Models of Computation II, Exercises 2: The Universal RM

1. 

\( B_0 = 2^{2 \times 1 + 1} \langle 1, 6 \rangle = 2^{2 \times 6 + 1} - 1 = 2^5 (2 \times 25 + 1) = 8 \times 51 = 408 \)

\( B_1 = 2 \times 2 + 1 \langle 2, 4 \rangle = 2^5 (2 \times 4 + 1) - 1 = 2^5 (2 \times 35 + 1) = 32 \times 71 = 2272 \)

\( B_2 = 2 \times 0 \langle 3 \rangle = 2^5 (2 \times 3 + 1) = 7 \)

\( B_3 = 2 \times 3 + 1 \langle 5, 0 \rangle = 2^5 (2 \times 0 + 1) - 1 = 2^5 (2 \times 31 + 1) = 8064 \)

\( B_4 = 2 \times 2 + 1 \langle 4, 1 \rangle = 2^4 (2 \times 4 + 1) = 144 \)

\( B_5 = 2 \times 2 \langle 4, 4 \rangle = 2^4 (2 \times 4 + 1) = 144 \)

\( B_6 = 0 \)

2. (a) To decode the program, we first decode it as a list \( l \) and then decode each individual instruction. First, observe that

\[ 2^{216} \times 833 = 2^{216} (2 \times 416 + 1) = \langle 216, 416 \rangle \]

so \( l = 216 :: l_1 \) for some list \( l_1 \) with \( \langle 1 \rangle = 416 \).

To decode \( l_1 \), we need to find \( x, y \) such that \( 2^x (2y + 1) = 416 \). We should therefore work out what power of 2 divides 416, which we can do by repeatedly factoring out 2:

\[ 416 = 2 \times 208 = 2^2 \times 104 = 2^3 \times 52 = 2^4 \times 26 = 2^5 \times 13 \]

Now, 2 does not divide 13, but \( 2 \times 6 + 1 = 13 \). Therefore \( \langle 1 \rangle = 2^5 (2 \times 6 + 1) = \langle 5, 6 \rangle \).

Consequently, \( l_1 = 5 :: l_2 \) for some list \( l_2 \) with \( \langle 1 \rangle = 6 \).

To decode \( l_2 \), we need to find \( x, y \) such that \( 2^x (2y + 1) = 6 \). It is easy to see that the solution is \( x = y = 1 \), so \( l_2 = 1 :: l_3 \) for some list \( l_3 \) with \( \langle 1 \rangle = 1 \).

Now \( 1 = 2^0 (2 \times 0 + 1) \), so \( l_3 = 0 :: l_4 \) for some list \( l_4 \) with \( \langle 1 \rangle = 0 \). It must be that \( l_4 = [ ] \).

Thus, we have

\[ l = [216, 5, 1, 0] \]

Now we have to decode each instruction.
216 is non-zero, so it represents either an increment or decrement instruction. To find out which, we should decode it as \( \langle x, y \rangle = 2^x(2y + 1) \):

\[
216 = 2 \times 108 = 2^2 \times 54 = 2^3 \times 27 = 2^3(2 \times 13 + 1) = \langle 3, 13 \rangle
\]

Now 3 = 2 \times 1 + 1, so the instruction is a decrement to register 1. To determine which labels the instruction goes to, we need to decode 13 as \( \langle j, k \rangle = 2^j(2k + 1) - 1 \):

\[
13 = 14 - 1 = 2 \times 7 - 1 = 2^1(2 \times 3 + 1) - 1 = \langle 1, 3 \rangle
\]

We have determined that 216 = \( \langle 2 \times 1 + 1, \langle 1, 3 \rangle \rangle \), so we have

\[
L_0 : R_1^- \rightarrow L_1, L_3
\]

5 is also non-zero.

\[
5 = 2^0(2 \times 2 + 1) = \langle 0, 2 \rangle = \langle 2 \times 0, 2 \rangle
\]

This therefore represents the instruction that increments \( R_0 \) and jumps to \( L_2 \):

\[
L_1 : R_0^+ \rightarrow L_2
\]

1 is also non-zero.

\[
1 = 2^0(2 \times 0 + 1) = \langle 0, 0 \rangle = \langle 2 \times 0, 0 \rangle
\]

This therefore represents the instruction that increments \( R_0 \) and jumps to \( R_0 \):

\[
L_2 : R_0^+ \rightarrow L_0
\]

0 represents the halting instruction:

\[
L_3 : HALT
\]

Consequently, the complete decoded program is:

\[
L_0 : R_1^- \rightarrow L_1, L_3
L_1 : R_0^+ \rightarrow L_2
L_2 : R_0^+ \rightarrow L_0
L_3 : HALT
\]

(b) The program increments \( R_0 \) by twice the initial value of \( R_1 \). If \( f \) is the function of one argument computed by the register machine with this program, \( f(x) \) is the final value of \( R_0 \) when the machine is run from the initial state with \( R_0 = 0 \) and \( R_1 = x \). Therefore \( f(x) = 2x \) — the machine computes the doubling function.

3. (a) “test \( L = 0 \)”: 

(b) \(Z \leftarrow L\):

(c) \(L \leftarrow Z/2\):

(d) \(L ::= L/2\):
(e) “$\langle X, L \rangle ::= L$”:

(f) If you have made the same design choices as here, you should get something like:

This should be familiar as the implementation of the “pop $L$ to $X$” graph component from the lectures.

4. No solution, but a hint: it is possible to simulate three registers using two registers by representing the values $x, y, z$ of the three registers as $2^x3^y5^z$. 