Quantum Computation (CO484)

Quantum Measurement and Registers

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Basic Measurement Principle

The values α and β describing a qubit are often called **probability amplitudes.** If we measure a qubit

$$|\phi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

in the **computational basis** $\{|0\rangle\,, |1\rangle\}$ then we observe state $|0\rangle$ with probability $|\alpha|^2$ and $|1\rangle$ with probability $|\beta|^2$.

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Furthermore, the state $|\phi\rangle$ changes: it **collapses** into state $|0\rangle$ with probability $|\alpha|^2$ or $|1\rangle$ with probability $|\beta|^2$, respectively.

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Probability to observe λ_k in state $|x\rangle = \sum_i \alpha_i\,|i\rangle$ is

$$Pr(A = \lambda_k, |x\rangle) = |\alpha_k|^2$$

Physicist refer to α_k as probability amplitude.

Spectrum

The set of eigen-values $\{\lambda_1, \lambda_2, \ldots\}$ of an operator **T** is called its spectrum $\sigma(\mathbf{T})$.

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Terminology: "eigen" means "self" or "own" in German (cf also Italian "auto-valore"), it **characterises** a matrix/operator.

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Birkhoff-von Neumann: Projections on Hilbert space form an (ortho-)lattice which gives rise to non-classical “quantum logic”.

Outer Product

The outer product $|x\rangle\langle y|$ for vectors $|x\rangle = (x_1, \ldots, x_n)^T$ and $\langle y| = (y_1, \ldots, y_n)$ is an operator/matrix (actually: $|x\rangle \otimes \langle y|$):

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It could be treated just as a formal combination, e.g. we can express the identity as $\mathbf{I} = |0\rangle\langle 0| + |1\rangle\langle 1|$ because

$$\begin{aligned} (|0\rangle\langle 0| + |1\rangle\langle 1|)\,|\psi\rangle &= (|0\rangle\langle 0| + |1\rangle\langle 1|)(\alpha\,|0\rangle + \beta\,|1\rangle) \\ &= \alpha\,|0\rangle\langle 0||0\rangle + \alpha\,|1\rangle\langle 1||0\rangle + \\ &\quad\ \beta\,|0\rangle\langle 0||1\rangle + \beta\,|1\rangle\langle 1||1\rangle \\ &= \alpha\,|0\rangle + \beta\,|1\rangle \end{aligned}$$

Spectral Theorem

In the bra-ket notation we can represent a projection onto the sub-space generated by $|x\rangle$ by the outer product $\mathbf{P}_x = |x\rangle\langle x|$.

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Theorem

A self-adjoint operator $\mathbf{A}$ *(on a finite dimensional Hilbert space, e.g.* $\mathbb{C}^n$*) can be represented uniquely as a linear combination*

$$\mathbf{A} = \sum_i \lambda_i \mathbf{P}_i$$

with $\lambda_i \in \mathbb{R}$ *and* $\mathbf{P}_i$ *the (orthogonal) projection onto the eigen-space generated by the eigen-vector* $|i\rangle$*, i.e.*

$$\mathbf{P}_i = |i\rangle\langle i|$$

Measurement Process

If we perform a measurement of the observable represented by:

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with eigen-values λ_i and eigen-vectors $|i\rangle$ in a state $|x\rangle$ we have to decompose the state according to the observable, i.e.

$$|x\rangle = \sum_i \mathbf{P}_i |x\rangle = \sum_i |i\rangle\langle i|x\rangle = \sum_i \langle i|x\rangle |i\rangle = \sum_i \alpha_i |i\rangle$$

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- The measurement instrument will the **display** λ_i.
- The state $|x\rangle$ **collapses** to $|i\rangle$.

Do-It-Yourself Observable

We can take any (orthonormal) basis $\{|i\rangle\}_0^n$ of $\mathbb{C}^{n+1}$ to act as **computational basis**. We are free to choose (different) measurement results λ_i to indicate different states in $\{|i\rangle\}$.

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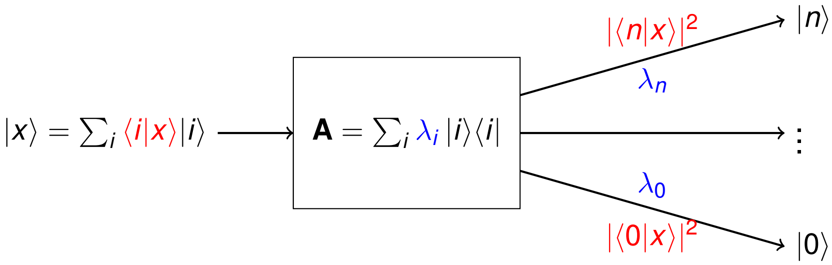

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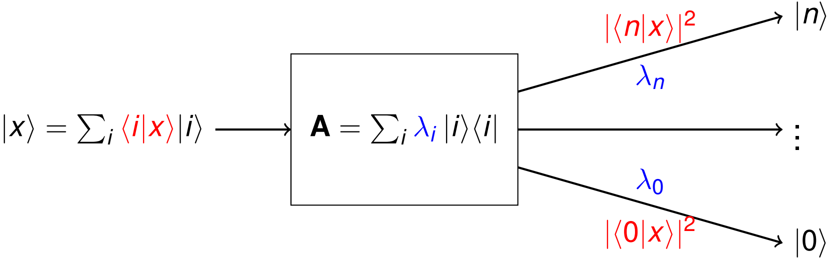

The “display” values λ_i are **essential** for physicists, in a quantum computing context they are just **side-effects.**

Reversibility

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i.e. the quantum measurement is not **reversible.** However

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i.e. the quantum measurement is **idempotent.**

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Though it might one day be physically more realistic/cheaper to build quantum devices based on not just binary basic states, even then it will be necessary to combine these larger "Qubits".

Free Vector Spaces

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For vector spaces there is similar construction: Take any (finite) set of objects B and "declare" it a base. The free vector space is the set of all linear combinations of elements in $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots\}$, i.e.

$$\mathcal{V}(B) = \left\{ \sum_i \lambda_i \mathbf{b}_i \mid \lambda_i \in \mathbb{C} \text{ and } \mathbf{b}_i \in B \right\}$$

or

$$\mathcal{V}(B) = \left\{ \sum_i \lambda_i \, |i\rangle \ \mid \lambda_i \in \mathbb{C} \text{ and } \ |i\rangle \in B \right\}$$

with the obvious algebraic operations (incl. inner product).

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The state space of a **two qubit** system is given by

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i.e. the base vectors are (in the standard base):

$$B_2 = \{(0,0),(1,0),(0,1),(1,1)\}$$

or we use a "short-hand" notation $B_2 = \{00, 01, 10, 11\}$

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Issue: What about $\mathcal{V}(B \times B \times B)$? What is its dimension, or how many base vectores are there in B_3?

Tensor Product

Given a $n \times m$ **matrix A** and a $k \times l$ matrix **B**:

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} b_{11} & \dots & b_{1l} \\ \vdots & \ddots & \vdots \\ b_{k1} & \dots & b_{kl} \end{pmatrix}$$

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Special cases are **square matrices** ($n = m$ and $k = l$) and **vectors** (row $n = k = 1$, column $m = l = 1$).

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$$(\alpha \mathbf{v} + \alpha' \mathbf{v}') \otimes (\beta \mathbf{w} + \beta' \mathbf{w}') =$$
$$= \alpha\beta(\mathbf{v} \otimes \mathbf{w}) + \alpha\beta'(\mathbf{v} \otimes \mathbf{w}') + \alpha'\beta(\mathbf{v}' \otimes \mathbf{w}) + \alpha'\beta'(\mathbf{v}' \otimes \mathbf{w}')$$

with $\alpha, \alpha', \beta, \beta' \in \mathbb{C}$, and $\mathbf{v}, \mathbf{v}' \in \mathbb{C}^k$, $\mathbf{w}, \mathbf{w}' \in \mathbb{C}^l$.

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2. For $\mathbf{v}, \mathbf{v}' \in \mathbb{C}^k$ and $\mathbf{w}, \mathbf{w}' \in \mathbb{C}^l$ we have:

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3. We denote by $\mathbf{b}_i^m \in B_m \subseteq \mathbb{C}^m$ the i'th basis vector in $\mathbb{C}^m$ then

$$\mathbf{b}_i^k \otimes \mathbf{b}_j^l = \mathbf{b}_{(i-1)l+j}^{kl}$$

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$\alpha, \alpha', \beta, \beta' \in \mathbb{C}$, $\mathbf{M}, \mathbf{M}'$ $m \times m$ matrices $\mathbf{N}, \mathbf{N}'$ $n \times n$ matrices.

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   $\alpha, \alpha', \beta, \beta' \in \mathbb{C}$, $\mathbf{M}, \mathbf{M}'$ $m \times m$ matrices $\mathbf{N}, \mathbf{N}'$ $n \times n$ matrices.

2. We have, with $\mathbf{v} \in \mathbb{C}^m$ and $\mathbf{w} \in \mathbb{C}^n$:

$$(\mathbf{M} \otimes \mathbf{N})(\mathbf{v} \otimes \mathbf{w}) = (\mathbf{M}\mathbf{v}) \otimes (\mathbf{N}\mathbf{w})$$
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Tensor Product of Matrices

For the tensor product of square matrices we also have:

1. The **bilinearity** property:

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3. If $\mathbf{M}$ and $\mathbf{N}$ are unitary (or invertible) so is $\mathbf{M} \otimes \mathbf{N}$, and:

$$(\mathbf{M} \otimes \mathbf{N})^T = \mathbf{M}^T \otimes \mathbf{N}^T \text{ and } (\mathbf{M} \otimes \mathbf{N})^\dagger = \mathbf{M}^\dagger \otimes \mathbf{N}^\dagger$$

The Two Qubit States

Given two Hilbert spaces $\mathcal{H}_1$ with basis B_1 and $\mathcal{H}_2$ with basis B_2 we can define the tensor product of **spaces** as

$$\mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{V}(\{\mathbf{b}_i \otimes \mathbf{b}_j \mid \mathbf{b}_i \in B_1, \mathbf{b}_j \in B_2\})$$

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Using the notation $|i\rangle \otimes |j\rangle = |i\rangle\,|j\rangle = |ij\rangle$ the standard base of the state space of a two qubit system $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$ are:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

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Often one also represents them using a “decimal” notation, i.e. $|00\rangle \equiv |0\rangle$, $|01\rangle \equiv |1\rangle$, $|10\rangle \equiv |2\rangle$, and $|11\rangle \equiv |3\rangle$.

Entanglement

The important relation between $\mathcal{V}(B)$, e.g. $\mathcal{V}(\{0,1\})$, and $\mathcal{V}(B^n)$, e.g. $\mathcal{V}(\{0,1\}^n)$ is given by $\mathcal{V}(B^n) = (\mathcal{V}(B))^{\otimes n}$, i.e.:

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Every n qubit state in $\mathbb{C}^{2^n}$ can be represented as a linear combination of the base vectors $|0\ldots00\rangle, |0\ldots10\rangle, \ldots,$ $|1\ldots11\rangle$ or decimal $|0\rangle, |1\rangle, |2\rangle, \ldots, \ldots, |2^n-1\rangle$.

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A two-qubit quantum state $|\psi\rangle \in \mathbb{C}^{2^2}$ is said to be separable iff there exist two single-qubit states $|\psi_1\rangle$ and $|\psi_2\rangle$ in $\mathbb{C}^2$ such that

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If $|\psi\rangle$ is not separable then we say that $|\psi\rangle$ is entangled.

Entanglement and Classical Probabilities

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If the events in Ω_{d_1} and Ω_{d_2} are independent ("uncorrelated") then their joint distribution is given as a product of distributions on Ω_{d_1} and Ω_{d_1}, i.e. $\rho = \rho_1 \otimes \rho_2$ or $P(\omega_i \wedge \omega_j) = P(\omega_i) \cdot P(\omega_j)$.

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$$\rho = \frac{1}{2}(1,0) \otimes (1,0)^T + \frac{1}{2}(0,1) \otimes (0,1)^T \neq \rho_1 \otimes \rho_2$$

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Consider random input $n \in \{1, 2, 3\}$ to the factorial, i.e. $P(n = 1) = P(n = 2) = P(n = 3) = \frac{1}{3}$. Determine the probability that $n!$ is **even** or **odd**.

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Entanglement represents correlation (non-independence):

$$P(\text{parity(n!)} \mid n) \neq P(\text{parity}(n!)) \otimes P(n).$$