

Exercises

Program Analysis (CO70020)

Sheet 4

Exercise 1 Consider the following FUN expression

```
(let f1 = (fn x1 => x1)
 in  (let f2 = (fn x2 => x2)
      in ((f1 f2) (fn y => y))))
```

1. Label this FUN expression.
2. Write down the constraints that are generated for the Control Flow Analysis of this function.
3. Give a walkthrough of the algorithm to develop a solution to the constraints.

Solution The labelled expression is:

```
e = (let f1 = (fn x1 => x11)2
      in  (let f2 = (fn x2 => x23)4
            in ((f15 f26)7 (fn y => y8)9)10)11)12
```

The constraints generated for e are:

$$\begin{aligned} \mathcal{C}_*[[e]] = \{ & \mathcal{C}(2) \subseteq r(f_1), \mathcal{C}(11) \subseteq \mathcal{C}(12), \{id_{x_1}\} \subseteq \mathcal{C}(2), r(x_1) \subseteq \mathcal{C}(1), \\ & \mathcal{C}(4) \subseteq r(f_2), \mathcal{C}(10) \subseteq \mathcal{C}(11), \{id_{x_2}\} \subseteq \mathcal{C}(4), r(x_2) \subseteq \mathcal{C}(3), \\ & r(f_1) \subseteq \mathcal{C}(5), r(f_2) \subseteq \mathcal{C}(6), \{id_y\} \subseteq \mathcal{C}(9), r(y) \subseteq \mathcal{C}(8), \\ & \{id_{x_1}\} \subseteq \mathcal{C}(5) \Rightarrow \mathcal{C}(6) \subseteq r(x_1), \{id_{x_1}\} \subseteq \mathcal{C}(5) \Rightarrow \mathcal{C}(1) \subseteq \mathcal{C}(7), \\ & \{id_{x_2}\} \subseteq \mathcal{C}(5) \Rightarrow \mathcal{C}(6) \subseteq r(x_2), \{id_{x_2}\} \subseteq \mathcal{C}(5) \Rightarrow \mathcal{C}(3) \subseteq \mathcal{C}(7), \\ & \{id_y\} \subseteq \mathcal{C}(5) \Rightarrow \mathcal{C}(6) \subseteq r(y), \{id_y\} \subseteq \mathcal{C}(5) \Rightarrow \mathcal{C}(8) \subseteq \mathcal{C}(7), \\ & \{id_{x_1}\} \subseteq \mathcal{C}(7) \Rightarrow \mathcal{C}(9) \subseteq r(x_1), \{id_{x_1}\} \subseteq \mathcal{C}(7) \Rightarrow \mathcal{C}(1) \subseteq \mathcal{C}(10), \\ & \{id_{x_2}\} \subseteq \mathcal{C}(7) \Rightarrow \mathcal{C}(9) \subseteq r(x_2), \{id_{x_2}\} \subseteq \mathcal{C}(7) \Rightarrow \mathcal{C}(3) \subseteq \mathcal{C}(10), \\ & \{id_y\} \subseteq \mathcal{C}(7) \Rightarrow \mathcal{C}(9) \subseteq r(y), \{id_y\} \subseteq \mathcal{C}(7) \Rightarrow \mathcal{C}(8) \subseteq \mathcal{C}(10) \} \end{aligned}$$

The worklist is initialised to $W_0 = [\mathcal{C}(2), \mathcal{C}(4), \mathcal{C}(9)]$ after processing the constraints $\{id_{x_1}\} \subseteq \mathcal{C}(2)$, $\{id_{x_2}\} \subseteq \mathcal{C}(4)$ and $\{id_y\} \subseteq \mathcal{C}(9)$.

Nodes	Edges
C(1)	$\{id_{x_1}\} \subseteq C(5) \Rightarrow C(1) \subseteq C(7), \{id_{x_1}\} \subseteq C(7) \Rightarrow C(1) \subseteq C(10)$
C(2)	$C(2) \subseteq r(f_1)$
C(3)	$\{id_{x_2}\} \subseteq C(5) \Rightarrow C(3) \subseteq C(7), \{id_{x_2}\} \subseteq C(7) \Rightarrow C(3) \subseteq C(10)$
C(4)	$C(4) \subseteq r(f_2)$
C(5)	$\{id_{x_1}\} \subseteq C(5) \Rightarrow C(6) \subseteq r(x_1), \{id_{x_1}\} \subseteq C(5) \Rightarrow C(1) \subseteq C(7),$ $\{id_{x_2}\} \subseteq C(5) \Rightarrow C(6) \subseteq r(x_2), \{id_{x_2}\} \subseteq C(5) \Rightarrow C(3) \subseteq C(7),$ $\{id_y\} \subseteq C(5) \Rightarrow C(6) \subseteq r(y), \{id_y\} \subseteq C(5) \Rightarrow C(8) \subseteq C(7),$
C(6)	$\{id_{x_1}\} \subseteq C(5) \Rightarrow C(6) \subseteq r(x_1), \{id_{x_2}\} \subseteq C(5) \Rightarrow C(6) \subseteq r(x_2), \{id_y\} \subseteq C(5) \Rightarrow C(6) \subseteq r(y)$
C(7)	$\{id_{x_1}\} \subseteq C(7) \Rightarrow C(9) \subseteq r(x_1), \{id_{x_1}\} \subseteq C(7) \Rightarrow C(1) \subseteq C(10),$ $\{id_{x_2}\} \subseteq C(7) \Rightarrow C(9) \subseteq r(x_2), \{id_{x_2}\} \subseteq C(7) \Rightarrow C(3) \subseteq C(10),$ $\{id_y\} \subseteq C(7) \Rightarrow C(9) \subseteq r(y), \{id_y\} \subseteq C(7) \Rightarrow C(8) \subseteq C(10)$
C(8)	$\{id_y\} \subseteq C(5) \Rightarrow C(8) \subseteq C(7), \{id_y\} \subseteq C(7) \Rightarrow C(8) \subseteq C(10)$
C(9)	$\{id_{x_1}\} \subseteq C(7) \Rightarrow C(9) \subseteq r(x_1), \{id_{x_2}\} \subseteq C(7) \Rightarrow C(9) \subseteq r(x_2), \{id_y\} \subseteq C(7) \Rightarrow C(9) \subseteq r(y)$
C(10)	$C(10) \subseteq C(11)$
C(11)	$C(11) \subseteq C(12)$
C(12)	\square
$r(f_1)$	$r(f_1) \subseteq C(5)$
$r(f_2)$	$r(f_2) \subseteq C(6)$
$r(x_1)$	$r(x_1) \subseteq C(1)$
$r(x_2)$	$r(x_2) \subseteq C(3)$
$r(y)$	$r(y) \subseteq C(8)$

Using $W_0 = [C(2), C(4), C(9)]$:

W	W_0	$[r(f_1), \dots]$	$[C(5), C(4), C(9)]$	$[r(f_2), C(9)]$	$[C(6), C(9)]$	$[r(x_1), C(9)]$	$[C(1), C(9)]$
C(1)	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	$\{id_{x_2}\}$
C(2)	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$
C(3)	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
C(4)	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$
C(5)	\emptyset	\emptyset	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$
C(6)	\emptyset	\emptyset	\emptyset	\emptyset	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$
C(7)	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
C(8)	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
C(9)	$\{id_y\}$	$\{id_y\}$	$\{id_y\}$	$\{id_y\}$	$\{id_y\}$	$\{id_y\}$	$\{id_y\}$
C(10)	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
C(11)	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
C(12)	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
$r(f_1)$	\emptyset	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$
$r(f_2)$	\emptyset	\emptyset	\emptyset	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$
$r(x_1)$	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	$\{id_{x_2}\}$	$\{id_{x_2}\}$
$r(x_2)$	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
$r(y)$	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset

W	$[C(7), C(9)]$	$[r(x_2), C(9)]$	$[C(3), C(9)]$	$[C(10), C(9)]$	$[C(11), C(9)]$	$[C(11), C(9)]..[]$
$C(1)$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$
$C(2)$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$
$C(3)$	\emptyset	\emptyset	$\{id_y\}$	$\{id_y\}$	$\{id_y\}$	$\{id_y\}$
$C(4)$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$
$C(5)$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$
$C(6)$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$
$C(7)$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$
$C(8)$	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
$C(9)$	$\{id_y\}$	$\{id_y\}$	$\{id_y\}$	$\{id_y\}$	$\{id_y\}$	$\{id_y\}$
$C(10)$	\emptyset	\emptyset	\emptyset	$\{id_y\}$	$\{id_y\}$	$\{id_y\}$
$C(11)$	\emptyset	\emptyset	\emptyset	\emptyset	$\{id_y\}$	$\{id_y\}$
$C(12)$	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	$\{id_y\}$
$r(f_1)$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$	$\{id_{x_1}\}$
$r(f_2)$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$
$r(x_1)$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$	$\{id_{x_2}\}$
$r(x_2)$	\emptyset	$\{id_y\}$	$\{id_y\}$	$\{id_y\}$	$\{id_y\}$	$\{id_y\}$
$r(y)$	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset

Exercise 2 Consider the (labelled) FUN expression e :

$$\begin{aligned}
& (\text{let } f = (\text{fn } x \Rightarrow \text{if } (x^1 > 0^2)^3 \text{ then } (\text{fn } y \Rightarrow y^4)^5 \\
& \qquad \qquad \qquad \text{else } (\text{fn } z \Rightarrow 25^6)^7)^8)^9 \\
& \text{in } ((f^{10}3^{11})^{12}0^{13})^{14})^{15}
\end{aligned}$$

Write down the constraints that get generated by the combined Control and Data Flow. Solve them to show that the identity is the only closure that is produced at label 12.

Solution There are three funtions in the program, which we call:

$$\begin{aligned}
f_8 &= \text{fn } x \Rightarrow e_8 \\
f_4 &= \text{fn } y \Rightarrow y^4 \text{ (identity)} \\
f_6 &= \text{fn } z \Rightarrow 25^6 \\
&\text{where } e_8 = (\text{if } (x^1 > 0^2)^3 \text{ then } (\text{fn } y \Rightarrow y^4)^5 \text{else } (\text{fn } z \Rightarrow 25^6)^7)^8
\end{aligned}$$

By applying the definition of $C_\star[\llbracket \cdot \rrbracket]$ for let-expressions to the program we get:

$$\begin{aligned}
C_\star[\llbracket e \rrbracket] &= \{C(14) \subseteq C(15), C(9) \subseteq r(f)\} \cup \\
& C_\star[\llbracket (\text{fn } x \Rightarrow e_8)^9 \rrbracket] \cup C_\star[\llbracket ((f^{10}3^{11})^{12}0^{13})^{14} \rrbracket]
\end{aligned}$$

Note that, since we are performing a data flow (sign) analysis together with the control flow analysis, the sets associated to entries of the abstract cache and abstract environment i.e. abstract values, will contain elements of $\mathbf{Data}_{\text{sign}} = \{\text{tt}, \text{ff}, -, 0, +\}$. For example, the constraint generated for 0^2 is $\{0\} \subseteq C(2)$ and the constraint generated for 3^{11} is $\{+\} \subseteq C(11)$. The treatment of data by the

analysis requires the definition of abstract versions of the binary operators e.g. $+$, $>$. For this particular example we need only define the behaviour of abstract operator $\widehat{>}$ when applied to arguments $\{+\}$ and $\{0\}$. Thus, we have:

$$\{+\}\widehat{>}\{0\} = \{\text{tt}\}$$

The result above is needed for the evaluation of $C(1) \widehat{>} C(2) \subseteq C(3)$ (constraint generated for $(x^1 > 0^2)^3$) during the constraint resolution phase. The constraints generated for the right-hand side of the **let**-assignment are:

$$\begin{aligned} \mathcal{C}_\star[\langle \text{fn } x \Rightarrow e_8 \rangle^9] &= \{ \{f_8\} \subseteq C(9) \} \cup \mathcal{C}_\star[\langle e_8 \rangle] \\ &\quad \{ \{f_8\} \subseteq C(9), \\ &\quad r(x) \subseteq C(1), \{0\} \subseteq C(2), C(1) \widehat{>} C(2) \subseteq C(3), \\ &\quad \{\text{tt}\} \subseteq C(3) \Rightarrow \mathcal{C}_\star[\langle \text{fn } y \Rightarrow y^4 \rangle^5] \cup \{C(5) \subseteq C(8)\} \quad (A) \\ &\quad \{\text{ff}\} \subseteq C(3) \Rightarrow \mathcal{C}_\star[\langle \text{fn } z \Rightarrow 25^6 \rangle^7] \cup \{C(7) \subseteq C(8)\} \end{aligned}$$

The use of data makes the analysis of **if**-expressions more precise. For example, the constraints generated for the **true** branch of the **if**-expression e_8 are checked only if **tt** flows to the test expression, as described by the conditional constraint (A) above. The constraints generated for $\langle \text{fn } y \Rightarrow y^4 \rangle^5$ are:

$$\mathcal{C}_\star[\langle \text{fn } y \Rightarrow y^4 \rangle^5] = \{ \{f_4\} \subseteq C(5), r(y) \subseteq C(4) \}$$

so we can re-write the conditional constraints for the **true** branch to:

$$(A) = \{ \{\text{tt}\} \subseteq C(3) \Rightarrow \{f_4\} \subseteq C(5), \{\text{tt}\} \subseteq C(3) \Rightarrow r(y) \subseteq C(4), \{\text{tt}\} \subseteq C(3) \Rightarrow C(5) \subseteq C(8) \}$$

Now we continue with the body of the **let**. There are two applications in $((f^{10}3^{11})^{12}0^{13})^{14}$. By applying the rule for applications twice we get:

$$\begin{aligned} \mathcal{C}_\star[\langle (f^{10}3^{11})^{12}0^{13} \rangle^{14}] &= \{ \{0\} \subseteq C(13), \\ &\quad \{f_8\} \subseteq C(12) \Rightarrow C(8) \subseteq C(14), \{f_8\} \subseteq C(12) \Rightarrow C(13) \subseteq r(x), \\ &\quad \{f_4\} \subseteq C(12) \Rightarrow C(4) \subseteq C(14), \{f_4\} \subseteq C(12) \Rightarrow C(13) \subseteq r(y), \\ &\quad \{f_6\} \subseteq C(12) \Rightarrow C(6) \subseteq C(14), \{f_6\} \subseteq C(12) \Rightarrow C(13) \subseteq r(z), \\ &\quad r(f) \subseteq C(10), \{+\} \subseteq C(11), \\ &\quad \{f_8\} \subseteq C(10) \Rightarrow C(8) \subseteq C(12), \{f_8\} \subseteq C(10) \Rightarrow C(11) \subseteq r(x), \\ &\quad \{f_4\} \subseteq C(10) \Rightarrow C(4) \subseteq C(12), \{f_4\} \subseteq C(10) \Rightarrow C(11) \subseteq r(y), \\ &\quad \{f_6\} \subseteq C(10) \Rightarrow C(6) \subseteq C(12), \{f_6\} \subseteq C(10) \Rightarrow C(11) \subseteq r(z) \quad \} \end{aligned}$$

By putting everything together we get:

$$\mathcal{C}_*[[e]] = \{ \begin{array}{l} C(14) \subseteq C(15), C(9) \subseteq r(f) \\ \{f_8\} \subseteq C(9), \\ r(x) \subseteq C(1), \{0\} \subseteq C(2), C(1) \succ C(2) \subseteq C(3), \\ \{tt\} \subseteq C(3) \Rightarrow \{f_4\} \subseteq C(5), \{tt\} \subseteq C(3) \Rightarrow r(y) \subseteq C(4), \{tt\} \subseteq C(3) \Rightarrow C(5) \subseteq C(8), \\ \{ff\} \subseteq C(3) \Rightarrow \mathcal{C}_*[[\text{fn } z \Rightarrow 25^6]^7] \cup \{C(7) \subseteq C(8)\} \\ \{0\} \subseteq C(13), \\ \{f_8\} \subseteq C(12) \Rightarrow C(8) \subseteq C(14), \{f_8\} \subseteq C(12) \Rightarrow C(13) \subseteq r(x), \\ \{f_4\} \subseteq C(12) \Rightarrow C(4) \subseteq C(14), \{f_4\} \subseteq C(12) \Rightarrow C(13) \subseteq r(y), \\ \{f_6\} \subseteq C(12) \Rightarrow C(6) \subseteq C(14), \{f_6\} \subseteq C(12) \Rightarrow C(13) \subseteq r(z), \\ r(f) \subseteq C(10), \{+\} \subseteq C(11), \\ \{f_8\} \subseteq C(10) \Rightarrow C(8) \subseteq C(12), \{f_8\} \subseteq C(10) \Rightarrow C(11) \subseteq r(x), \\ \{f_4\} \subseteq C(10) \Rightarrow C(4) \subseteq C(12), \{f_4\} \subseteq C(10) \Rightarrow C(11) \subseteq r(y), \\ \{f_6\} \subseteq C(10) \Rightarrow C(6) \subseteq C(12), \{f_6\} \subseteq C(10) \Rightarrow C(11) \subseteq r(z) \end{array} \}$$

Algorithm: The worklist is initialised to $W_0 = [C(9), C(2), C(13), C(11)]$ after processing the constraints $\{f_8\} \subseteq C(9)$, $\{0\} \subseteq C(2)$, $\{0\} \subseteq C(13)$ and $\{+\} \subseteq C(11)$. The rest of the constraints are used to create the table of edges:

Nodes	Edges
C(1)	$C(1) \succ C(2) \subseteq C(3)$
C(2)	$C(1) \succ C(2) \subseteq C(3)$
C(3)	$\{tt\} \subseteq C(3) \Rightarrow \{f_4\} \subseteq C(5), \{tt\} \subseteq C(3) \Rightarrow r(y) \subseteq C(4), \{tt\} \subseteq C(3) \Rightarrow C(5) \subseteq C(8),$ $\{ff\} \subseteq C(3) \Rightarrow \mathcal{C}_*[[\text{fn } z \Rightarrow 25^6]^7] \cup \{C(7) \subseteq C(8)\}$
C(4)	$\{f_4\} \subseteq C(12) \Rightarrow C(4) \subseteq C(14), \{f_4\} \subseteq C(10) \Rightarrow C(4) \subseteq C(12)$
C(5)	$C(5) \subseteq C(8)$
C(8)	$\{f_8\} \subseteq C(12) \Rightarrow C(8) \subseteq C(14), \{f_8\} \subseteq C(10) \Rightarrow C(8) \subseteq C(12)$
C(9)	$C(9) \subseteq r(f)$
C(10)	$\{f_8\} \subseteq C(10) \Rightarrow C(8) \subseteq C(12), \{f_8\} \subseteq C(10) \Rightarrow C(11) \subseteq r(x),$ $\{f_4\} \subseteq C(10) \Rightarrow C(4) \subseteq C(12), \{f_4\} \subseteq C(10) \Rightarrow C(11) \subseteq r(y),$ $\{f_6\} \subseteq C(10) \Rightarrow C(6) \subseteq C(12), \{f_6\} \subseteq C(10) \Rightarrow C(11) \subseteq r(z)$
C(11)	$\{f_8\} \subseteq C(10) \Rightarrow C(11) \subseteq r(x), \{f_4\} \subseteq C(10) \Rightarrow C(11) \subseteq r(y), \{f_6\} \subseteq C(10) \Rightarrow C(11) \subseteq r(z)$
C(12)	$\{f_8\} \subseteq C(12) \Rightarrow C(8) \subseteq C(14), \{f_8\} \subseteq C(12) \Rightarrow C(13) \subseteq r(x),$ $\{f_4\} \subseteq C(12) \Rightarrow C(4) \subseteq C(14), \{f_4\} \subseteq C(12) \Rightarrow C(13) \subseteq r(y),$ $\{f_6\} \subseteq C(12) \Rightarrow C(6) \subseteq C(14), \{f_6\} \subseteq C(12) \Rightarrow C(13) \subseteq r(z),$
C(13)	$\{f_8\} \subseteq C(12) \Rightarrow C(13) \subseteq r(x), \{f_4\} \subseteq C(12) \Rightarrow C(13) \subseteq r(y), \{f_6\} \subseteq C(12) \Rightarrow C(13) \subseteq r(z)$
C(14)	$C(14) \subseteq C(15)$
C(15)	–
$r(f)$	$r(f) \subseteq C(10)$
$r(x)$	$r(x) \subseteq C(1)$
$r(y)$	$\{tt\} \subseteq C(3) \Rightarrow r(y) \subseteq C(4)$

The iterations are described by the two tables of Figure 1. Note the last iteration actually corresponds to four iterations: the constraints that needed to be checked for the elements of the worklist $[C(15), C(2), C(13), C(11)]$ are true and no new elements are added to the worklist. We see that $C(12) = \{f_4\}$, where $f_4 = \text{fn } y \Rightarrow y^4$, the identity function.

W	W_0	$[r(f), \dots]$	$[C(10), \dots]$	$[r(x), \dots]$	$[C(1), \dots]$	$[C(3), \dots]$	$[C(5), \dots]$	$[C(8), \dots]$
C(1)	ϕ	ϕ	ϕ	ϕ	{+}	{+}	{+}	{+}
C(2)	{0}	{0}	{0}	{0}	{0}	{0}	{0}	{0}
C(3)	ϕ	ϕ	ϕ	ϕ	ϕ	{tt}	{tt}	{tt}
C(4)	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
C(5)	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	{ f_4 }	{ f_4 }
C(6)	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
C(7)	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
C(8)	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	{ f_4 }
C(9)	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }
C(10)	ϕ	ϕ	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }
C(11)	{+}	{+}	{+}	{+}	{+}	{+}	{+}	{+}
C(12)	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
C(13)	{0}	{0}	{0}	{0}	{0}	{0}	{0}	{0}
C(14)	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
C(15)	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$r(f)$	ϕ	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }
$r(x)$	ϕ	ϕ	ϕ	{+}	{+}	{+}	{+}	{+}
$r(y)$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$r(z)$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ

W	$[C(12), \dots]$	$[r(y), \dots]$	$[C(4), \dots]$	$[C(14), \dots]$	$[C(15), C(2), C(13), C(11)]$	[]
C(1)	{+}	{+}	{+}	{+}	{+}	{+}
C(2)	{0}	{0}	{0}	{0}	{0}	{0}
C(3)	{tt}	{tt}	{tt}	{tt}	{tt}	{tt}
C(4)	ϕ	ϕ	{0}	{0}	{0}	{0}
C(5)	{ f_4 }	{ f_4 }	{ f_4 }	{ f_4 }	{ f_4 }	{ f_4 }
C(6)	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
C(7)	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
C(8)	{ f_4 }	{ f_4 }	{ f_4 }	{ f_4 }	{ f_4 }	{ f_4 }
C(9)	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }
C(10)	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }
C(11)	{+}	{+}	{+}	{+}	{+}	{+}
C(12)	{ f_4 }	{ f_4 }	{ f_4 }	{ f_4 }	{ f_4 }	{ f_4 }
C(13)	{0}	{0}	{0}	{0}	{0}	{0}
C(14)	ϕ	ϕ	ϕ	{0}	{0}	{0}
C(15)	ϕ	ϕ	ϕ	ϕ	{0}	{0}
$r(f)$	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }	{ f_8 }
$r(x)$	{+}	{+}	{+}	{+}	{+}	{+}
$r(y)$	ϕ	{0}	{0}	{0}	{0}	{0}
$r(z)$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ

Figure 1: Iterations Question 1

Exercise 3 We consider the language (i.e. the pure λ calculus):

$$L ::= x \mid \lambda x.L \mid L_1L_2$$

where $\lambda x.L$ represents a function with a single bound variable x and L_1L_2 denotes application of L_1 to L_2 . You are asked to develop a 0-CFA for this language by the following steps:

- Write down a suitably labelled version of the syntax and define the components needed (e.g. cache) for your 0-CFA specification.
- Write down the specification of the 0-CFA.

Solution

- Labelling:

$$\begin{aligned} E &::= L^\ell \\ L &::= x \mid \lambda x.L \mid L_1L_2 \end{aligned}$$

Abstract values, \widehat{Val} , are a set of lambda expressions $(\lambda x.L)$.

We also need an environment, $\widehat{\rho} : Var \rightarrow \widehat{Val}$, and a cache, $\widehat{C} : Lab \rightarrow \widehat{Val}$.

- Specification:

$$\begin{aligned} (\widehat{C}, \widehat{\rho}) &\models x^\ell \text{ iff } \widehat{\rho}(x) \subseteq \widehat{C}(\ell) \\ (\widehat{C}, \widehat{\rho}) &\models (\lambda x.L)^\ell \\ &\text{iff } (\lambda x.L) \subseteq \widehat{C}(\ell) \wedge (\widehat{C}, \widehat{\rho}) \models L \\ (\widehat{C}, \widehat{\rho}) &\models (L_1^{\ell_1} L_2^{\ell_2})^\ell \\ &\text{iff } (\widehat{C}, \widehat{\rho}) \models L_1^{\ell_1} \wedge (\widehat{C}, \widehat{\rho}) \models L_2^{\ell_2} \wedge \\ &\quad \forall (\lambda x.L_0^{\ell_0}) \in \widehat{C}(\ell_1) : \widehat{C}(\ell_2) \subseteq \widehat{\rho}(x) \wedge \widehat{C}(\ell_0) \subseteq \widehat{C}(\ell) \end{aligned}$$