\sigma}_1(x) \sqcup \hat{\sigma}_2(x)
\]
CP State Evaluation

\[ A_{\text{CP}} : \text{AExp} \to (\widehat{\text{State}}_{\text{CP}} \to Z_{\top}) \]
CP State Evaluation

\[ \mathcal{A}_{CP} : AExp \rightarrow (\widehat{\text{State}}_{CP} \rightarrow \mathbb{Z}^\top) \]

\[ \mathcal{A}_{CP}[x]^{\hat{\sigma}} = \begin{cases} \bot & \text{if } \hat{\sigma} = \bot \\ \hat{\sigma}(x) & \text{otherwise} \end{cases} \]

\[ \mathcal{A}_{CP}[n]^{\hat{\sigma}} = \begin{cases} \bot & \text{if } \hat{\sigma} = \bot \\ n & \text{otherwise} \end{cases} \]

\[ \mathcal{A}_{CP}[a_1 \ op_a a_2]^{\hat{\sigma}} = \mathcal{A}_{CP}[a_1]^{\hat{\sigma}} \ op_a \mathcal{A}_{CP}[a_2]^{\hat{\sigma}} \]
CP State Evaluation

\[ \mathcal{A}_{\text{CP}} : A\text{Exp} \rightarrow (\widehat{\text{State}}_{\text{CP}} \rightarrow \mathbb{Z}^\top) \]

\[ \mathcal{A}_{\text{CP}}[x]\widehat{\sigma} = \begin{cases} \bot & \text{if } \widehat{\sigma} = \bot \\ \widehat{\sigma}(x) & \text{otherwise} \end{cases} \]

\[ \mathcal{A}_{\text{CP}}[n]\widehat{\sigma} = \begin{cases} \bot & \text{if } \widehat{\sigma} = \bot \\ n & \text{otherwise} \end{cases} \]

\[ \mathcal{A}_{\text{CP}}[a_1 \text{ op}_a a_2]\widehat{\sigma} = \mathcal{A}_{\text{CP}}[a_1]\widehat{\sigma} \text{ op}_a \mathcal{A}_{\text{CP}}[a_2]\widehat{\sigma} \]

The operations on \( \mathbb{Z} \) are lifted to \( \mathbb{Z}^\top = \mathbb{Z} \cup \{\bot, \top\} \) by taking

\[ z_1 \text{ op}_a z_2 = z_1 \text{ op}_a z_2 \text{ if } z_1, z_2 \in \mathbb{Z} \] (and where \( \text{ op}_a \) is the corresponding arithmetic operation on \( \mathbb{Z} \)), \[ z_1 \text{ op}_a z_2 = \bot \text{ if } z_1 = \bot \text{ or } z_2 = \bot \text{ and } z_1 \text{ op}_a z_2 = \top \text{ otherwise.} \]
$\mathcal{F}_{\text{CP}} = \{ f \mid f \text{ is a monotone function on } \widehat{\text{State}}_{\text{CP}} \}$
\[ F_{CP} = \{ f \mid f \text{ is a monotone function on } \widehat{\text{State}}_{CP} \} \]

\[
\begin{align*}
[x := a]^{\ell} : \quad f^{CP}_{\ell}(\widehat{\sigma}) &= \begin{cases} 
\bot & \text{ if } \widehat{\sigma} = \bot \\
\widehat{\sigma}[x \mapsto A_{CP}[a]\widehat{\sigma}] & \text{ otherwise}
\end{cases} \\
[\text{skip}]^{\ell} : \quad f^{CP}_{\ell}(\widehat{\sigma}) &= \widehat{\sigma} \\
[b]^{\ell} : \quad f^{CP}_{\ell}(\widehat{\sigma}) &= \widehat{\sigma}
\end{align*}
\]
Constant Propagation (CP) is a forward analysis, so for the program $S_\star$ we take the flow, $F$, to be $\text{flow}(S_\star)$.

The extremal labels, $E$, are given by $\{\text{init}(S_\star)\}$, and the extremal value, $\nu_{CP}$, is $\lambda x. \top$. The property lattice $L$ and transfer function $\mathcal{F}_{CP}$ as above.
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**Lemma:** Constant Propagation is a Monotone Framework that is *not* a Distributive Framework.
To show that it is not a Distributive Framework consider the transfer function $f_{\ell}^{CP}$ for $[y := x \ast x]^{\ell}$ and let $\hat{\sigma}_1$ and $\hat{\sigma}_2$ be such that $\hat{\sigma}_1(x) = 1$ and $\hat{\sigma}_2(x) = -1$. 
Distributive Framework

To show that it is not a Distributive Framework consider the transfer function \( f_{CP}^\ell \) for \([y := x \ast x]^\ell\) and let \( \hat{\sigma}_1 \) and \( \hat{\sigma}_2 \) be such that \( \hat{\sigma}_1(x) = 1 \) and \( \hat{\sigma}_2(x) = -1 \).

Then \( \hat{\sigma}_1 \sqcup \hat{\sigma}_2 \) maps \( x \) to \( \top \) and thus \( f_{CP}^\ell(\hat{\sigma}_1 \sqcup \hat{\sigma}_2) \) maps \( y \) to \( \top \) and hence fails to record that \( y \) has the constant value 1.
Distributive Framework

To show that it is not a Distributive Framework consider the transfer function $f^{\text{CP}}_\ell$ for $[y := x \ast x]_\ell$ and let $\hat{\sigma}_1$ and $\hat{\sigma}_2$ be such that $\hat{\sigma}_1(x) = 1$ and $\hat{\sigma}_2(x) = -1$.

Then $\hat{\sigma}_1 \sqcup \hat{\sigma}_2$ maps $x$ to $\top$ and thus $f^{\text{CP}}_\ell(\hat{\sigma}_1 \sqcup \hat{\sigma}_2)$ maps $y$ to $\top$ and hence fails to record that $y$ has the constant value 1.

However, both $f^{\text{CP}}_\ell(\hat{\sigma}_1)$ and $f^{\text{CP}}_\ell(\hat{\sigma}_2)$ map $y$ to 1 and so does $f^{\text{CP}}_\ell(\hat{\sigma}_1) \sqcup f^{\text{CP}}_\ell(\hat{\sigma}_2)$. 
Transfer Functions and Information Collection

Consider transfer function for the block $y := x \ast x^\ell$. 
Transfer Functions and Information Collection

Consider transfer function for the block \([ y := x \ast x ]^\ell \).

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>T</td>
</tr>
<tr>
<td>+1</td>
<td>1</td>
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</tbody>
</table>
Consider transfer function for the block $[y := x \ast x]_\ell$. 

\[
\begin{array}{c|c}
  x & y \\
  \hline
  -1 & T \\
\end{array}
\begin{array}{c|c}
  x & y \\
  \hline
  +1 & 1 \\
\end{array}
\begin{array}{c|c}
  x & y \\
  \hline
  T & T \\
\end{array}
\]
Consider transfer function for the block \( y := x \times x \).
Consider transfer function for the block \([y := x \ast x]^{\ell}\).
Transfer Functions and Information Collection

Consider transfer function for the block \([y := x \ast x]^\ell\).
Consider transfer function for the block \( y := x \ast x \).
The MFP Solution (1)

INPUT: An instance of a Monotone Framework: $(L, \mathcal{F}, F, E, \iota, f)$

OUTPUT: $MFP_\circ, MFP_\bullet$
The MFP Solution (1)

INPUT: An instance of a Monotone Framework: 
\((L, F, F, E, \iota, f.)\)

OUTPUT: \(MFP_\circ, MFP_\bullet\)

Step 1: Initialisation (of \(W\) and Analysis)

\[W := \text{nil};\]

for all \((\ell, \ell')\) in \(F\) do 
\[W := \text{cons}((\ell, \ell'), W);\]

for all \(\ell\) in \(F\) or \(E\) do 
\[\text{if } \ell \in E \text{ then Analysis}[\ell] := \iota \]
\[\text{else Analysis}[\ell] := \perp_L;\]
The MFP Solution (2&3)

Step 2: Iteration (updating $W$ and Analysis)
while $W \neq \text{nil}$ do
    $\ell := \text{fst}(\text{head}(W))$; $\ell' = \text{snd}(\text{head}(W))$
    $W := \text{tail}(W)$;
    if $f_\ell(\text{Analysis}[\ell]) \not\sqsubseteq \text{Analysis}[\ell']$ then
        $\text{Analysis}[\ell'] := \text{Analysis}[\ell'] \sqcup f_\ell(\text{Analysis}[\ell])$
    for all $(\ell', \ell'')$ in $F$ do $W := \text{cons}((\ell', \ell''), W)$;
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\]
for all $(\ell', \ell'')$ in $F$ do $W := \text{cons}((\ell', \ell''), W);$ 

Step 3: Presenting the result ($MFP_\circ$ and $MFP_\bullet$)
for all $\ell$ in $F$ or $E$ do
\[
MFP_\circ(\ell) := \text{Analysis}[\ell];
\]
\[
MFP_\bullet(\ell) := f_\ell(\text{Analysis}[\ell])
\]
Given an instance of a Monotone Framework \((L, \mathcal{F}, F, E, \nu, f.)\) with a property lattice \(L\) fullfilling the ACC/DCC.

Starting from \(\bot\) and using iterative (approximation) methods like Chaotic Iteration or the Worklist Algorithm (which optimises the iterations by only considering updates when “necessary”) we can compute solutions \(Analysis\circ\) and \(Analysis\bullet\).
MFP Termination

Given an instance of a Monotone Framework \((L, \mathcal{F}, F, E, \nu, f.)\) with a property lattice \(L\) fullfilling the ACC/DCC.

Starting from \(\bot\) and using iterative (approximation) methods like Chaotic Iteration or the Worklist Algorithm (which optimises the iterations by only considering updates when “necessary”) we can compute solutions \(\text{Analysis}_\circ\) and \(\text{Analysis}_\bullet\).

**Lemma:** The iterative construction of a solution (using chaotic iteration, worklist algorithm) always terminates and it computes the least MFP solution (more precisely \(MFP_\circ\) and \(MFP_\bullet\)) to the instance of the framework.
MFP Complexity

Assume that the flow $F$ is represented in such a way that all $(\ell', \ell'')$ emanating from $\ell'$ can be found in time proportional to their number. Suppose that $E$ and $F$ contain at most $b \geq 1$ distinct labels, that $F$ contains at most $e \geq b$ pairs, and that $L$ has finite height at most $h \geq 1$. 
MFP Complexity

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Then steps 1 and 3 perform at most $O(b + e)$ basic operations. In step 2 a pair is placed on the worklist at most $O(h)$ times, and each time it takes only a constant number of basic steps to process it; this yields at most $O(e \cdot h)$ basic operations for step 2. Since $h \geq 1$ and $e \geq b$ this gives at most $O(e \cdot h)$ basic operations for the algorithm.
Consider the Reaching Definitions Analysis and suppose that there are at most $v \geq 1$ variables and $b \geq 1$ labels in the program, $S_*$, being analysed. Since $L = \mathcal{P}(\text{Var}_* \times \text{Lab}_*)$, it follows that $h \leq v \cdot b$ and thus we have an $O(v \cdot b^3)$ upper bound on the number of basic operations.
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Better: If \( S_* \) is label consistent then the variable of the pairs \((x, \ell)\) of \( P(\text{Var}_* \times \text{Lab}_*) \) will always be uniquely determined by the label \( \ell \) so we get an \( O(b^3) \) upper bound on the number of basic operations. Furthermore, \( F \) is \( \text{flow}(S_*) \) and inspection of the equations for \( \text{flow}(S_*) \) shows that for each label \( \ell \) we construct at most two pairs with \( \ell \) in the first component. This means that \( e \leq 2 \cdot b \) and we get an \( O(b^2) \) upper bound on the number of basic operations.
Consider an instance \((L, \mathcal{F}, F, E, \iota, f)\) of a Monotone Framework.
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We shall use the notation \(\vec{\ell} = [\ell_1, \cdots, \ell_n]\) for a sequence of \(n \geq 0\) labels.
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The paths up to \textit{but not} including \(\ell\) are:

\[
\text{path}_\circ(\ell) = \{[\ell_1, \cdots, \ell_{n-1}] \mid n \geq 1 \land \forall i < n : (\ell_i, \ell_{i+1}) \in F \land \ell_n = \ell \land \ell_1 \in E\}
\]

The paths up to \textit{and} including \(\ell\) are:

\[
\text{path}_\bullet(\ell) = \{[\ell_1, \cdots, \ell_n] \mid n \geq 1 \land \forall i < n : (\ell_i, \ell_{i+1}) \in F \land \ell_n = \ell \land \ell_1 \in E\}
\]
MOP Solutions

For a path $\vec{\ell} = [\ell_1, \cdots, \ell_n]$ we define the transfer function

$$f_{\vec{\ell}} = f_{\ell_n} \circ \cdots \circ f_{\ell_1} \circ \text{id}$$

so that for the empty path we have $f_{[]} = \text{id}$ where $\text{id}$ is the identity function.
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The MOP solutions are then given by:

$$MOP_\circ(\ell) = \bigsqcup \{ f_{\vec{\ell}}(\iota) \mid \vec{\ell} \in path_\circ(\ell) \}$$

$$MOP_\bullet(\ell) = \bigsqcup \{ f_{\vec{\ell}}(\iota) \mid \vec{\ell} \in path_\bullet(\ell) \}$$
Unfortunately, the MOP solution sometimes cannot be computable (meaning that it is undecidable what the solution is) even though the MFP solution is always easily computable (because of the property space satisfying the Ascending Chain Condition); the following result establishes one such result:
MOP Solution: Termination

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**Lemma:** The MOP solution for the Constant Propagation Analysis is undecidable.
MFP and MOP Solutions

**Lemma:** Consider the MFP and the MOP solutions to an instance \((L, \mathcal{F}, F, B, \iota, f.)\) of a Monotone Framework; then:

\[
MFP \circ \subseteq MOP \circ \quad \text{and} \quad MFP \bullet \subseteq MOP \bullet.
\]
MFP and MOP Solutions

**Lemma:** Consider the MFP and the MOP solutions to an instance \((L, \mathcal{F}, F, B, \iota, f.)\) of a Monotone Framework; then:

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\text{MFP}_\circ \supseteq \text{MOP}_\circ \text{ and } \text{MFP}_\bullet \supseteq \text{MOP}_\bullet
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If the framework is a Distributive Framework and if \(\text{path}_\circ(\ell) \neq \emptyset\) for all \(\ell\) in \(E\) and \(F\) then:

\[
\text{MFP}_\circ = \text{MOP}_\circ \text{ and } \text{MFP}_\bullet = \text{MOP}_\bullet
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**MFP and MOP Solutions**

**Lemma:** Consider the MFP and the MOP solutions to an instance \((L, \mathcal{F}, F, B, \iota, f.)\) of a **Monotone Framework**; then:

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If the framework is a **Distributive Framework** and if \(\text{path}_\circ(\ell) \neq \emptyset\) for all \(\ell\) in \(E\) and \(F\) then:

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It is always possible to formulate the MOP solution as an MFP solution over a different property space (like \(\mathcal{P}(L)\)) and therefore little is lost by focusing on the fixed point approach to Monotone Frameworks.