Program Analysis (70020)

Abstract Interpretation

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Live Variable Analysis

A variable is *live* at the exit from a label if there exists a path from the label to a use of the variable that does not re-define the variable. The *Live Variables Analysis* will determine:

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This analysis might be used as the basis for *Dead Code Elimination*. If the variable is not live at the exit from a label then, if the elementary block is an assignment to the variable, the elementary block can be eliminated.

Parity Analysis

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This analysis might be used as the basis for . . . (saving a bit?).

LV Analysis: Property Space

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Important fact: Information we are interested in is in $\mathcal{P}(\mathbf{Var}_\star)$.

LV Equations and Transfer Functions

$$\mathsf{LV}_{exit}(\ell) = \begin{cases} \emptyset, \text{if } \ell \in \mathit{final}(S_\star) \\ \bigcup\{\mathsf{LV}_{entry}(\ell') \mid (\ell', \ell) \in \mathit{flow}^R(S_\star)\}, \text{otherwise} \end{cases}$$

$$\begin{aligned} \mathsf{LV}_{entry}(\ell) &= (\mathsf{LV}_{exit}(\ell) \backslash \mathit{kill}_{\mathsf{LV}}([B]^\ell) \cup \mathit{gen}_{\mathsf{LV}}([B]^\ell) \\ &\quad \text{where } [B]^\ell \in \mathit{blocks}(S_\star) \end{aligned}$$

with

$$\begin{aligned} \mathit{kill}_{\mathsf{LV}}([\, x := a\,]^\ell) &= \{x\} \\ \mathit{kill}_{\mathsf{LV}}([\, \textbf{skip}\,]^\ell) &= \emptyset \\ \mathit{kill}_{\mathsf{LV}}([b]^\ell) &= \emptyset \end{aligned}$$

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Questions: How to modify parity information locally and how to combine it, e.g. maybe $\{(x, e), (x, o), (y, e)\} \cup \{(x, e), (y, e)\}$.

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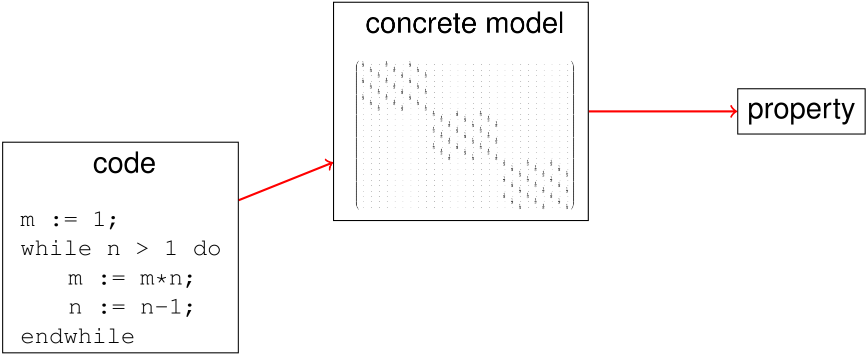
The central element is the simplification of the concrete semantics in order to obtain an abstract one as an optimal approximation.

Concrete Semantics vs Abstract Semantics



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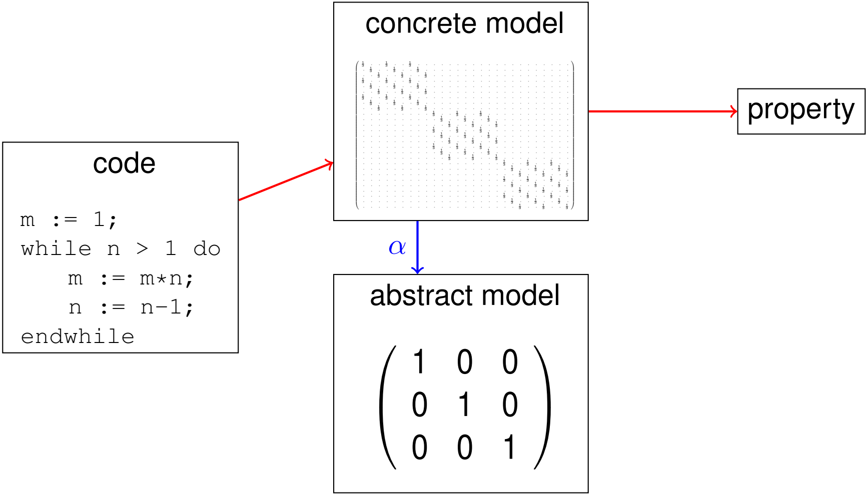

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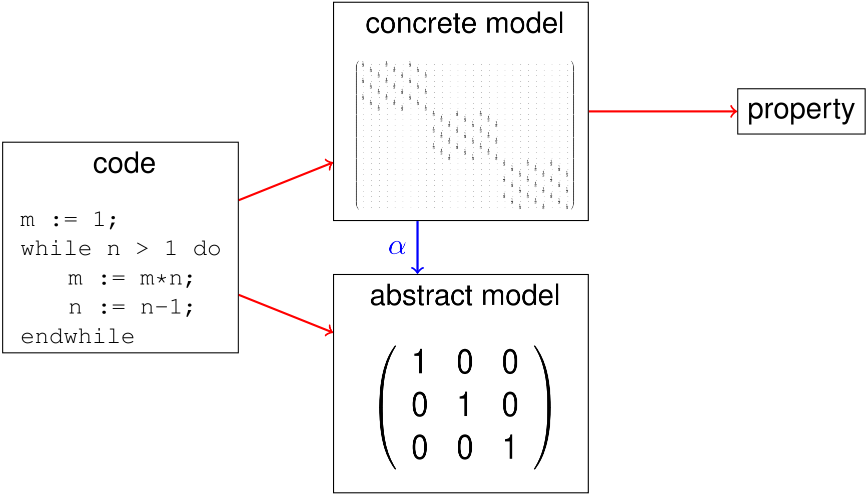

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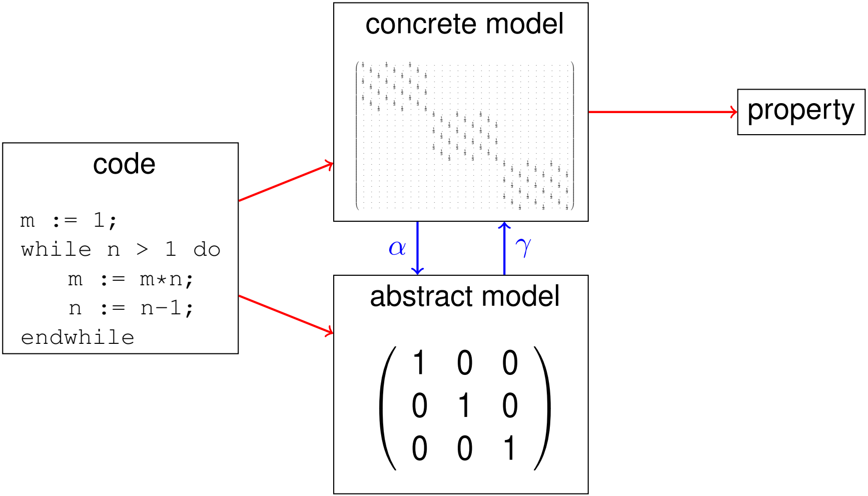

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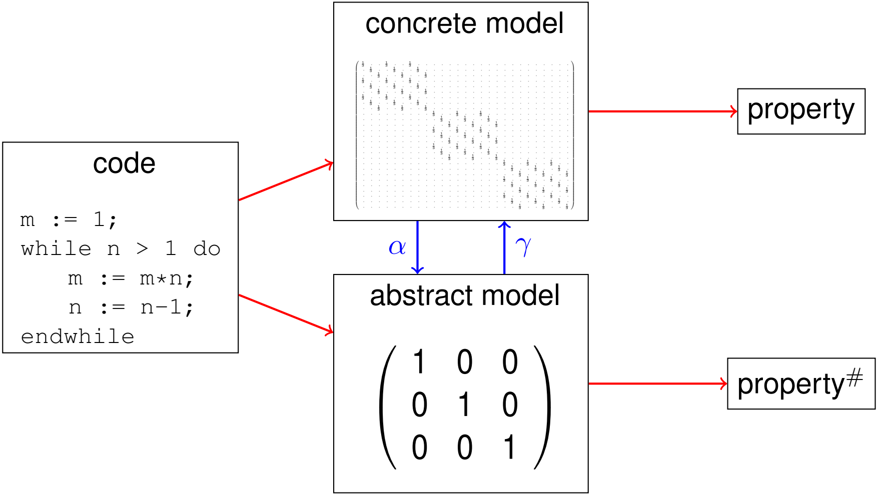

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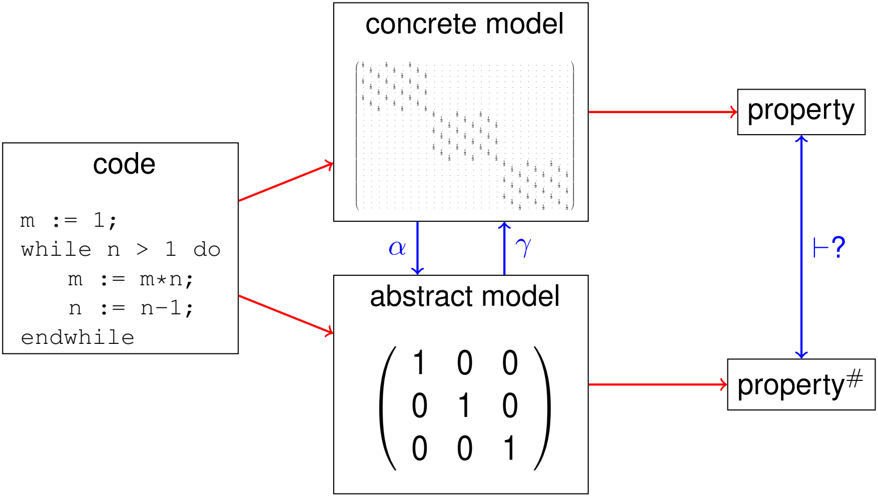

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Note that there are false positives, cf also [1] and [2].

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Abstract Interpretation also uses other techniques, like widening/narrowing, which we will not cover here.

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In quantitative, vector space structures we want
Close Approximations

$$\|s - s^*\| = \min_x \|s - x\|$$

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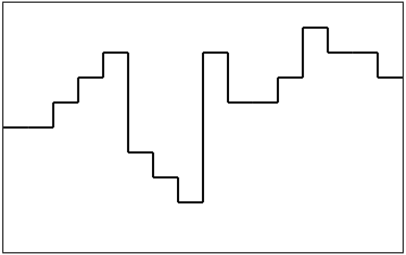

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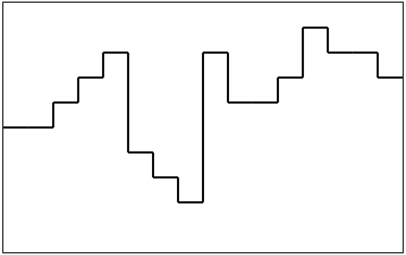
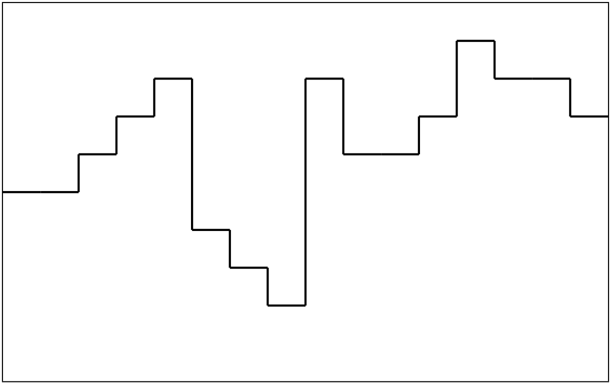


The concrete function needs n data points, its abstraction or approximation should need less, i.e. from $\mathbb{R}^n$ to $\mathbb{R}^m$ with $m < m$.

Close Approximations

Approximate $f \in \mathbb{R}^{16}$ by “least square” simplifications



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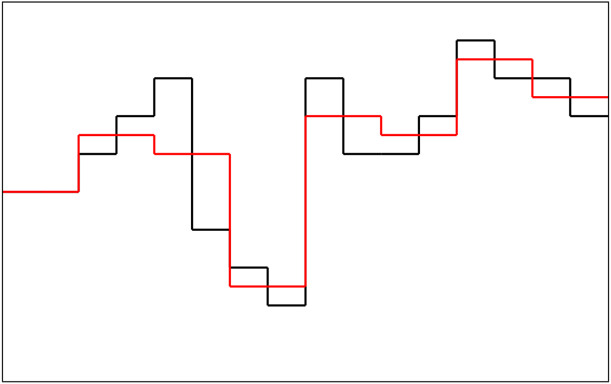

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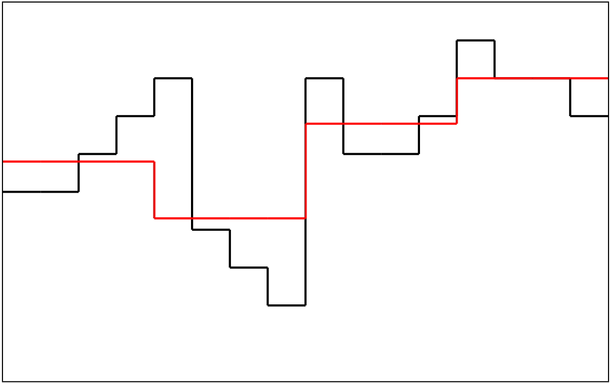

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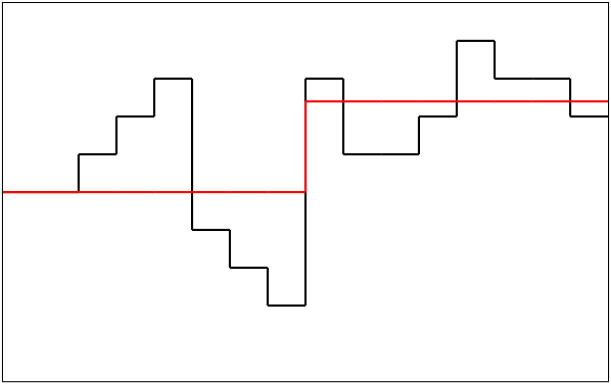

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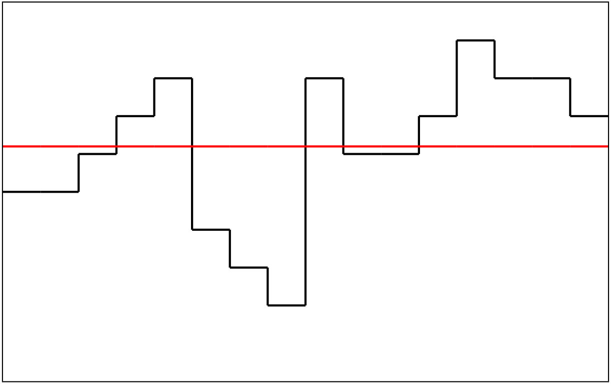

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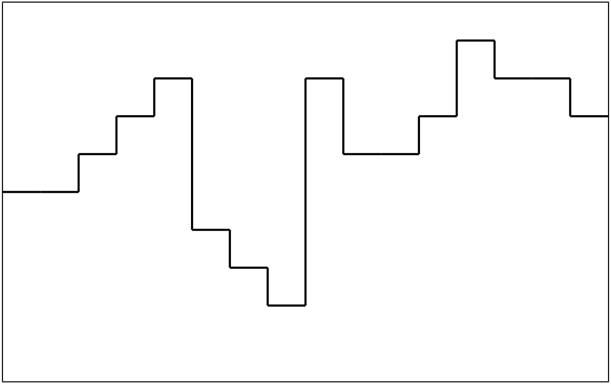

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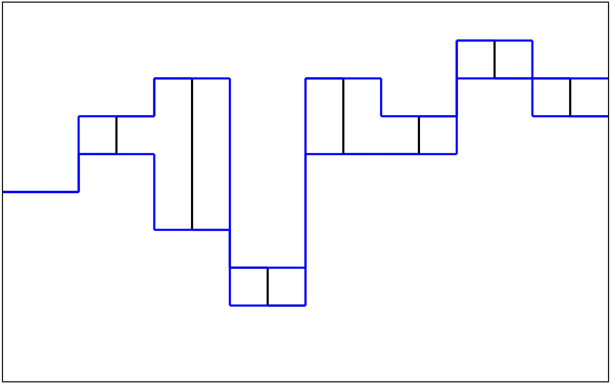

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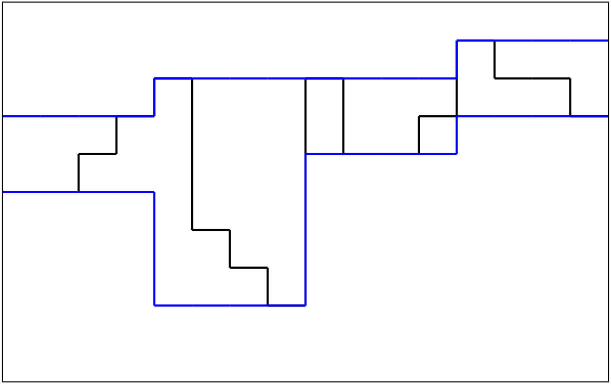

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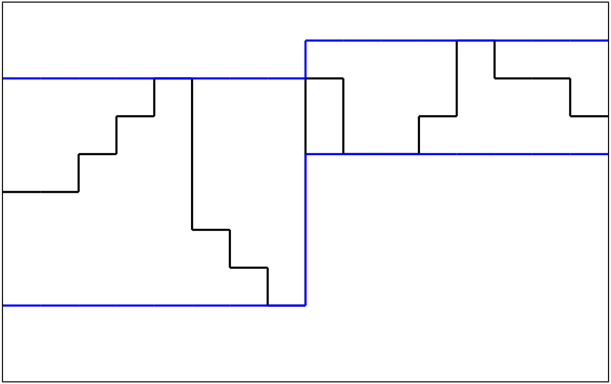

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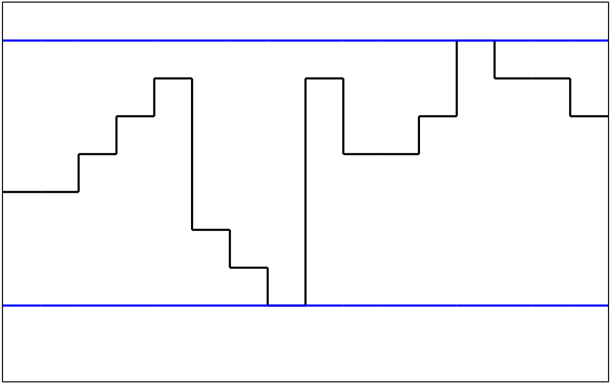

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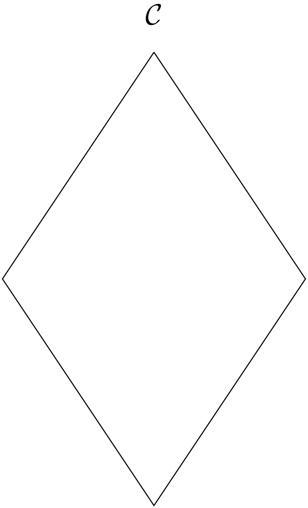
Definition

Let $\mathcal{C} = (\mathcal{C}, \leq_{\mathcal{C}})$ and $\mathcal{D} = (\mathcal{D}, \leq_{\mathcal{D}})$ be two partially ordered sets. If there are two functions $\alpha : \mathcal{C} \to \mathcal{D}$ and $\gamma : \mathcal{D} \to \mathcal{C}$ such that for all $c \in \mathcal{C}$ and all $d \in \mathcal{D}$:

$$c \leq_{\mathcal{C}} \gamma(d) \text{ iff } \alpha(c) \leq_{\mathcal{D}} d,$$

then $(\mathcal{C}, \alpha, \gamma, \mathcal{D})$ form a Galois connection.

Relating Concrete and Abstract Properties



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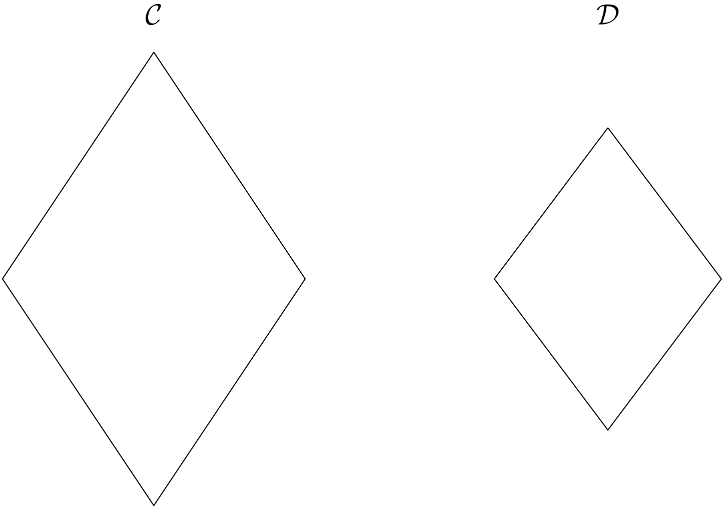

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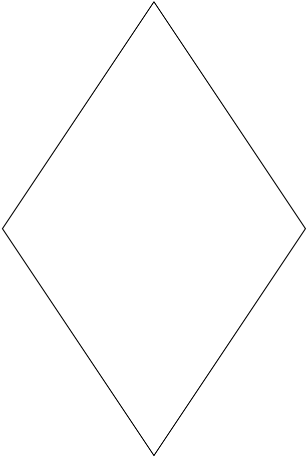

e.g. $\mathcal{P}(\mathbb{Z})$

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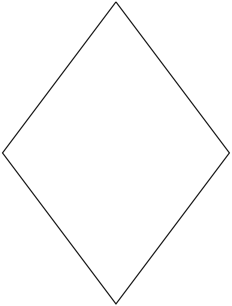

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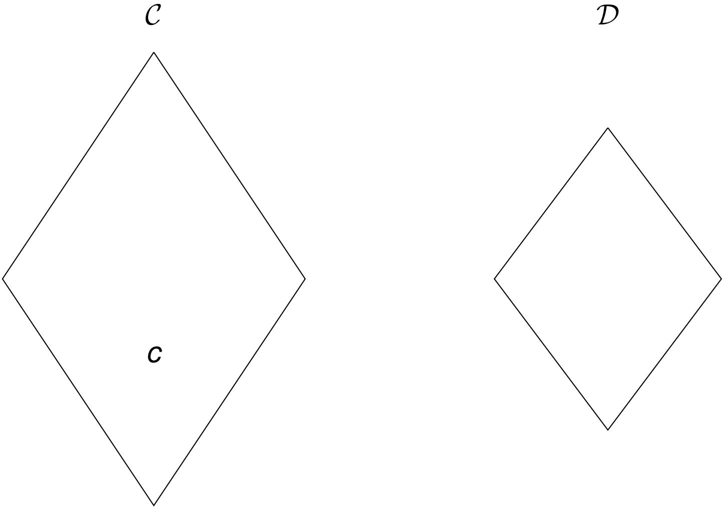

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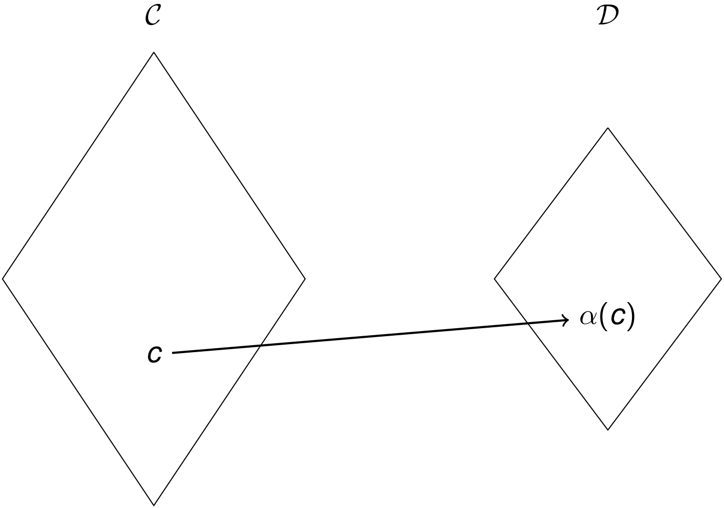

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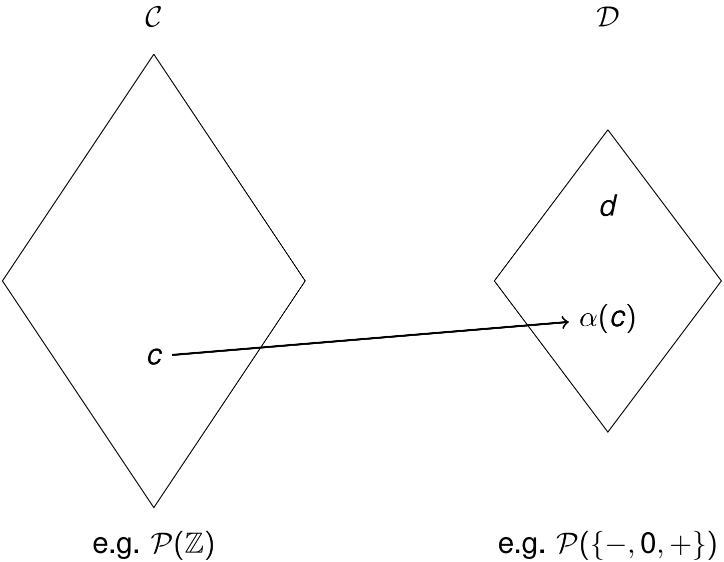
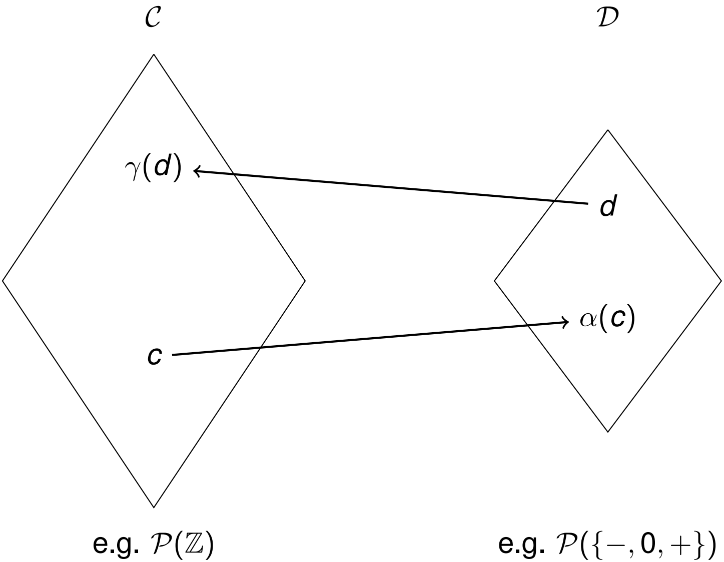


Relating Concrete and Abstract Properties



Galois Connections

Definition

Let $\mathcal{C} = (\mathcal{C}, \leq_{\mathcal{C}})$ and $\mathcal{D} = (\mathcal{D}, \leq_{\mathcal{D}})$ be two partially ordered sets with two order-preserving functions $\alpha : \mathcal{C} \mapsto \mathcal{D}$ and $\gamma : \mathcal{D} \mapsto \mathcal{C}$. Then $(\mathcal{C}, \alpha, \gamma, \mathcal{D})$ form a Galois connection iff

(i) $\alpha \circ \gamma$ is reductive i.e. $\forall d \in \mathcal{D}$, $\alpha \circ \gamma(d) \leq_{\mathcal{D}} d$,

(ii) $\gamma \circ \alpha$ is extensive i.e. $\forall c \in \mathcal{C}$, $c \leq_{\mathcal{C}} \gamma \circ \alpha(c)$.

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Proposition

Let $(\mathcal{C}, \alpha, \gamma, \mathcal{D})$ be a Galois connection. Then α and γ are quasi-inverse, i.e.

(i) $\alpha \circ \gamma \circ \alpha = \alpha$ *and* *(ii)* $\gamma \circ \alpha \circ \gamma = \gamma$

Uniqueness and Duality

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Let $(\mathcal{C}, \alpha, \gamma, \mathcal{D})$ be a Galois connection, then

(i) α uniquely determines γ by

$$\gamma(d) = \bigsqcup\{c \mid \alpha(c) \leq_{\mathcal{D}} d\},$$

and γ uniquely determines α via

$$\alpha(c) = \bigsqcap\{d \mid c \leq_{\mathcal{C}} \gamma(d)\}.$$

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For a proof see e.g. [3] Lemma 4.22.

Correctness and Optimality

Proposition

Given $\alpha : \mathcal{P}(\mathbb{Z}) \to \mathcal{D}$ and $\gamma : \mathcal{D} \to \mathcal{P}(\mathbb{Z})$ a Galois connection with $\mathcal{D}$ some property lattice. Consider an operation $op : \mathbb{Z} \to \mathbb{Z}$ on $\mathbb{Z}$ which is lifted to $\widehat{op} : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$ via

$$\widehat{op}(X) = \{op(x) \mid x \in X\},$$

then $op^\# : \mathcal{D} \to \mathcal{D}$ defined as $op^\# = \alpha \circ \widehat{op} \circ \gamma$ is the most precise function on $\mathcal{D}$ satisfying for all $Z \subseteq \mathbb{Z}$:

$$\alpha(\widehat{op}(Z)) \sqsubseteq op^\#(\alpha(Z))$$

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It is enough to consider so-called Galois Insertions.
See [1] Lemma 2.3.2.

General Construction

The general construction of correct (and optimal) abstractions $f^{\#}$ of concrete function f is as follows:

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\begin{tikzcd}
\mathcal{A} \arrow[r, shift left, "\alpha"] \arrow[d, "f"'] & \mathcal{A}^{\#} \arrow[l, shift left, "\gamma"] \arrow[d, "f^{\#}"] \\
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Induced semantics:

$$f^{\#} = \alpha' \circ f \circ \gamma.$$

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$\times^{\#}$	$\bot$	**even**	**odd**	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
even	$\bot$	**even**	**even**	**even**
odd	$\bot$	**even**	**odd**	$\top$
$\top$	$\bot$	**even**	$\top$	$\top$

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even	$\perp$	**even**	**even**	**even**
odd	$\perp$	**even**	**odd**	$\top$
$\top$	$\perp$	**even**	$\top$	$\top$

Abstract Interpretation – introduced by Patrick Cousot and Radhia Cousot in 1977 – allows to "compute" abstractions which are correct by construction.

Parity (again)

Consider concrete $\mathcal{C} = \mathcal{P}(\mathbb{Z})$ and abstract $\mathcal{D} = \mathcal{P}(\{\textbf{even}, \textbf{odd}\})$.

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The abstraction $\alpha : \mathcal{C} \to \mathcal{D}$ is given by for $X \subseteq \mathbb{Z}$:

$$\begin{aligned}
\alpha(\emptyset) &= \bot = \emptyset \\
\alpha(X) &= \textbf{even} \text{ iff } \forall x \in X \, \exists k : x = 2k \\
\alpha(X) &= \textbf{odd} \text{ iff } \forall x \in X \, \exists k : x = 2k + 1 \\
\alpha(X) &= \top = \{\textbf{even}, \textbf{odd}\} \text{ otherwise}
\end{aligned}$$

Parity (again)

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The concretisation $\gamma : \mathcal{D} \to \mathcal{C}$ then needs to be:

$$\begin{array}{rcl} \gamma(\bot) & = & \emptyset \\ \gamma(\textbf{even}) & = & \{x \in \mathbb{Z} \mid \exists k : x = 2k\} = E \\ \gamma(\textbf{odd}) & = & \{x \in \mathbb{Z} \mid \exists k : x = 2k+1\} = O \\ \gamma(\top) & = & \top = \mathbb{Z} \text{ otherwise} \end{array}$$

Parity: From $\times$ to $\times^{\#}$

To construct $\times^{\#}$ using α and γ we need to lift $. \times . : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ to $.\widehat{\times}. : \mathcal{P}(\mathbb{Z}) \times \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$.

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$$X \widehat{\times} Y = \{x \times y \mid x \in X \text{ and } y \in Y\}$$

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- etc.

Therefore, **even** $\times^{\#}$ **even** $=$ **even**, **even** $\times^{\#}$ **odd** $=$ **even**, etc.

Concrete Semantics $\rightarrow$ and Abstract Semantics $\rightsquigarrow$

Imagine some programming language, e.g. WHILE. Its concrete semantics identifies values in $\mathcal{V}$ (e.g. states) and specifies how a program S transforms v_1 into v_2;

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Unlike for general semantics, it is customary to require $\rightsquigarrow$ to be deterministic and thus define a function; this allows us to write:

$$f_S(l_1) = l_2 \text{ to mean } S \vdash l_1 \rightsquigarrow l_2.$$

Situation in While

We have SOS transitions $\langle S, s\rangle \Rightarrow \langle S', s'\rangle$ with S and S' programs and $s, s' \in \mathbf{State} = (\mathbf{Var} \to \mathbf{Z})$, e.g.

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translates to just an evaluation of the state:

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The fact that this also holds for the (abstract) parity means:

$$z := 2 \times z \vdash \textbf{even}(z) \rightsquigarrow \textbf{even}(z)$$

and also $z := 2 \times z \vdash \textbf{odd}(z) \rightsquigarrow \textbf{even}(z)$.

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For a class of (so-called first-order) program analyses this is established by directly relating properties to values using a correctness relation:

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The intention is that "$v \ \rhd\ l$" formalises our claim that the value v is described by the property l (or v abstracts to l).

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This property is also expressed by the following diagram:

$$\begin{array}{cccccc} S & \vdash & v_1 & \to & v_2 \\ & & \vdots & & \vdots \\ & & \triangleright & \Rightarrow & \triangleright \\ & & \vdots & & \vdots \\ S & \vdash & l_1 & \leadsto & l_2 \end{array}$$

Correctness of Parity

0	$\triangleright$	**even**	1	$\triangleright$	**odd**
2	$\triangleright$	**even**	3	$\triangleright$	**odd**
4	$\triangleright$	**even**	5	$\triangleright$	**odd**
	...			...	

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$$\begin{array}{c|c} z := 2 \times z \vdash [z \mapsto 1] \to [z \mapsto 2] & \textbf{odd}(z) \leadsto \textbf{even}(z) \\ z := 2 \times z \vdash [z \mapsto 2] \to [z \mapsto 4] & \textbf{even}(z) \leadsto \textbf{even}(z) \\ z := 2 \times z \vdash [z \mapsto 3] \to [z \mapsto 6] & \textbf{odd}(z) \leadsto \textbf{even}(z) \\ \dots & \dots \end{array}$$

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$$\begin{array}{c} 1 \rhd \textbf{odd} \;\wedge\; p \vdash 1 \to 2 \;\wedge\; p \vdash \textbf{odd} \rightsquigarrow \textbf{even} \Rightarrow 2 \rhd \textbf{even} \\ 2 \rhd \textbf{even} \;\wedge\; p \vdash 2 \to 4 \;\wedge\; p \vdash \textbf{even} \rightsquigarrow \textbf{even} \Rightarrow 4 \rhd \textbf{even} \\ 3 \rhd \textbf{odd} \;\wedge\; p \vdash 3 \to 6 \;\wedge\; p \vdash \textbf{odd} \rightsquigarrow \textbf{even} \Rightarrow 6 \rhd \textbf{even} \\ \ldots \end{array}$$

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Thus it is correct: "$p \equiv z := 2 \times z$ always produces an **even** z".

Abstract Interpretation and Correctness

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The most common scenario is when $\mathcal{L} = (\mathcal{L}, \sqsubseteq, \sqcup, \sqcap, \perp, \top)$ is a complete lattice with partial ordering $\sqsubseteq$.

We then impose the following relationship between $\rhd$ and $\mathcal{L}$:

$$v \rhd l_1 \;\wedge\; l_1 \sqsubseteq l_2 \quad \Rightarrow \quad v \rhd l_2 \tag{1}$$

$$\forall l \in \mathcal{L}' \subseteq \mathcal{L} : v \rhd l \quad \Rightarrow \quad v \rhd \bigsqcap \mathcal{L}' \tag{2}$$

Condition (1)

Consider the first of these conditions:

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$$v \rhd l_1 \;\wedge\; l_1 \sqsubseteq l_2 \Rightarrow v \rhd l_2$$

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- This is an "arbitrary" decision in the sense that we could instead have decided that the larger the property is, the better it is, as is indeed the case in much of the literature on Data Flow Analysis; luckily the principle of duality from lattice theory tells us that this difference is only cosmetic.

Condition (2)

Looking at the second condition describing correctness:

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- The condition has two immediate consequences:

$$v \rhd \top$$

$$v \rhd l_1 \wedge v \rhd l_2 \Rightarrow v \rhd (l_1 \sqcap l_2)$$

Again: Parity Example

The abstract properties **even** and **odd** do themselves not form a lattice $\mathcal{L}$, but we can use – as usual: $\mathcal{L} = \mathcal{P}(\{\textbf{even}, \textbf{odd}\})$, where $\{\textbf{even}\}$ represents the definitive fact **even** and $\{\textbf{odd}\}$ the precise property **odd**; while the empty set $\bot = \emptyset$ represents an undefined parity and $\top = \{\textbf{even}, \textbf{odd}\}$ stands for any parity.

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(2) The most precise parity is valid, e.g.

$$(2 \triangleright \{\textbf{even}\} \ \wedge \ 2 \triangleright \top) \Rightarrow 2 \triangleright (\{\textbf{even}\} \sqcap \top)$$

i.e. $(2 \triangleright \{\textbf{even}\} \ \wedge \ 2 \triangleright \top) \Rightarrow 2 \triangleright \{\textbf{even}\}$

Preservation of Correctness via Abstraction

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With a (semantical transfer) function function f_S we have:

$$v_1 \triangleright l_1 \;\wedge f_S(v_1) = v_2 \wedge f_S^{\#}(l_1) = l_2 \Rightarrow v_2 \triangleright l_2$$

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This property is also expressed by the following diagram:

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "f_S"] \arrow[d, shift right, "\alpha"'] & \mathcal{V} \arrow[d, shift left, "\alpha"] \\
\mathcal{L} \arrow[r, "f_S^{\#}"'] \arrow[u, shift right, "\gamma"'] & \mathcal{L} \arrow[u, shift left, "\gamma"]
\end{tikzcd}$$

Representation and Extraction Functions

We can use a representation function $\beta : \mathcal{V} \to \mathcal{L}$ to induce a Galois connection $(\mathcal{P}(\mathcal{V}), \alpha, \gamma, \mathcal{L})$ via

$$\begin{array}{rcl} \alpha(V) & = & \bigsqcup\{\beta(v) \mid v \in V\} \\ \gamma(l) & = & \{v \in V \mid \beta(v) \sqsubseteq l\} \end{array}$$

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For $\mathcal{L} = \mathcal{P}(\mathcal{D})$ with $\mathcal{D}$ being some set of "abstract values" we can also use an extraction function, $\eta : \mathcal{V} \to \mathcal{D}$ defined as

$$\begin{array}{rcl} \alpha(V) & = & \{\eta(v) \mid v \in V\} \\ \gamma(D) & = & \{v \mid \eta(v) \in D\} \end{array}$$

in order to construct a Galois connection.

Example: Parity

A repesentation function $\beta : \mathbf{Z} \rightarrow \mathcal{P}(\{\mathbf{even}, \mathbf{odd}\})$ is easily defined by:

$$\beta(n) = \begin{cases} \{\mathbf{even}\} & \text{if } \exists k \in \mathbf{Z} \text{ s.t. } n = 2k \\ \{\mathbf{odd}\} & \text{otherwise} \end{cases}$$

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Correctness implies that the abstract properties are dominated by the actual ones, e.g. $\beta(4) = \{\mathbf{even}\} \sqsubseteq \top = \{\mathbf{even}, \mathbf{odd}\}$ is acceptable.

This means that we also could use as a representation function

$$\beta(n) = \top = \{\mathbf{even}, \mathbf{odd}\}$$

for all $n \in \mathbf{Z}$. Though this would be valid it would also be rather imprecise.

References Abstract Interpretation

[1] Neil D. Jones and Flemming Nielson: *Abstract Interpretation: A semantics-based tool for program analysis*. in: Handbook of Logic in Computer Science (Vol. 4), pp 527–636, Oxford University Press,1995.

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