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Monty Hall

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The game show proceeds as follows: First the contestant is invited to pick one of three doors (behind one is the prize) but the door is not yet opened.

Instead, the host – legendary Monty Hall – opens one of the other doors which is empty.

After that the contestant is given a last chance to stick with his/her door or to switch to the other closed one.

Note that the host (knowing where the prize is) has always at least one door he can open.
Monty Hall - Stick $H_t$

```
var
d :{0,1,2}; g :{0,1,2}; o :{0,1,2};
begin
  d ?= {0,1,2}; # Pick winning door
  g ?= {0,1,2}; # Pick guessed door
  o ?= {0,1,2}; # Open empty door
  while ((o == g) || (o == d)) do
    o := (o+1)%3; od;
  # Stick with guess
  stop; # looping
end
```
Monty Hall - Switch $H_w$

```plaintext
var
d :\{0,1,2\}; g :\{0,1,2\}; o :\{0,1,2\};
begin
d \?= \{0,1,2\}; # Pick winning door
g \?= \{0,1,2\}; # Pick guessed door
o \?= \{0,1,2\}; # Open empty door
while ((o == g) || (o == d)) do
  o := (o+1)%3; od;
g := (g+1)%3; # Switch guess
while (g == o) do
  g := (g+1)%3; od;
stop; # looping
end
```
Monty Hall – Labelling $H_w$

var
    d :\{0,1,2\};   g :\{0,1,2\};   o :\{0,1,2\};
begin
    \[d \xleftarrow{\text{\{0,1,2\}}} \]
    \[g \xleftarrow{\text{\{0,1,2\}}} \]
    \[o \xleftarrow{\text{\{0,1,2\}}} \]
while \[((o == g)\lor(o == d))\] do
    \[o := (o+1)\mod{3} \]
od;
\[g := (g+1)\mod{3} \]
while \[(g == o)\] do
    \[g := (g+1)\mod{3} \]
od;
\[\text{stop} \]
end

Labelling for $H_t$ is a “sub-labelling” (with labels 1, 2, 3, 4, 5, 6, $\equiv$ 9).
Monty Hall – Labelling $H_w$

```plaintext
var
d : {0,1,2};
g : {0,1,2};
o : {0,1,2};
begin
[d := {0,1,2}]
1;
[g := {0,1,2}]
2;
[o := {0,1,2}]
3;
while (((o == g) || (o == d))]
4 do
    [o := (o+1)%3]
5;
end;
[g := (g+1)%3]
6;
while [(g == o)]
7 do
    [g := (g+1)%3]
8;
end;
[stop]
9;
end
```

Labelling for $H_t$ is a “sub-labelling” (with labels $1\ldots6 \equiv 9$).
blocks($H_w$) =

= \{ [d \Leftarrow \{ 0, 1, 2 \}]^1, [g \Leftarrow \{ 0, 1, 2 \}]^2, [o \Leftarrow \{ 0, 1, 2 \}]^3, [((o == g) || (o == d))]^4, [o := ((o + 1) \% 3)]^5, [g := ((g + 1) \% 3)]^6, [(g == o)]^7, [g := ((g + 1) \% 3)]^8, [stop]^9 \}
blocks(Hw) =

= \{ [d \equiv \{ 0, 1, 2 \}]^1, [g \equiv \{ 0, 1, 2 \}]^2, \\
[ o \equiv \{ 0, 1, 2 \}]^3, [((o == g) || (o == d))]^4, \\
[ o \equiv ((o + 1) \% 3)]^5, [g \equiv ((g + 1) \% 3)]^6, \\
[ (g == o)]^7, [g \equiv ((g + 1) \% 3)]^8, [\text{stop}]^9 \}

flow(Hw) =

= \{(1, 1, 2), (2, 1, 3), (3, 1, 4), (4, 1, 5), (5, 1, 4), (4, 1, 6), \\
(6, 1, 7), (7, 1, 8), (8, 1, 7), (7, 1, 9), (9, 1, 9)\}
blocks($H_w$) =

= $\{ [d \ ?= \{ 0, 1, 2 \}]^1, [g \ ?= \{ 0, 1, 2 \}]^2,$
  $[o \ ?= \{ 0, 1, 2 \}]^3, [((o == g) || (o == d))]^4,$
  $[o := ((o + 1) \ % \ 3)]^5, [g := ((g + 1) \ % \ 3)]^6,$
  $[(g == o)]^7, [g := ((g + 1) \ % \ 3)]^8, [\text{stop}]^9 \}$

flow($H_w$) =

= $\{(1, 1, 2), (2, 1, 3), (3, 1, 4), (4, 1, 5), (5, 1, 4), (4, 1, 6),$
  $(6, 1, 7), (7, 1, 8), (8, 1, 7), (7, 1, 9), (9, 1, 9)\}$

Again for $H_t$ consider a restricted version of flow($H_w$).
Monty Hall – Stick $H_t$

$$T(H_t) = \frac{1}{3} (U(d \leftarrow 0) + U(d \leftarrow 1) + U(d \leftarrow 2)) \otimes E(1, 2) + \frac{1}{3} (U(g \leftarrow 0) + U(g \leftarrow 1) + U(g \leftarrow 2)) \otimes E(2, 3) + \frac{1}{3} (U(o \leftarrow 0) + U(o \leftarrow 1) + U(o \leftarrow 2)) \otimes E(3, 4) + P((o == g) || (o == d) = tt) \otimes E(4, 5) + P((o == g) || (o == d) = ff) \otimes E(4, 6) + I \otimes E(6, 6)$$
Monty Hall – Switch $H_w$

\[
T(H_w) = \frac{1}{3} \left( U(d \leftarrow 0) + U(d \leftarrow 1) + U(d \leftarrow 2) \right) \otimes E(1, 2) + \\
\frac{1}{3} \left( U(g \leftarrow 0) + U(g \leftarrow 1) + U(g \leftarrow 2) \right) \otimes E(2, 3) + \\
\frac{1}{3} \left( U(o \leftarrow 0) + U(o \leftarrow 1) + U(o \leftarrow 2) \right) \otimes E(3, 4) + \\
P((o == g) || (o == d) = \text{tt}) \otimes E(4, 5) + \\
P((o == g) || (o == d) = \text{ff}) \otimes E(4, 6) + \\
U(g \leftarrow (g + 1) \mod 3) \otimes E(6, 7) + \\
P((g == o) = \text{tt}) \otimes E(7, 8) + \\
P((g == o) = \text{ff}) \otimes E(7, 9) + \\
U(g \leftarrow (g + 1) \mod 3) \otimes E(6, 7) + I \otimes E(9, 9)
\]
Monty Hall – Enumeration of States

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<tr>
<td>27</td>
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<td>(d $\mapsto$ 2, g $\mapsto$ 2, o $\mapsto$ 2)</td>
</tr>
</tbody>
</table>
Monty Hall – $T(1,2)$

\[
T(1,2) = \begin{pmatrix}
\frac{1}{3} & \cdots & \frac{1}{3} & \cdots & \frac{1}{3} & \cdots \\
\frac{1}{3} & \cdots & \frac{1}{3} & \cdots & \frac{1}{3} & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\frac{1}{3} & \cdots & \frac{1}{3} & \cdots & \frac{1}{3} & \cdots \\
\frac{1}{3} & \cdots & \frac{1}{3} & \cdots & \frac{1}{3} & \cdots \\
\frac{1}{3} & \cdots & \frac{1}{3} & \cdots & \frac{1}{3} & \cdots \\
\end{pmatrix} \otimes E(1,2)
\]
Monty Hall – $T(2, 3)$

$$T(2, 3) = \mathcal{E}(2, 3)$$
Monty Hall – $T(3, 4)$

$$T(3, 4) = \bigotimes E(3, 4)$$
Monty Hall – $\mathbf{T}(4, 5)$ and $\mathbf{T}(4, 6)$

The test operators or filters at label 4 are diagonal matrices. Note that $\mathbf{T}(4, 5) + \mathbf{T}(4, 6) = \mathbf{I}_{27}$.

$\mathbf{T}(4, 5) = \text{diag}(1 0 0 1 1 0 1 0 1 1 1 0 0 1 0 0 1 1 1 0 1 0 1 1 0 0 1) \otimes \mathbf{E}(4, 5)$

$\mathbf{T}(4, 6) = \text{diag}(0 1 1 0 0 1 0 1 0 0 0 1 1 0 1 1 0 0 0 1 0 1 0 0 1 1 0) \otimes \mathbf{E}(4, 6)$
Monty Hall – $T(5, 4)$

\[
T(5, 4) = \left( \begin{array}{ccccccc}
1 & & & & & & \\
1 & 1 & & & & & \\
1 & & 1 & & & & \\
1 & & & 1 & & & \\
1 & & & & 1 & & \\
1 & & & & & 1 & \\
1 & & & & & & 1
\end{array} \right) \otimes E(5, 4)
\]
Monty Hall – $\mathbf{T}(6, 7)$

$$
\mathbf{T}(6, 7) = \begin{pmatrix}
\begin{array}{ccccccc}
1 & & & & & & \\
& 1 & & & & & \\
& & 1 & & & & \\
& & & 1 & & & \\
& & & & 1 & & \\
& & & & & 1 & \\
& & & & & & 1
\end{array}
\end{pmatrix} \otimes \mathbf{E}(6, 7)
$$
Monty Hall – $T(7, 8)$ and $T(7, 9)$

The test operators or filters at label 7 are also diagonal matrices, and again $T(7, 8) + T(7, 9) = I_{27}$.

$T(7, 8) = \text{diag} (1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1) \otimes E(7, 8)$

$T(7, 9) = \text{diag} (0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1) \otimes E(7, 9)$
Monty Hall – $T(8, 7)$

$$T(8, 7) = \begin{pmatrix}
\vdots & 1 & \vdots & & \vdots & & \vdots & & 1 & \vdots \\
1 & & 1 & & 1 & & 1 & & 1 & \\
1 & \vdots & 1 & & \vdots & & \vdots & & 1 & \\
& & 1 & & 1 & & 1 & & 1 & \\
& & & & 1 & & 1 & & 1 & \\
& & & & & & 1 & & 1 & \\
& & & & & & & & 1 & \\
& & & & & & & & & & 1
\end{pmatrix} \otimes E(8, 7)$$
Monty Hall – $\mathbf{T}(H_s)$ and $\mathbf{T}(H_s)$

\[
\mathbf{T}(H_t) = \mathbf{T}(1, 2) + \mathbf{T}(2, 3) + \mathbf{T}(3, 4) + \mathbf{T}(4, 5) + \mathbf{T}(5, 4) + \mathbf{T}(4, 6) + \\
+ \mathbf{I} \otimes \mathbf{E}(6, 6)
\]
Monty Hall – $\mathbf{T}(H_s)$ and $\mathbf{T}(H_t)$

$\mathbf{T}(H_t) = \mathbf{T}(1, 2) + \mathbf{T}(2, 3) + \mathbf{T}(3, 4) + \mathbf{T}(4, 5) + \mathbf{T}(5, 4) + \mathbf{T}(4, 6) + I \otimes \mathbf{E}(6, 6)$

$\mathbf{T}(H_w) = \mathbf{T}(1, 2) + \mathbf{T}(2, 3) + \mathbf{T}(3, 4) + \mathbf{T}(4, 5) + \mathbf{T}(5, 4) + \mathbf{T}(4, 6) + \mathbf{T}(6, 7) + \mathbf{T}(7, 8) + \mathbf{T}(8, 7) + \mathbf{T}(7, 9) + I \otimes \mathbf{E}(9, 9)$
Monty Hall – $\mathbf{T}(H_S)$ and $\mathbf{T}(H_S)$

$$\mathbf{T}(H_t) = \mathbf{T}(1, 2) + \mathbf{T}(2, 3) + \mathbf{T}(3, 4) + \mathbf{T}(4, 5) + \mathbf{T}(5, 4) + \mathbf{T}(4, 6) +$$
$$+ \mathbf{I} \otimes \mathbf{E}(6, 6)$$

$$\mathbf{T}(H_w) = \mathbf{T}(1, 2) + \mathbf{T}(2, 3) + \mathbf{T}(3, 4) + \mathbf{T}(4, 5) + \mathbf{T}(5, 4) + \mathbf{T}(4, 6) +$$
$$+ \mathbf{T}(6, 7) + \mathbf{T}(7, 8) + \mathbf{T}(8, 7) + \mathbf{T}(7, 9) + \mathbf{I} \otimes \mathbf{E}(9, 9)$$

$$\dim(\mathbf{T}(H_t)) = 27 \cdot 6 = 162 \quad \text{and} \quad \dim(\mathbf{T}(H_w)) = 27 \cdot 9 = 243$$
Initial configuration is 162 or 243 dimensional (last: \( \text{dim} = 6/9 \))

\[ x_0 = \left( \begin{array} {ccc} 1 & 0 & 0 \\ \end{array} \right) \otimes \left( \begin{array} {ccc} 1 & 0 & 0 \\ \end{array} \right) \otimes \left( \begin{array} {ccc} 1 & 0 & 0 \\ \end{array} \right) \otimes \left( \begin{array} {ccc} 1 & 0 & 0 & \ldots & 0 \\ \end{array} \right) \]
Initial configuration is 162 or 243 dimensional (last: \( \text{dim} = 6/9 \))

\[ x_0 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} \]

Final configurations of the same dimension, non-zero entries:

For \( H_t \)

\[
\begin{align*}
x_{12} &= 0.074074 \\
x_{18} &= 0.037037 \\
x_{36} &= 0.11111 \\
x_{48} &= 0.11111 \\
x_{72} &= 0.11111 \\
x_{78} &= 0.037037 \\
x_{90} &= 0.074074 \\
x_{96} &= 0.11111 \\
x_{120} &= 0.11111 \\
x_{132} &= 0.11111 \\
x_{150} &= 0.074074 \\
x_{156} &= 0.037037
\end{align*}
\]

For \( H_w \)

\[
\begin{align*}
x_{18} &= 0.11111 \\
x_{27} &= 0.11111 \\
x_{54} &= 0.037037 \\
x_{72} &= 0.074074 \\
x_{108} &= 0.074074 \\
x_{117} &= 0.11111 \\
x_{135} &= 0.11111 \\
x_{144} &= 0.037037 \\
x_{180} &= 0.037037 \\
x_{198} &= 0.074074 \\
x_{225} &= 0.11111 \\
x_{234} &= 0.11111
\end{align*}
\]
Consider the terminal configurations

\[ x_t = \lim_{n \to \infty} (T(H_t))^n x_0 \quad \text{and} \quad x_w = \lim_{n \to \infty} (T(H_w))^n x_0 \]

Abstract relevant information with \( A = I \) and \( A_f = (1, 1, \ldots, 1)^t \):

\[
\begin{align*}
    x_t \cdot (I \otimes I \otimes A_f \otimes A_f) &= \\
    &\begin{pmatrix}
        0.11 & 0.11 & 0.11 & 0.11 & 0.11 & 0.11 & 0.11 & 0.11 & 0.11 \\
        0.22 & 0.04 & 0.07 & 0.07 & 0.22 & 0.04 & 0.04 & 0.07 & 0.22
    \end{pmatrix}
\end{align*}
\]
Monty Hall – Optimal Strategy

With a further abstraction (first column = winning):

\[ A^t_w = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \]

we get

\[ x_t \cdot (A_w \otimes A_f \otimes A_f) = \begin{pmatrix} 0.33333 & 0.66667 \end{pmatrix} \]

and

\[ x_w \cdot (A_w \otimes A_f \otimes A_f) = \begin{pmatrix} 0.66667 & 0.33333 \end{pmatrix} \]