

Exercises

Program Analysis (CO70020)

Sheet 2

Exercise 1 Consider the following **while** program:

```
[x := 1]1;  
while [y>0]2 do [x := x-1]3;  
[x := 2]4;
```

Perform a Live Variables Analysis for this program (state equations and construct solutions).

Solution The auxiliary functions $kill_{LV}$ and gen_{LV} are given as:

$$\begin{aligned} kill_{LV}(1) &= \{x\} \\ kill_{LV}(2) &= \emptyset \\ kill_{LV}(3) &= \{x\} \\ kill_{LV}(4) &= \{x\} \end{aligned}$$

$$\begin{aligned} gen_{LV}(1) &= \emptyset \\ gen_{LV}(2) &= \{y\} \\ gen_{LV}(3) &= \{x\} \\ gen_{LV}(4) &= \emptyset \end{aligned}$$

The corresponding equations are therefore:

$$\begin{aligned} LV_{exit}(4) &= \emptyset \\ LV_{exit}(3) &= LV_{entry}(2) \\ LV_{exit}(2) &= LV_{entry}(3) \cup LV_{entry}(4) \\ LV_{exit}(1) &= LV_{entry}(2) \end{aligned}$$

$$\begin{aligned} LV_{entry}(4) &= \emptyset \\ LV_{entry}(3) &= (LV_{exit}(3) \setminus \{x\}) \cup \{x\} \\ LV_{entry}(2) &= LV_{exit}(2) \cup \{y\} \\ LV_{entry}(1) &= LV_{exit}(1) \setminus \{x\} \end{aligned}$$

Thus the solution is the final column in the following table:

$\text{LV}_{exit}(4) = \emptyset$	\emptyset	\emptyset	\emptyset	\dots
$\text{LV}_{exit}(3) = \emptyset$	\emptyset	$\{x, y\}$	$\{x, y\}$	\dots
$\text{LV}_{exit}(2) = \emptyset$	$\{x\}$	$\{x, y\}$	$\{x, y\}$	\dots
$\text{LV}_{exit}(1) = \emptyset$	$\{x, y\}$	$\{x, y\}$	$\{x, y\}$	\dots
$\text{LV}_{entry}(4) = \emptyset$	\emptyset	\emptyset	\emptyset	\dots
$\text{LV}_{entry}(3) = \emptyset$	$\{x\}$	$\{x, y\}$	$\{x, y\}$	\dots
$\text{LV}_{entry}(2) = \emptyset$	$\{x, y\}$	$\{x, y\}$	$\{x, y\}$	\dots
$\text{LV}_{entry}(1) = \emptyset$	$\{y\}$	$\{y\}$	$\{y\}$	\dots

Exercise 2 Construct/specify all elements of $\mathcal{P}(\{x, y, z\})$, i.e. the power set of $\{x, y, z\}$. Describe the sub-set relation on $\mathcal{P}(\{x, y, z\})$, i.e. which sub-set is a sub-set of another sub-set. What is the maximum number of sub-sets a sub-set of $\{x, y, z\}$ can be included in and/or the height of $\mathcal{P}(\{x, y, z\})$?

Solution This gives a “partially ordered set”, more precisely a (prototypical) complete lattice; more details in the lectures on Lattice Theory. This set set has $s^3 = 8$ elements. A maximal inclusion “chain” is for example $\emptyset \subseteq \{x\} \subseteq \{x, y\} \subseteq \{x, y, z\}$ which is of length 3.

Exercise 3 Construct/specify all elements of $\mathcal{P}(\{x, y\} \times \{1, 2, 3\})$, i.e. the power set of the cartesian product $\{x, y\} \times \{1, 2, 3\}$. Describe the sub-set relation on $\mathcal{P}(\{x, y\} \times \{1, 2, 3\})$. What is the maximum number of sub-sets any sub-set of $\mathcal{P}(\{x, y\} \times \{1, 2, 3\})$ can be included in?

Solution This gives a “partially ordered set”, more precisely a complete lattice; more details in the lectures on Lattice Theory. There are 6 elements in $\{x, y\} \times \{1, 2, 3\}$ Thus $\emptyset \subseteq \{(x, 1)\} \subseteq \dots \subseteq \{x, y\} \times \{1, 2, 3\}$ is a maximal “chain” of length 6.

Exercise 4 Consider the following **While** program

while ($x > 0$) **do** $y := y - 1$

Describe the possible RD solutions at every program point. What is the size of this “property space” and how many possible solution are there for the RD analysis?

Solution Based on the fact that the the number of functions between finite sets is given by $|X \rightarrow Y| = |Y|^{|X|}$ we also have $\mathcal{P}(X) \equiv (X \rightarrow \{0, 1\})$ and thus $|\mathcal{P}(X)| = 2^{|X|}$, where $|\cdot|$ denotes the cardinality/size of sets. For the Cartesian product we have: $|X \times Y| = |X| \times |Y|$.

We can label the given program as:

while ($[x > 0]^1$) **do** $[y := y - 1]^2$

Then the “property space” is $L = \mathcal{P}(\{?, 1, 2\} \times \{\mathbf{x}, \mathbf{y}\})$.

The program has no isolated entry and one could argue that one needs to ‘extend’ it before we can perform the analysis, i.e. consider

$$[\mathbf{skip}]^0; \mathbf{while} \ ([\mathbf{x}>0])^1 \mathbf{do} \ [\mathbf{y}:=\mathbf{y}-1]^2$$

with property space $L = \mathcal{P}(\{?, 0, 1, 2\} \times \{\mathbf{x}, \mathbf{y}\})$.

The size of the property space is $|L| = 2^{l \times v}$ with $l = |\mathbf{Lab}|$ and $v = |\mathbf{Var}|$. Thus, $|L| = 2^{3 \times 2} = 2^6 = 64$ for the original program and $|L| = 2^{4 \times 2} = 2^8 = 256$ for the extended program.

There are l program points, thus we have $2 \times l$ equations, i.e. functions $\text{RD}_{\text{entry}} : \mathbf{Lab} \rightarrow L$ and $\text{RD}_{\text{exit}} : \mathbf{Lab} \rightarrow L$. For the original program we therefore have (arguably) $2 \times 64^2 = 8192$ potential solutions, for the extended program it’s maybe $2 \times 256^3 = 33554432$.

Exercise 5 Consider the set of all sets of the form:

$$\{*\}, \{*, \{*\}\}, \dots, \{*, \{*, \{*, \dots\}\}\}, \dots$$

i.e. $S_1 = \{*\}$ and $S_n = \{*\} \cup \{S_{n-1}\}$ where $*$ is some element/object. Describe the element relation on this set of sets. What is the maximum number of sets any set of can be included in (be element of)?

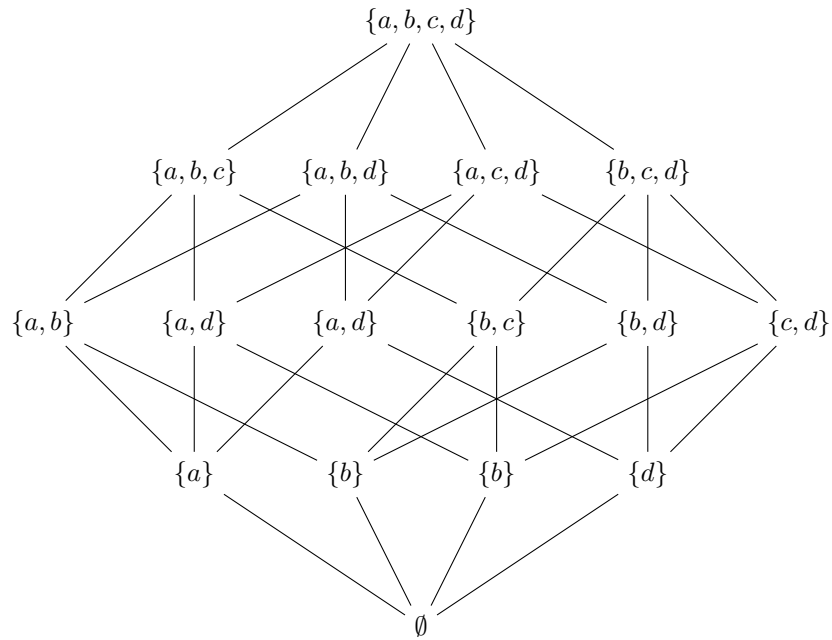
Solution This gives a “totally ordered” *non-complete* lattice, i.e. partial order, isomorphic to \mathbb{N} , more details in the lectures on Lattice Theory. There is no maximal “inclusion depth” (or you could say it is “ ∞ ”).

Note that the inclusions “ \subseteq ” is not transitive on $\{S_i\}$, e.g. $\{*\} \subseteq \{*, \{*\}\}$ and $\{*, \{*\}\} \subseteq \{*, \{*, \{*\}\}\}$ but $\{*\} \not\subseteq \{*, \{*, \{*\}\}\}$. However, “ \subseteq ” can be extended in the obvious way (so-called *transitive closure*) to a partial order which is equivalent to “ \leq ”.

Exercise 6 Consider the power set $\mathcal{P}(X)$ of $X = \{a, b, c, d\}$.

1. Draw the Hasse diagram.
2. Give a monotone map from $(\mathcal{P}(X), \subseteq)$ into (\mathbb{Z}, \leq) .
3. Give a monotone map from (\mathbb{Z}, \leq) into $(\mathcal{P}(X), \subseteq)$.

Solution The Hasse diagram is similar to the diagram in the lecture.



Any constant map is, of course, a monotone map. A slightly more complicated example $s : \mathcal{P}(X) \mapsto \mathbb{Z}$ is given by:

$$s(S) = \text{number of elements in } S$$

and for $f : \mathbb{Z} \mapsto \mathcal{P}(X)$

$$f(x) = \begin{cases} \emptyset & \text{if } x \leq 0 \\ \{a\} & 1 \\ \{a, b\} & 2 \\ \{a, b, c\} & 3 \\ X & \text{if } 4 \leq x \end{cases}$$