The Tangled Derivative Logic of the Real Line and Zero-Dimensional Spaces

Robert Goldblatt

School of Mathematics and Statistics, Victoria University of Wellington
sms.vuw.ac.nz/~rob

Ian Hodkinson

Department of Computing, Imperial College London
www.doc.ic.ac.uk/~imh

Abstract

In a topological setting in which the diamond modality is interpreted as the derivative (set of limit points) operator, we study a 'tangled derivative' connective that assigns to any finite set of propositions the largest set in which all those propositions are strictly dense. Building on earlier work of ourselves and others we axiomatise the resulting logic of the real line. We then show that the logic of any zero-dimensional dense-in-itself metric space is the 'tangled' extension of KD4, eliminating an assumption of separability in previous results for zero-dimensional spaces. This requires new kinds of 'dissection lemma' in the sense of McKinsey-Tarski. We extend the analysis to include the universal modality, and also show that the tangled extension of KD4 has a strong completeness result for topological models that fails for its Kripke semantics.

Keywords: derivative operator, dense-in-itself metric space, modal logic, finite model property, zero-dimensional, strong completeness

1 Introduction

The *tangle* connective applies to a finite set Γ of modal formulas to give a new formula $\langle t\rangle\Gamma$ with the following semantics in a model on Kripke frame (W, R):

> $\langle t\rangle\Gamma$ is true at w iff there is an endless R-path $wRw_1 \cdots w_nRw_{n+1} \cdots\cdots$ in W with each member of Γ being true at w_n for infinitely many n.

This pertains to arbitrary models, but in a finite transitive frame the truth condition equivalently means that w can access a cluster (maximal R-clique) in which each member of Γ is true at some point. Denoting by $[\![\varphi]\!]$ the set of points at which a formula φ is true, $[\![\langle t\rangle\Gamma]\!]$ can be shown to be equal to the union

$$\bigcup\{S \subseteq W : S \subseteq \bigcap\nolimits_{\gamma\in\Gamma} R^{-1}([\![\gamma]\!] \cap S)\}. \tag{1.1}$$

This connective was introduced by Dawar and Otto [3], who showed that over the class of finite transitive frames, the bisimulation-invariant fragment of monadic second-order logic collapses to that of first-order logic, with both fragments being expressively equivalent to the language $\mathcal{L}_\Box^{\langle t\rangle}$ that adds $\langle t\rangle$ to the language $\mathcal{L}_\Box$ of the basic modal logic of a unary modality $\Box$.

Now $\mathcal{L}_\Box^{\langle t\rangle}$ is translatable into the language $\mathcal{L}_\Box^{\mu}$ of the modal mu-calculus, since $\langle t\rangle$ has the same meaning as the $\mathcal{L}_\Box^{\mu}$-formula

$$\nu p \bigwedge_{\gamma\in\Gamma} \Diamond(\gamma \wedge p),$$

where ν is the greatest fixed point operator, $\Diamond$ is the dual modality to $\Box$, and p is a fresh propositional atom not occurring in Γ. But the mu-calculus is expressively equivalent to the bisimulation-invariant fragment of monadic second-order logic [7], so the upshot is that $\mathcal{L}_\Box^{\langle t\rangle}$ is expressively equivalent to the seemingly more powerful $\mathcal{L}_\Box^{\mu}$ over finite transitive frames.

The name 'tangle' was coined by Fernández-Duque [4,5], who developed the following topological interpretation of $\langle t\rangle$. A collection $\mathcal{G}$ of subsets of a topological space X is said to be *tangled in a subset* S of X if, for all $G \in \mathcal{G}$, $G\cap S$ is dense in S. In other words, each point of S is in the closure $\mathrm{cl}_X(G\cap S)$ of $G\cap S$. There is a largest subset of X in which $\mathcal{G}$ is tangled, and this is called the *tangled closure* of $\mathcal{G}$. In a model on X, $[\![\langle t\rangle\Gamma]\!]$ is defined to be the tangled closure of $\{[\![\gamma]\!] : \gamma \in \Gamma\}$, which can be described as the set

$$\bigcup\{S \subseteq X : S \subseteq \bigcap\nolimits_{\gamma\in\Gamma} \mathrm{cl}_X([\![\gamma]\!] \cap S)\}. \tag{1.2}$$

Interpreting $\Diamond\varphi$ as the closure $\mathrm{cl}_X[\![\varphi]\!]$ of $[\![\varphi]\!]$, and $\Box\varphi$ as the interior $\mathrm{int}_X[\![\varphi]\!]$, Fernańdez-Duque axiomatised the resulting $\mathcal{L}_\Box^{\langle t\rangle}$-logic as an extension of S4, and showed it has the finite model property. [1] In this S4 setting (1.1) is an instance of (1.2), because an S4-frame, having R reflexive and transitive, is a topological space under the Alexandroff topology generated by the sets $R(w) = \{v : wRv\}$ for all $w \in W$, and in this topology the closure $\mathrm{cl}(S)$ of S is just $R^{-1}(S)$.

The present paper is a continuation of our work in [6], where the equivalence of $\mathcal{L}_\Box^{\langle t\rangle}$ and $\mathcal{L}_\Box^{\mu}$ over finite transitive frames was lifted to the class of all topological spaces, and the finite model property over frames was established for a range of logics having the tangle connective, including some having the universal modality $\forall$ as well. We also studied the more expressive interpretation of a modal diamond as the *derivative operator* $\langle d_X\rangle$ of a space X. For that semantics the diamond and its dual box are written as $\langle d\rangle$ and $[d]$, with $[\![\langle d\rangle\varphi]\!]$ being the set $\langle d_X\rangle[\![\varphi]\!]$ of *limit points* of $[\![\varphi]\!]$. Then $\Diamond\varphi$ is spatially equivalent to $\varphi \vee \langle d\rangle\varphi$, and $\Box\varphi$ to $\varphi \wedge [d]\varphi$. Here we write $[d]^*\varphi$ for the formula $\varphi \wedge [d]\varphi$.

For this derivative interpretation we write the tangle connective as $\langle dt\rangle$. It has exactly the same 'endless R-path' meaning as $\langle t\rangle$ in frames, where $[\![\langle dt\rangle\Gamma]\!]$

[1] The notation $\langle t\rangle$ is ours. In [5] $\langle t\rangle\Gamma$ is written $\Diamond^*\Gamma$, or just $\Diamond\Gamma$, justified because in finite S4 models the $\mathcal{L}_\Box^{\langle t\rangle}$-formula $\Diamond^*\{\varphi\}$ has the same meaning as the $\mathcal{L}_\Box$-formula $\Diamond\varphi$.

continues to be the set (1.1). But what changes are the frames themselves, which no longer require all points to be reflexive. So we interpret $\langle dt\rangle$ over K4-frames rather than S4-frames. For the spatial interpretation of $\langle dt\rangle$ we replace the closure operator cl_X by $\langle d_X\rangle$, and define $[\![\langle dt\rangle\Gamma]\!]$ to be the set

$$\bigcup\{S \subseteq X : S \subseteq \bigcap\nolimits_{\gamma\in\Gamma} \langle d_X\rangle([\![\gamma]\!] \cap S)\}. \tag{1.3}$$

The inclusion $S \subseteq \langle d_X\rangle([\![\gamma]\!] \cap S)$ says that every point of S is a limit point of $[\![\gamma]\!] \cap S$. Since in general $\langle d_X\rangle Y \subseteq \mathrm{cl}_X Y$, and indeed $\mathrm{cl}_X Y = Y \cup \langle d_X\rangle Y$, this is a stricter form of density of $[\![\gamma]\!] \cap S$ in S. So according to (1.3), $[\![\langle dt\rangle\Gamma]\!]$ is the union of all sets S in which $\{[\![\gamma]\!] : \gamma \in \Gamma\}$ is *strictly tangled*, and may be called the *tangled derivative* of $\{[\![\gamma]\!] : \gamma \in \Gamma\}$.

In spaces in which the derivative $\langle d_X\rangle\{x\}$ of any point x is closed (so-called T_D spaces), $\langle t\rangle$ is definable from $\langle dt\rangle$, since $\langle t\rangle\Gamma$ is equivalent to the formula $\bigwedge\Gamma \vee \langle d\rangle \bigwedge\Gamma \vee \langle dt\rangle\Gamma$ (see [6, Lemma 6.5]).

Shehtman's seminal paper [13] axiomatised the logic of some classical spaces in the language $\mathcal{L}_{[d]}$ with the derivative interpretation. It proved that the $\mathcal{L}_{[d]}$-logic of any separable zero-dimensional dense-in-itself metric space is KD4, and the logic of the Euclidean space $\mathbb{R}^n$ is $\mathrm{KD4G_1}$ for all $n \geq 2$. It also conjectured that the logic of the real line $\mathbb{R}$ is $\mathrm{KD4G_2}$, which was later verified by Shehtman [12] and Lucero-Bryan [8].

Our purpose in this paper is to lift these results to the language $\mathcal{L}_{[d]}^{\langle dt\rangle}$ with the tangled derivative connective. We use the name Lt for the $\mathcal{L}_{[d]}^{\langle dt\rangle}$-logic defined by adding to the axiomatisation of some $\mathcal{L}_{[d]}$-logic named L the 'fixed point' axioms

Fix: $\langle dt\rangle\Gamma \to \langle d\rangle(\gamma \wedge \langle dt\rangle\Gamma)$, all $\gamma \in \Gamma$

Ind: $[d]^*(\varphi \to \bigwedge_{\gamma\in\Gamma} \langle d\rangle(\gamma \wedge \varphi)) \to (\varphi \to \langle dt\rangle\Gamma)$.

We already dealt with $\mathbb{R}^n$ for $n \geq 2$ in [6, Theorem 9.3] which showed that if X is any dense-in-itself metric space, then the $\mathcal{L}_{[d]}^{\langle dt\rangle}$-logic of X is included in $\mathrm{KD4G_1}t$, and is exactly $\mathrm{KD4G_1}t$ if X validates $\mathrm{G_1}$.[2] In particular this holds when $X = \mathbb{R}^n$.

Here we will prove that the $\mathcal{L}_{[d]}^{\langle dt\rangle}$-logic of $\mathbb{R}$ is $\mathrm{KD4G_2}t$. The proof uses a result of [8] about the existence of *d-morphisms* from $\mathbb{R}$ onto finite $\mathrm{KD4G_2}$-frames. We show that these morphisms preserve validity of $\mathcal{L}_{[d]}^{\langle dt\rangle}$-formulas.

We then turn to zero-dimensional spaces and generalize the result of [13] by eliminating the restriction to separable spaces, and showing that the $\mathcal{L}_{[d]}^{\langle dt\rangle}$-logic of each zero-dimensional dense-in-itself metric space is KD4t. This requires certain 'dissection' lemmas about the partitioning of an open set into subsets with properties that allow them to be used to represent the structure of finite frames. A variant of the dissection lemma of McKinsey and Tarski [9] allows

[2] This result also holds when restricted to $\mathcal{L}_{[d]}$ and $\mathrm{KD4G_1}$, answering another question from [13].

any dense-in-itself metric space to represent finite rooted S4-frames. But KD-frames may have irreflexive points, and we need a further dissection result to handle them (Theorem 7.3), as well as one for zero-dimensional spaces about dissection into special open sets (Theorem 7.5).

We extend our results to the language with the universal modality, showing that the $\mathcal{L}^{\langle dt\rangle}_{[d]\forall}$-logic of any zero-dimensional dense-in-itself metric space is KD4t.U, and of $\mathbb{R}$ is KD4G$_2t$.UC, where C is Shehtman's axiom expressing topological connectedness.

Finally, we give a topological strong completeness result for KD4t, showing that any countable KD4t-consistent set of formulas is satisfiable at any point of any zero-dimensional dense-in-itself metric space X. By contrast, this strong completeness fails for KD4t over its Kripke semantics.

2 Formulas and Frames

We assume a set Var of *propositional variables* or *atoms*. Formulas are constructed from these variables by the standard Boolean connectives $\top$, $\neg$, $\wedge$; the unary modality $[d]$; and the *tangle* connective $\langle dt\rangle$ which assigns a formula $\langle dt\rangle\Gamma$ to each finite non-empty set Γ of formulas. The other Boolean connectives $\bot$, $\vee$, $\to$ are introduced as standard abbreviations, and the dual $\langle d\rangle$ of $[d]$ is defined to be $\neg[d]\neg$. We write $[d]^*\varphi$ for the formula $\varphi \wedge [d]\varphi$, and $\langle d\rangle^*\varphi$ for $\varphi \vee \langle d\rangle\varphi$. We denote the set of all formulas by $\mathcal{L}^{\langle dt\rangle}_{[d]}$, and the set of formulas with no occurrence of $\langle dt\rangle$ by $\mathcal{L}_{[d]}$.

A *frame* is a pair $\mathcal{F} = (W, R)$, where W is a non-empty set and R is a binary relation on W. We may write any of $R(w, v)$, Rwv, and wRv to denote that $(w, v) \in R$. We let $R(w)$ denote the set $\{v \in W : wRv\}$ An element w is called *reflexive* if wRw, and *irreflexive* otherwise.

We restrict ourselves throughout the paper to frames that have *transitive* R. Then if $R^* = R \cup id_W$, where id_W is the identity relation on W, we get that R^* is the reflexive transitive closure of R, and putting $R^*(w) = \{v \in W : wR^*v\}$ we get that $R^*(w) = \{w\} \cup R(w)$. Observe that w is reflexive iff $R^*(w) = R(w)$.

If R^{-1} is the inverse relation to R, then each subset $W' \subseteq W$ has the R-*inverse image* $R^{-1}(W') = \{w \in W : \exists v \in W'(wRv)\} = \{w : R(w) \cap W' \neq \emptyset\}$. For singleton subsets we write $R^{-1}(\{v\})$ just as $R^{-1}(v)$.

A *model* $(\mathcal{F}, h)$ on $\mathcal{F}$ is given by an *assignment* h, which is a function from Var into the powerset $\wp W$ of W. The notion $(\mathcal{F}, h), w \models \varphi$ of a formula φ being true, or satisfied, at w in model $(\mathcal{F}, h)$ is defined by induction on the formation of φ, by the following clauses.

(i) $(\mathcal{F}, h), w \models p$ iff $w \in h(p)$, for $p \in \mathsf{Var}$.

(ii) $(\mathcal{F}, h), w \models \top$.

(iii) $(\mathcal{F}, h), w \models \neg\varphi$ iff $(\mathcal{F}, h), w \not\models \varphi$.

(iv) $(\mathcal{F}, h), w \models \varphi \wedge \psi$ iff $(\mathcal{F}, h), w \models \varphi$ and $(\mathcal{F}, h), w \models \psi$.

(v) $(\mathcal{F}, h), w \models [d]\varphi$ iff $(\mathcal{F}, h), v \models \varphi$ for every $v \in R(w)$.

(vi) $(\mathcal{F}, h), w \models \langle dt \rangle \Gamma$ iff there is a sequence $w = w_0, w_1, \ldots$ in W with $w_n R w_{n+1}$ for each $n < \omega$ and such that for each $\gamma \in \Gamma$ there are infinitely many $n < \omega$ with $(\mathcal{F}, h), w_n \models \gamma$.

The sequence $\{w_n : n < \omega\}$ in the last clause could be described as an *endless R-path satisfying each member of Γ infinitely often.*

A formula φ is *satisfiable in frame $\mathcal{F}$* if $(\mathcal{F}, h), w \models \varphi$ for some h and some w. φ is *valid in $\mathcal{F}$* if $\neg\varphi$ is not satisfiable in $\mathcal{F}$, i.e. if φ is true at every point in every model on $\mathcal{F}$.

In any model $(\mathcal{F}, h)$ each formula defines the 'truth-set' $[\![\varphi]\!]_h = \{w \in W : (\mathcal{F}, h), w \models \varphi\}$. In particular $[\![p]\!]_h = h(p)$ for $p \in \mathsf{Var}$. The semantic clause 5 above for $[d]$ states that $[\![[d]\varphi]\!]_h = \{w : R(w) \subseteq [\![\varphi]\!]_h\}$. The truth condition for the dual modality $\langle d \rangle$ gives

$$[\![\langle d \rangle \varphi]\!]_h = \{w : R(w) \cap [\![\varphi]\!]_h \neq \emptyset\} = R^{-1}[\![\varphi]\!]_h. \tag{2.1}$$

Lemma 2.1 *In any model $(\mathcal{F}, h)$ on a frame, and any $w \in W$, we have $w \in [\![\langle dt \rangle \Gamma]\!]_h$ iff there is a subset S of W such that*

$$w \in S \subseteq \bigcap\nolimits_{\gamma \in \Gamma} R^{-1}([\![\gamma]\!]_h \cap S). \tag{2.2}$$

Proof. Suppose $w \in [\![\langle dt \rangle \Gamma]\!]_h$, and let $S = \{w_n : n < \omega\}$ be the resulting sequence given by clause 6 above. Then $w = w_0 \in S$, and for any $w_n \in S$ and any $\gamma \in \Gamma$ there is an $m > n$ such that $(\mathcal{F}, h), w_m \models \gamma$, so as R is transitive, $w_n R w_m \in [\![\gamma]\!]_h \cap S$, showing $w_n \in R^{-1}([\![\gamma]\!]_h \cap S)$. This proves (2.2).

Conversely, if (2.2) holds for some S, then for each $v \in S$ and each $\gamma \in \Gamma$ there is some $u \in S$ such that vRu and $(\mathcal{F}, h), u \models \gamma$. Using this, if $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ we can iteratively choose a sequence $\{w_n : n < \omega\}$ of members of S with $w_0 = w$ and $w_n R w_{n+1}$ and every index $mk + i$ with $1 \leq i \leq m$ having $(\mathcal{F}, h), w_{mk+i} \models \gamma_i$. This shows that $(\mathcal{F}, h), w \models \langle dt \rangle \Gamma$. □

Thus $[\![\langle dt \rangle \Gamma]\!]_h$ is the union of all subsets S of W such that $S \subseteq \bigcap_{\gamma \in \Gamma} R^{-1}([\![\gamma]\!]_h \cap S)$. To put this in perspective we invoke the Knaster-Tarski Theorem on fixed points of monotone functions on complete lattices [15]. Define $F(S) = \bigcap_{\gamma \in \Gamma} R^{-1}([\![\gamma]\!]_h \cap S)$. Then F is a function on the lattice of all subsets of W that is monotone for inclusion: $S \subseteq S'$ implies $F(S) \subseteq F(S')$. The Knaster-Tarski Theorem states that such a function has a *largest fixed point* S_0, namely if $S_0 = \bigcup\{S \subseteq X : S \subseteq F(S)\}$, then $S_0 = F(S_0)$ and $S \subseteq S_0$ whenever $F(S) = S$. But here $\bigcup\{S \subseteq X : S \subseteq F(S)\} = [\![\langle dt \rangle \Gamma]\!]_h$. So $[\![\langle dt \rangle \Gamma]\!]_h$ is the largest fixed point of F.

This fixed point interpretation and the 'endless R-path' interpretation of $\langle dt \rangle$ each have their uses. In Section 4 we use the fixed point approach to show that validity of formulas is preserved by certain morphisms. Here we note that the endless path approach makes it very easy to see that validity is preserved by *generated* subframes. Recall that $\mathcal{F}' = (W', R')$ is a generated subframe of $\mathcal{F} = (W, R)$ if $W' \subseteq W$, R' is the restriction of R to W', and W' is R-closed in the sense that $R(w) \subseteq W'$ for all $w \in W'$. Then given models $(\mathcal{F}, h)$ and

$(\mathcal{F}', h')$ with $h'(p) = h(p) \cap W'$ for $p \in \mathsf{Var}$, it is a standard fact that for all $\mathcal{L}_{[d]}$-formulas φ and all $w \in W'$,

$$(\mathcal{F}, h), w \models \varphi \quad \text{iff} \quad (\mathcal{F}', h'), w \models \varphi. \tag{2.3}$$

In other words, $[\![\varphi]\!]_{h'} = [\![\varphi]\!]_h \cap W'$. But this result extends readily to all $\mathcal{L}_{[d]}^{\langle dt \rangle}$-formulas, with the inductive case of a formula $\langle dt \rangle \Gamma$ holding because any endless R'-path is an R-path and, crucially, any endless R-path that starts in W' remains in W' by the R-closure of W' and so is an R'-path.

We write $\mathcal{F}^*(w)$ for the subframe of $\mathcal{F}$ based on $R^*(w)$, which is a generated subframe as R is transitive. If $W = R^*(w)$, then $\mathcal{F} = \mathcal{F}^*(w)$ and we say that w is a *root* of $\mathcal{F}$. From the result (2.3) a standard argument gives

Theorem 2.2 *A $\mathcal{L}_{[d]}^{\langle dt \rangle}$-formula is valid in a frame $\mathcal{F}$ iff it is valid in every rooted subframe $\mathcal{F}^*(w)$ of $\mathcal{F}$.* □

3 Spaces

Let X be a topological space. We do not name the topology of X, but just refer to various subsets of X as being open or closed in X. An open set O containing a point x is called an *open neighbourhood* of x. Then $O \setminus \{x\}$ is a *punctured neighbourhood* of x. We write $\mathrm{cl}_X\, S$ for the closure (smallest closed superset) of a subset $S \subseteq X$, and $\mathrm{int}_X\, S$ for the interior (largest open subset) of S. $\langle d_X \rangle S$ denotes the *derivative* or set of limit points of S. Then we have

$$\mathrm{int}_X S = \{x \in X : O \subseteq S \text{ for some open neighbourhood } O \text{ of } x\}. \tag{3.1}$$

$$\mathrm{cl}_X S = \{x \in X : S \cap O \neq \emptyset \text{ for all open neighbourhoods } O \text{ of } x\}. \tag{3.2}$$

$$\langle d_X \rangle S = \{x \in X : S \cap O \setminus \{x\} \neq \emptyset \text{ for all open neighbourhoods } O \text{ of } x\}. \tag{3.3}$$

X is called *dense-in-itself* if $\langle d_X \rangle X = X$, i.e. if every point x of X is a limit point of X and so $\{x\}$ is not open. We record some standard facts about these operators:

Lemma 3.1

(1) $\langle d_X \rangle$ *is additive:* $\langle d_X \rangle (S \cup T) = \langle d_X \rangle S \cup \langle d_X \rangle T$.

(2) $\mathrm{cl}_X\, S = S \cup \langle d_X \rangle S$.

(3) *S is closed iff it contains all its limit points (i.e. $\langle d_X \rangle S \subseteq S$).* □

A *model* (X, h) on X is given by an *assignment* $h : \mathsf{Var} \to \wp X$. The truth/satisfaction relation $(X, h), x \models \varphi$, and associated truth-sets $[\![\varphi]\!]_h = \{x \in X : (X, h), x \models \varphi\}$, are defined inductively as follows.

1. $(X, h), x \models p$ iff $x \in h(p)$, for $p \in \mathsf{Var}$.
2. $(X, h), x \models \top$.
3. $(X, h), x \models \neg\varphi$ iff $(X, h), x \not\models \varphi$.
4. $(X, h), x \models \varphi \wedge \psi$ iff $(X, h), x \models \varphi$ and $(X, h), x \models \psi$.

5. $(X,h), x \models [d]\varphi$ iff there is an open neighbourhood O of x with $(X,h), y \models \varphi$ for every $y \in O \setminus \{x\}$.
6. $(X,h), x \models \langle dt \rangle \Gamma$ iff there is some $S \subseteq X$ such that $x \in S \subseteq \bigcap_{\gamma\in\Gamma} \langle d_X \rangle (\llbracket \gamma \rrbracket_h \cap S)$.

By clause 5, $[d]\varphi$ is true at x iff x has a punctured neighbourhood included in $\llbracket \varphi \rrbracket_h$. This implies that

$(X,h), x \models \langle d \rangle \varphi$ iff every open neighbourhood O of x has $(X,h), y \models \varphi$ for some $y \in O \setminus \{x\}$.

From (3.3) it then follows that $\llbracket \langle d \rangle \varphi \rrbracket_h = \langle d_X \rangle \llbracket \varphi \rrbracket_h$, the set of all limit points of $\llbracket \varphi \rrbracket_h$. Using (3.1) and Lemma 3.1(2) we get that $\llbracket [d]^* \varphi \rrbracket_h = \mathrm{int}_X \llbracket \varphi \rrbracket_h$ and $\llbracket \langle d \rangle^* \varphi \rrbracket_h = \mathrm{cl}_X \llbracket \varphi \rrbracket_h$.

A set Y is *strictly dense* in a set S containing Y if every member of S is a limit point of Y, i.e. $S \subseteq \langle d_X \rangle Y$. In a model (X,h) a finite set of formulas Γ will be called *strictly tangled in* S if $\llbracket \gamma \rrbracket_h \cap S$ is strictly dense in S for all $\gamma \in \Gamma$, i.e. if $S \subseteq \bigcap_{\gamma\in\Gamma} \langle d_X \rangle (\llbracket \gamma \rrbracket_h \cap S)$. Thus the truth condition for $\langle dt \rangle \Gamma$ gives that $\llbracket \langle dt \rangle \Gamma \rrbracket_h$ is the union of all the sets S such that Γ is strictly tangled in S.

Theorem 3.2 *Γ is tangled in $\llbracket \langle dt \rangle \Gamma \rrbracket_h$, so $\llbracket \langle dt \rangle \Gamma \rrbracket_h$ is the largest set in which Γ is strictly tangled. Moreover $\llbracket \langle dt \rangle \Gamma \rrbracket_h = \bigcap_{\gamma\in\Gamma} \langle d_X \rangle (\llbracket \gamma \rrbracket_h \cap \llbracket \langle dt \rangle \Gamma \rrbracket_h)$.*

Proof. This is another instance of the Knaster-Tarski Theorem (see the paragraph after Lemma 2.1). If $F(S) = \bigcap_{\gamma\in\Gamma} \langle d_X \rangle (\llbracket \gamma \rrbracket_h \cap S)$, then Γ is strictly tangled in S iff $S \subseteq F(S)$. Now F is a monotone function on the powerset lattice of W, so the Knaster-Tarski Theorem states that F has *largest fixed point* $\bigcup \{S \subseteq X : S \subseteq F(S)\} = \llbracket \langle dt \rangle \Gamma \rrbracket_h$. □

Given the interpretation of $\langle d \rangle$ as $\langle d \rangle_X$ in spaces and as R^{-1} in frames (2.1), we now see that the semantics of $\langle dt \rangle$ in frames and spaces is formally the same: in both cases $\llbracket \langle dt \rangle \Gamma \rrbracket_h$ is the largest solution of the equation $S = \bigcap_{\gamma\in\Gamma} \mathbf{d}(\llbracket \gamma \rrbracket_h \cap S)$, where $\mathbf{d}$ is the relevant interpretation of $\langle d \rangle$.

A formula φ is *satisfiable in space* X if $(X,h), x \models \varphi$ for some h and some x, and is *valid in* X if $\neg\varphi$ is not satisfiable in X.

Theorem 3.3 *In any topological space, all instances of the following formula schemes are valid.*

Fix: $\langle dt \rangle \Gamma \to \langle d \rangle (\gamma \wedge \langle dt \rangle \Gamma)$, *with* $\gamma \in \Gamma$.

Ind: $[d]^* (\varphi \to \bigwedge_{\gamma\in\Gamma} \langle d \rangle (\gamma \wedge \varphi)) \to (\varphi \to \langle dt \rangle \Gamma)$.

Proof. Working in any model (X,h) on X, we show that these formulas are true at all points. The result for Fix is immediate from the previous theorem, which gives $\llbracket \langle dt \rangle \Gamma \rrbracket_h \subseteq \llbracket \langle d \rangle (\gamma \wedge \langle dt \rangle \Gamma) \rrbracket_h$.

For Ind, suppose $x \models [d]^* (\varphi \to \bigwedge_{\gamma\in\Gamma} \langle d \rangle (\gamma \wedge \varphi))$ and $x \models \varphi$. Then there is an open neighbourhood O of x such that for any $\gamma \in \Gamma$, $O \subseteq \llbracket \varphi \to \langle d \rangle (\gamma \wedge \varphi) \rrbracket_h$, hence $O \cap \llbracket \varphi \rrbracket_h \subseteq \langle d_X \rangle (\llbracket \gamma \rrbracket_h \cap \llbracket \varphi \rrbracket_h)$.

But then $O \cap \llbracket \varphi \rrbracket_h \subseteq \langle d_X \rangle (\llbracket \gamma \rrbracket_h \cap O \cap \llbracket \varphi \rrbracket_h)$, because if $y \in O \cap \llbracket \varphi \rrbracket_h$, then for any open neighbourhood O' of y, $O \cap O' \setminus \{y\}$ intersects $\llbracket \gamma \rrbracket_h \cap \llbracket \varphi \rrbracket_h$, hence

$O' \setminus \{y\}$ intersects $[\![\gamma]\!]_h \cap O \cap [\![\varphi]\!]_h$.

This shows that Γ is strictly tangled in $O \cap [\![\varphi]\!]_h$, and hence that $O \cap [\![\varphi]\!]_h \subseteq [\![\langle dt\rangle\Gamma]\!]_h$. But $x \in O \cap [\![\varphi]\!]_h$, so then $x \models \langle dt\rangle\Gamma$, confirming that Ind is true at x. □

Corresponding to generated subframes we have the notion of Y being an *open subspace* of X, meaning that Y is a subspace of X that is itself open in the topology on X. The openness of Y ensures that for all $S \subseteq X$,

$$\langle d_X\rangle S \cap Y = \langle d_Y\rangle(S \cap Y). \tag{3.4}$$

Theorem 3.4 *If Y is an open subspace of X, then any $\mathcal{L}^{\langle dt\rangle}_{[d]}$-formula valid in X is valid in Y.*

Proof. This follows from the result that if models (X, h) and (Y, h') have $h'(p) = h(p) \cap Y$ for $p \in \mathsf{Var}$, then for all $\mathcal{L}^{\langle dt\rangle}_{[d]}$-formulas φ and all $y \in Y$,

$$(X, h), y \models \varphi \quad \text{iff} \quad (Y, h'), y \models \varphi.$$

This result, which gives $[\![\varphi]\!]_{h'} = [\![\varphi]\!]_h \cap Y$, is standard for $\mathcal{L}_{[d]}$-formulas, with the inductive case of a formula $\langle d\rangle\varphi$ holding because (3.4) implies that $[\![\langle d\rangle\varphi]\!]_h \cap Y = \langle d_Y\rangle([\![\varphi]\!]_h \cap Y) = [\![\langle d\rangle\varphi]\!]_{h'}$.

For the case of a formula $\langle dt\rangle\Gamma$, assuming inductively that the result holds for all members of Γ, let $y \in Y$ and $(Y, h'), y \models \langle dt\rangle\Gamma$. Then y belongs to some $S \subseteq Y$ in which Γ is strictly tangled in (Y, h'). So for any $\gamma \in \Gamma$, $S \subseteq \langle d_Y\rangle([\![\gamma]\!]_{h'} \cap S) \subseteq \langle d_X\rangle([\![\gamma]\!]_h \cap S)$. This shows that Γ is strictly tangled in S in (X, h), hence $(X, h), y \models \langle dt\rangle\Gamma$.

Conversely, if $(X, h), y \models \langle dt\rangle\Gamma$, then y belongs to some $S \subseteq X$ in which Γ is strictly tangled in (X, h). Then for any $\gamma \in \Gamma$, $S \subseteq \langle d_X\rangle([\![\gamma]\!]_h \cap S)$, and so

$$S \cap Y \subseteq \langle d_X\rangle([\![\gamma]\!]_h \cap S) \cap Y = \langle d_Y\rangle([\![\gamma]\!]_h \cap S \cap Y) = \langle d_Y\rangle([\![\gamma]\!]_{h'} \cap S \cap Y).$$

Hence Γ is strictly tangled in $S \cap Y$ in (Y, h'). Since $y \in S \cap Y$, $(Y, h'), y \models \langle dt\rangle\Gamma$ follows as required. □

4 d-Morphisms

A function $\rho : X \to W$ is called a *d-morphism* from a space X to a frame $\mathcal{F} = (W, R)$ if every subset $S \subseteq W$ has $\langle d_X\rangle\rho^{-1}(S) = \rho^{-1}(R^{-1}(S))$. When W is finite, for this to hold it is sufficient to require that it hold whenever S is a singleton, i.e. $\langle d_X\rangle\rho^{-1}(w) = \rho^{-1}(R^{-1}(w))$ for all $w \in W$. This is because the operators $\langle d_X\rangle$, ρ^{-1} and R^{-1} distribute across finite unions, as was observed in [13] where maps of this kind were first studied. [3]

It is shown in [2, Corollary 2.9] that surjective d-morphisms preserve validity of all $\mathcal{L}_{[d]}$-formulas, regardless of whether $\mathcal{F}$ is finite or infinite. We will extend that result to the present language with $\langle dt\rangle$, making use of the two results in the next Lemma, which follow from [2, Theorem 2.7].

[3] In [6] a d-morphism with W finite is called a *representation* of the frame $\mathcal{F}$ over the space X.

Lemma 4.1 *Let $\rho : X \to W$ be a d-morphism from space X to frame $\mathcal{F} = (W, R)$.*

(1) *For all $w \in W$, $\rho^{-1}(R^*(w))$ is an open subset of X.*

(2) *For every* ***irreflexive*** *$w \in W$, the preimage $\rho^{-1}(w)$ is a discrete subspace of X.* □

Theorem 4.2 *Let $\rho : X \to W$ be a d-morphism from space X to frame $\mathcal{F} = (W, R)$, and let (X, h) and $(\mathcal{F}, h')$ be models having $[\![p]\!]_h = \rho^{-1}[\![p]\!]_{h'}$ for all $p \in \mathsf{Var}$. Then $[\![\varphi]\!]_h = \rho^{-1}[\![\varphi]\!]_{h'}$ for all $\mathcal{L}^{\langle dt \rangle}_{[d]}$-formulas φ.*

Proof. By induction on the formation of φ, with the result holding by assumption if $\varphi \in \mathsf{Var}$; the inductive cases of the Boolean connectives being standard, and the inductive case of a formula beginning with $\langle d \rangle$ following from the definition of d-morphism.

Consider the case of a formula $\langle dt \rangle \Gamma$, on the inductive assumption that the result holds for all members of Γ. If $x \in \rho^{-1}[\![\langle dt \rangle \Gamma]\!]_{h'}$, then by Lemma 2.1 $\rho(x)$ belong to some $S \subseteq W$ such that $S \subseteq \bigcap_{\gamma \in \Gamma} R^{-1}([\![\gamma]\!]_{h'} \cap S)$. Let $Y = \rho^{-1}(S)$. If $y \in Y$, then for any $\gamma \in \Gamma$, $\rho(y) \in R^{-1}([\![\gamma]\!]_{h'} \cap S)$, so $y \in \rho^{-1}(R^{-1}([\![\gamma]\!]_{h'} \cap S))$. But as ρ is a d-morphism, this implies

$$y \in \langle d_X \rangle \rho^{-1}([\![\gamma]\!]_{h'} \cap S) = \langle d_X \rangle (\rho^{-1}[\![\gamma]\!]_{h'} \cap \rho^{-1}(S)) = \langle d_X \rangle ([\![\gamma]\!]_h \cap Y).$$

This shows that $Y \subseteq \langle d_X \rangle ([\![\gamma]\!]_h \cap Y)$ for all $\gamma \in \Gamma$, so Γ is strictly tangled in Y in (X, h). Since $x \in Y$, that gives $x \in [\![\langle dt \rangle \Gamma]\!]_h$, proving that $\rho^{-1}[\![\langle dt \rangle \Gamma]\!]_{h'} \subseteq [\![\langle dt \rangle \Gamma]\!]_h$.

For the converse inclusion, suppose $x \in [\![\langle dt \rangle \Gamma]\!]_h$. Then x belongs to some $S \subseteq X$ such that $S \subseteq \bigcap_{\gamma \in \Gamma} \langle d_X \rangle ([\![\gamma]\!]_h \cap S)$. Let $Z = R^*(\rho(x)) \cap \rho(S) \subseteq W$. Then $\rho(x) \in Z$. We will show that $Z \subseteq \bigcap_{\gamma \in \Gamma} R^{-1}([\![\gamma]\!]_{h'} \cap Z)$. For this, take any $w \in Z$. Then $w = \rho(y)$ for some $y \in S$. Let $O = \rho^{-1}(R^*(w))$, an open neighbourhood of y by Lemma 4.1(1).

Now if w is irreflexive, then by Lemma 4.1(2), y is isolated from the rest of $\rho^{-1}(w)$, so there is an open neighbourhood O' of y such that $O' \cap \rho^{-1}(w) = \{y\}$. Put $U = O \cap O'$. If however w is reflexive, put $U = O$. Either way, U is an open neighbourhood of y. Since $y \in S$, for any $\gamma \in \Gamma$ we have $y \in \langle d_X \rangle ([\![\gamma]\!]_h \cap S)$, so there is some $y' \in U$ with $y \neq y' \in [\![\gamma]\!]_h \cap S$. Then $\rho(y') \in \rho(S)$. Also as $y' \in [\![\gamma]\!]_h$, the induction hypothesis on γ gives $\rho(y') \in [\![\gamma]\!]_{h'}$.

Since $y' \in U \subseteq O$, we get $\rho(y') \in R^*(w)$. If w is reflexive, this immediately gives $wR\rho(y')$. But if w is irreflexive, then $y' \in O'$ and $y' \neq y$, so $y' \notin \rho^{-1}(w)$, hence $\rho(y') \neq w$, and therefore again $wR\rho(y')$. Thus in any case $wR\rho(y')$, so as $w \in R^*(\rho(x))$, transitivity of R gives $\rho(y') \in R^*(\rho(x))$. Now we have $\rho(y') \in [\![\gamma]\!]_{h'} \cap R^*(\rho(x)) \cap \rho(S) = [\![\gamma]\!]_{h'} \cap Z$. Hence $w \in R^{-1}([\![\gamma]\!]_{h'} \cap Z)$ as required.

This complete the proof that $\rho(x) \in Z \subseteq \bigcap_{\gamma \in \Gamma} R^{-1}([\![\gamma]\!]_{h'} \cap Z)$, which by Lemma 2.1 implies $\rho(x) \in [\![\langle dt \rangle \Gamma]\!]_{h'}$, hence $x \in \rho^{-1}[\![\langle dt \rangle \Gamma]\!]_{h'}$. □

Corollary 4.3 *If there exists a surjective d-morphism $\rho : X \to \mathcal{F}$, then any $\mathcal{L}^{\langle dt \rangle}_{[d]}$-formula valid in X is valid in $\mathcal{F}$.*

Proof. If φ is not valid in $\mathcal{F}$, then there is a model $(\mathcal{F}, h')$ with $w \notin [\![\varphi]\!]_{h'}$ for some w. Then $w = \rho(x)$ for some $x \in X$ as ρ is surjective. Define a model (X, h) by putting $h(p) = \rho^{-1}[\![p]\!]_{h'}$ for all $p \in \mathsf{Var}$. Then $x \notin \rho^{-1}[\![\varphi]\!]_{h'} = [\![\varphi]\!]_h$ by the Theorem, making φ not valid in X. □

5 The Logic of $\mathbb{R}$

By the *logic of a space* X, in a given language, we mean the set of all formulas of that language that are valid in X. In this section we will identify the $\mathcal{L}^{\langle dt \rangle}_{[d]}$-logic of the real line $\mathbb{R}$, considered as a space under its standard Euclidean metric topology.

Any space validates the formula-scheme $\langle d \rangle\langle d \rangle\varphi \to \langle d \rangle^*\varphi$. The stronger scheme $\langle d \rangle\langle d \rangle\varphi \to \langle d \rangle\varphi$, which corresponds to transitivity of R in frames, is valid precisely in the T_D-spaces, which are those in which the derivative $\langle d_X \rangle\{x\}$ of any point is closed. This T_D condition, introduced in [1], is implied by the T_1 separation property, which is itself equivalent to $\langle d_X \rangle\{x\} = \emptyset$. Our concern here is with the logic of metric spaces, which have the even stronger T_2 property, so this justifies our restriction to transitive frames.

By a *tangle logic* we mean any set of $\mathcal{L}^{\langle dt \rangle}_{[d]}$-formulas that includes all instances of tautologies and of the schemes

K: $[d](\varphi \to \psi) \to ([d]\varphi \to [d]\psi)$

4: $\langle d \rangle\langle d \rangle\varphi \to \langle d \rangle\varphi$

Fix: $\langle dt \rangle\Gamma \to \langle d \rangle(\gamma \wedge \langle dt \rangle\Gamma)$, all $\gamma \in \Gamma$

Ind: $[d]^*(\varphi \to \bigwedge_{\gamma\in\Gamma} \langle d \rangle(\gamma \wedge \varphi)) \to (\varphi \to \langle dt \rangle\Gamma)$

and is closed under modus ponens and the $[d]$-modality. The members of the logic will be called its *theorems*. It is readily seen that if $\mathcal{F}$ is any transitive frame, then the set of all formulas valid in $\mathcal{F}$ is a tangle logic.

The smallest tangle logic will be denoted K4t, and the smallest tangle logic containing the D-axiom $\langle d \rangle\top$ with be denoted KD4t. The frame condition for validity of $\langle d \rangle\top$ is that R be *serial*, i.e. $\forall w \exists w'(wRw')$, so $R(w) \neq \emptyset$ for all w. A serial transitive frame will be referred to as a *KD4-frame*.

Theorem 5.1 (Soundness) *The $\mathcal{L}^{\langle dt \rangle}_{[d]}$-logic of any T_D-space includes K4t and the logic of any dense-in-itself T_D-space includes KD4t.*

Proof. It is standard that the set of formulas valid in a space includes axiom K and is closed under modus ponens and $[d]$. It includes Fix and Ind by Theorem 3.3, and includes axiom 4 when the space is T_D as just noted. The D-axiom is valid in space X iff $\langle d_X \rangle X = X$, which means that X is dense-in-itself. □

There is a natural topological distinction between $\mathbb{R}$ and the higher dimensional Euclidean spaces $\mathbb{R}^n$ for $n \geq 2$. If O is an open ball in $\mathbb{R}^n$, then a punctured subspace $O \setminus \{x\}$ is connected (indeed any two points of $O \setminus \{x\}$ are joined by a continuous path lying in $O \setminus \{x\}$). But if O is an open interval (a, b) in $\mathbb{R}$, then $O \setminus \{x\}$ is disconnected, with two connected components (a, x) and (x, b). This difference is captured with the help of certain formulas G_n for $n \geq 1$. G_n has the $n+1$ variables $p_0, \dots, p_n$. For $i \leq n$, put $Q_i = p_i \wedge \bigwedge_{i \neq j \leq n} \neg p_j$. Then G_n is

$$[d]\big(\bigvee_{i \leq n} [d]^* Q_i\big) \to \bigvee_{i \leq n} [d]\neg Q_i.$$

It asserts that if some punctured neighbourhood of x can be covered by the interiors of $n+1$ disjoint sets, then there must be a punctured neighbourhood of x that is disjoint from one of those sets.

The G_n's were introduced by Shehtman in [13], where he showed that the $\mathcal{L}_{[d]}$-logic of $\mathbb{R}^n$ for $n \geq 2$ is KD4G$_1$, the smallest extension of KD4 to include all substitution instances of G_1. He also conjectured that the $\mathcal{L}_{[d]}$-logic of $\mathbb{R}$ is KD4G$_2$. This was later verified by himself [12] with another proof given by Lucero-Bryan [8].

Let KD4G$_2t$ be the smallest tangle logic including the D-axiom and all substitution instances of G_2. In [6, Section 4.13] we proved that this logic has the finite model property for frames: if an $\mathcal{L}^{\langle dt \rangle}_{[d]}$-formula φ is not a KD4G$_2t$-theorem, then it is falsified by a model on some finite frame that validates KD4G$_2t$.

Theorem 5.2 *KD4G$_2t$ is the $\mathcal{L}^{\langle dt \rangle}_{[d]}$-logic of* $\mathbb{R}$.

Proof. *Soundness*: since $\mathbb{R}$ is a dense-in-itself T_D-space that validates G_2, its logic includes KD4G$_2t$. *Completeness*: let φ be an $\mathcal{L}^{\langle dt \rangle}_{[d]}$-formula that is not a KD4G$_2t$-theorem. Then by [6] there is a finite KD4-frame $\mathcal{F}$ falsifying φ that validates KD4G$_2t$. Hence by Theorem 2.2, φ is not valid in some rooted subframe $\mathcal{F}^*(w)$ of $\mathcal{F}$ that also validates K4DG$_2t$. But Lemma 4.4 of [8] showed that any finite rooted KD4G$_2$-frame is a d-morphic image of any interval (x, y) of $\mathbb{R}$ with $x < y$. So there exists such a d-morphism from (x, y) onto $\mathcal{F}^*(w)$. By Corollary 4.3 it follows that φ is not valid in the open subspace (x, y) of $\mathbb{R}$. Hence φ is not valid in $\mathbb{R}$ by Theorem 3.4. □

As already noted in the Introduction, we showed in [6, Theorem 9.3] that for $n \geq 2$, the $\mathcal{L}^{\langle dt \rangle}_{[d]}$-logic of $\mathbb{R}^n$ is KD4G$_1t$.

6 Universal Modality

We now extend the syntax to include the universal modality $\forall$, creating formulas $\forall\varphi$ with the spatial semantics $(X, h), x \models \forall\varphi$ iff for all $y \in X$, $(X, h), y \models \varphi$; and the frame-semantics $(\mathcal{F}, h), w \models \forall\varphi$ iff for all $v \in W$, $(\mathcal{F}, h), v \models \varphi$.

We denote the new set of formulas by $\mathcal{L}^{\langle dt \rangle}_{[d]\forall}$. It is straightforward to extend Corollary 4.3 to show that validity of these formulas is preserved by surjective d-morphisms.

A tangle logic for the new language is now required to include the S5 axioms and rules for $\forall$, and the scheme

U: $\forall\varphi \to [d]\varphi$,

all of which are valid in all frames and spaces. The smallest such logic is denoted K4t.U, and KD4t.U and KD4G$_n t$.U are defined similarly. All of these $\mathcal{L}^{\langle dt \rangle}_{[d]\forall}$-logics were shown in [6] to have the finite model property over their frames.

Let K4t.UC be the smallest tangle logic extending K4t.U in the language $\mathcal{L}^{\langle dt \rangle}_{[d]\forall}$ that includes the scheme

C: $\forall(\Box^*\varphi \vee \Box^*\neg\varphi) \to (\forall\varphi \vee \forall\neg\varphi)$,

which was introduced in [11]. The condition for validity of C in a frame is graphical connectedness: any two points are connected by an $(R \cup R^{-1})$-path. The validity condition for C in a space X is topological connectedness, i.e. that X cannot be partitioned into two non-empty open subsets.

In [6], the logics K4t.UC, KD4t.UC and KD4G$_n t$.UC were all shown to have the finite model property over their frames. Theorem 9.5 of [6] proved that if X is any dense-in-itself metric space then the $\mathcal{L}^{\langle dt \rangle}_{[d]\forall}$-logic of X is included in KD4G$_1 t$.UC, and is exactly KD4G$_1 t$.UC if X validates G$_1$. In particular the $\mathcal{L}^{\langle dt \rangle}_{[d]\forall}$-logic of $\mathbb{R}^n$ is KD4G$_1 t$.UC for all $n \geq 2$.

As to the real line $\mathbb{R}$, its $\mathcal{L}_{[d]\forall}$-logic was shown to be KD4G$_2$.UC in [8]. We extend this now to the language with $\langle dt \rangle$.

Theorem 6.1 *KD4G$_2 t$.UC is the $\mathcal{L}^{\langle dt \rangle}_{[d]\forall}$-logic of* $\mathbb{R}$.

Proof. *Soundness*: the axioms D, 4, G$_2$ and C are all valid in $\mathbb{R}$.

Completeness: let φ be an $\mathcal{L}^{\langle dt \rangle}_{[d]\forall}$-formula that is not a KD4G$_2 t$.UC-theorem. Then by [6, Section 4.13] there is a finite frame $\mathcal{F}$ that validates KD4G$_2 t$.UC and falsifies φ. But Theorem 5.25 of [8] proved that any KD4G$_2$.UC frame is a d-morphic image of $\mathbb{R}$. Since validity of $\mathcal{L}^{\langle dt \rangle}_{[d]\forall}$-formulas is preserved by surjective d-morphisms, φ is not valid in $\mathbb{R}$.

□

7 Zero-Dimensionality and Dissection

A metric space is *zero-dimensional* if its topology has a basis of clopen (closed and open) sets. Such a space is totally disconnected: distinct points can be separated by a clopen set. Examples of zero-dimensionality include the space of rationals $\mathbb{Q}$, the irrationals $\mathbb{R} - \mathbb{Q}$, the Cantor space, and the Baire space ω^ω. These are all dense-in-themselves and *separable*, i.e. they have a countable dense subset.

It was shown in [13] that the $\mathcal{L}_{[d]}$-logic of each separable zero-dimensional dense-in-itself metric space is KD4. Here we will generalise this, first by eliminating the restriction to separable spaces, and then by showing that the $\mathcal{L}^{\langle dt \rangle}_{[d]}$-logic of each zero-dimensional dense-in-itself metric space is KD4t. This section provides some prerequisite results about 'dissecting' an open set into special subsets with properties that allow us to use them to represent the structure of finite frames.

First some background. If X is a metric space, we write d for its metric, and $N_\varepsilon(x_0)$ for the open ball $\{x \in X : d(x, x_0) < \varepsilon\}$, where $x_0 \in X$, $\varepsilon \in \mathbb{R}$,

$\varepsilon > 0$. For non-empty $S \subseteq X$, define $d(x,S) = \inf\{d(x,y) : y \in S\}$. (We leave $d(x,\emptyset)$ undefined.) The following Lemma collects some standard facts.

Lemma 7.1 *Let X be a dense-in-itself metric space and $S \subseteq X$.*

(1) *If S is non-empty and open, then S is infinite.*

(2) $\langle d_X \rangle S = \{x \in X : S \cap O$ *is infinite for every open neighbourhood O of x*$\}$.

(3) $\mathrm{int}_X\, S \subseteq \langle d_X \rangle S$. *If S is open then $\langle d_X \rangle S = \mathrm{cl}_X\, S$.* □

Theorem 7.2 *Let X be a dense-in-itself metric space. If $\mathbb{G}$ is a non-empty open subset of X, then for $r, s < \omega$, $\mathbb{G}$ can be partitioned into non-empty open subsets $\mathbb{G}_1, \dots, \mathbb{G}_r$ and other non-empty sets $\mathbb{B}_0, \dots, \mathbb{B}_s$ such that, letting $D = \mathrm{cl}_X(\mathbb{G}) \setminus (\mathbb{G}_1 \cup \cdots \cup \mathbb{G}_r)$, we have $\mathrm{cl}_X(\mathbb{G}_i) \setminus \mathbb{G}_i = D$ for each $i = 1, \dots, r$, and $\langle d_X \rangle \mathbb{B}_j = D$ for each $j = 0, \dots, s$.* □

Rasiowa and Sikorski in [10, III, 7.1] proved the version of this theorem that has '$\mathrm{cl}_X\, \mathbb{B}_j = D$' in place of '$\langle d_X \rangle \mathbb{B}_j = D$'. That version follows from the above because if $\langle d_X \rangle \mathbb{B}_j = D$, then $\mathrm{cl}_X\, \mathbb{B}_j = \mathbb{B}_j \cup \langle d_X \rangle \mathbb{B}_j = \mathbb{B}_j \cup D = D$ since $\mathbb{B}_j \subseteq D$. But the two versions are equivalent, for by applying the version from [10] to r and $2s+1$ we first obtain disjoint sets $\mathbb{B}^i_j$ with $\mathrm{cl}_X\, \mathbb{B}^i_j = D$ for $j = 0, \dots, s$ and $i = 0, 1$, and define $\mathbb{B}_j = \mathbb{B}^0_j \cup \mathbb{B}^1_j$ for each j. Then it can be shown that $\langle d_X \rangle \mathbb{B}_j = D$ [6, Section 7.4].

Tarski introduced the first version of the cl-formulation of this theorem in [14, satz 3.10]. It had $s = 0$ and required X to be separable. He credited the proof to Samuel Eilenberg, noting that he had originally proven the result himself for $\mathbb{R}$ and its dense-in-themselves subspaces. The restriction to $s = 0$ was removed in [9, theorem 3.5], where the property of X that the theorem asserts was called 'dissectability'. The separability restriction was removed in [10].

Theorem 7.2 allows a suitable morphism to be constructed from $\mathbb{G}$ onto any finite rooted reflexive transitive frame. But KD-frames may have irreflexive points, and we need further dissection results to handle this. First we state a result that is an instance of Theorem 7.8(1) of [6].

Theorem 7.3 *Let X be a dense-in-itself metric space and $\mathbb{U}$ be a non-empty open subset of X. Then there are disjoint non-empty subsets $\mathbb{I}_0, \mathbb{I}_1 \subseteq \mathbb{U}$ satisfying $\langle d_X \rangle \mathbb{I}_0 = \langle d_X \rangle \mathbb{I}_1 = \mathrm{cl}_X(\mathbb{U}) \setminus \mathbb{U}$.* □

The next results require X to be zero-dimensional.

Lemma 7.4 *Let X be a zero-dimensional dense-in-itself metric space and $\mathbb{G}$ be a non-empty open subset of X. Then $\mathbb{G}$ can be partitioned into non-empty open subsets $\mathbb{G}_0, \mathbb{G}_1$ such that $\mathrm{cl}_X(\mathbb{G}) \setminus \mathbb{G} = \mathrm{cl}_X(\mathbb{G}_i) \setminus \mathbb{G}_i$ for each $i = 0, 1$.*

Proof. See page 15 in Appendix. □

The partitioning of $\mathbb{G}$ can now be extended to any finite number of cells.

Theorem 7.5 (dissection) *Let $\mathbb{G}$ be a non-empty open subset of a zero-dimensional dense-in-itself metric space X, and let $n < \omega$. Then $\mathbb{G}$ can be* *partitioned into non-empty open subsets* $\mathbb{G}_0, \ldots, \mathbb{G}_n$ *such that* $\mathrm{cl}_X(\mathbb{G}) \setminus \mathbb{G} = \mathrm{cl}_X(\mathbb{G}_i) \setminus \mathbb{G}_i$ *for each* $i \leq n$.

Proof. By induction on n. If $n = 0$ we let $\mathbb{G}_0 = \mathbb{G}$. Now we assume the result for n and prove it for $n+1$. By the inductive hypothesis, there is a partition $\mathbb{G}_0, \ldots, \mathbb{G}_n$ of $\mathbb{G}$ into non-empty open sets with $\mathrm{cl}_X(\mathbb{G}) \setminus \mathbb{G} = \mathrm{cl}_X(\mathbb{G}_i) \setminus \mathbb{G}_i$ for each $i \leq n$. By the preceding lemma, $\mathbb{G}_n$ can be partitioned into non-empty open sets $\mathbb{G}_n^0, \mathbb{G}_n^1$ with $\mathrm{cl}_X(\mathbb{G}_n) \setminus \mathbb{G}_n = \mathrm{cl}_X(\mathbb{G}_n^i) \setminus \mathbb{G}_n^i$ for each $i = 0, 1$. So $\mathrm{cl}_X(\mathbb{G}) \setminus \mathbb{G} = \mathrm{cl}_X(\mathbb{G}_n^i) \setminus \mathbb{G}_n^i$ for each $i = 0, 1$. The required partition is now $\mathbb{G}_0, \ldots, \mathbb{G}_{n-1}, \mathbb{G}_n^0, \mathbb{G}_n^1$.

□

8 d-Morphisms on Open Subspaces

Let X be a topological space, $\mathbb{G}$ a non-empty open subset of X, $\mathcal{F} = (W, R)$ a *finite* frame, and $\rho : \mathbb{G} \to W$ a map. Recall from Section 4 that ρ is a *d-morphism* from the space $\mathbb{G}$ to $\mathcal{F}$ when

$$\langle d_{\mathbb{G}} \rangle \rho^{-1}(w) = \rho^{-1}(R^{-1}(w)) \text{ for all } w \in W. \tag{8.1}$$

But the openness of $\mathbb{G}$ ensures (3.4) that $\langle d_{\mathbb{G}} \rangle \rho^{-1}(w) = \mathbb{G} \cap \langle d_X \rangle \rho^{-1}(w)$, so ρ is a d-morphism from $\mathbb{G}$ to $\mathcal{F}$ iff $\mathbb{G} \cap \langle d_X \rangle \rho^{-1}(w) = \rho^{-1}(R^{-1}(w))$ for all $w \in W$.

The following are some useful facts about the relations between d-morphisms on open subspaces. The proofs are left to the reader.

Lemma 8.1 *Let* $\mathcal{F}' = (W', R')$ *be a generated subframe of* $\mathcal{F} = (W, R)$*, let* $\mathbb{T}$ *and* $\mathbb{G}$ *be open subsets of a space* X *with* $\emptyset \neq \mathbb{T} \subseteq \mathbb{G}$*, and let* $\rho : \mathbb{G} \to W'$ *be a map.*

(1) ρ *is a d-morphism from* $\mathbb{G}$ *to* $\mathcal{F}$ *iff it is a d-morphism from* $\mathbb{G}$ *to* $\mathcal{F}'$.

(2) $\rho \upharpoonright \mathbb{T}$ *is a d-morphism from* $\mathbb{T}$ *to* $\mathcal{F}$ *iff* $\mathbb{T} \cap \langle d_X \rangle \rho^{-1}(w) = \mathbb{T} \cap \rho^{-1}(R^{-1}(w))$ *for every* $w \in W$.

(3) *If* $\mathbb{T} = \bigcup_{i \in I} \mathbb{G}_i$*, where for each* i*,* $\mathbb{G}_i$ *is non-empty and open and* $\rho \upharpoonright \mathbb{G}_i$ *is a d-morphism from* $\mathbb{G}_i$ *to* $\mathcal{F}$*, then* $\rho \upharpoonright \mathbb{T}$ *is a d-morphism from* $\mathbb{T}$ *to* $\mathcal{F}$. □

A map $\rho : \mathbb{G} \to W$ is said to be *full* if $\mathrm{cl}_X(\mathbb{G}) \setminus \mathbb{G} \subseteq \langle d_X \rangle \rho^{-1}(w)$ for all $w \in W$. The following key result on the existence of full d-morphisms is proven on page 16 in the Appendix.

Theorem 8.2 *Let* X *be a zero-dimensional dense-in-itself metric space, and* $\mathcal{F} = (W, R)$ *be a finite, rooted, KD4-frame (i.e.* R *is transitive and serial). If* $\mathbb{G} \subseteq X$ *is a non-empty open set, then there is a surjective full d-morphism* $\rho : \mathbb{G} \to W$ *from* $\mathbb{G}$ *to* $\mathcal{F}$. □

Corollary 8.3 *Let* X *be a zero-dimensional dense-in-itself metric space. Then for every finite KD4-frame* $\mathcal{F} = (W, R)$*, there is a surjective d-morphism from any non-empty open subset* $\mathbb{G}$ *of* X *to* $\mathcal{F}$.

Proof. Let $W = \{w_0, \ldots, w_n\}$. Then $W = R^*(w_0) \cup \cdots \cup R^*(w_n)$ and so $\mathcal{F}$ is the union of its rooted subframes $\mathcal{F}^*(w_0), \ldots, \mathcal{F}^*(w_n)$, which are also KD4-

frames. As an open subspace of X, $\mathbb{G}$ is infinite (Lemma 7.1(1)) and zero-dimensional, so can be partitioned into non-empty open subsets $\mathbb{G}_0, \dots, \mathbb{G}_n$. This follows from Theorem 7.5, but does not depend on it and is a standard fact that holds for any infinite zero-dimensional metric space. [4] For each $i \leq n$, by Theorem 8.2 there is a surjective d-morphism $\rho_i : \mathbb{G}_i \to R^*(w_i)$ from $\mathbb{G}_i$ to $\mathcal{F}^*(w_i)$. By Lemma 8.1(1), ρ_i is a d-morphism from $\mathbb{G}_i$ to $\mathcal{F}$. Put $\rho = \bigcup_{i \leq n} \rho_i$. Then ρ is a map $\mathbb{G} \to W$ that is surjective as the $R^*(w_i)$'s cover W and each ρ_i is surjective. For each i, $\rho \restriction \mathbb{G}_i$ is the d-morphism ρ_i from $\mathbb{G}_i$ to $\mathcal{F}$, so by Lemma 8.1(3), ρ is a d-morphism from $\mathbb{G}$ to $\mathcal{F}$. □

In [6, Section 4.8] we showed that KD4t has the finite model property over serial transitive frames. We can now apply that to zero-dimensional metric spaces.

Theorem 8.4 *If X is any zero-dimensional dense-in-itself metric space, then the $\mathcal{L}^{\langle dt \rangle}_{[d]}$-logic of X is KD4t.*

Proof. (A similar argument to Theorem 5.2.) *Soundness*: By Theorem 5.1, the $\mathcal{L}^{\langle dt \rangle}_{[d]}$-logic of any dense-in-itself metric space includes KD4t.

Completeness: if φ is an $\mathcal{L}^{\langle dt \rangle}_{[d]}$-formula that is not a KD4t-theorem, then by [6, Section 4.8] there is a finite frame $\mathcal{F}$ validating KD4t in which φ is not valid. Let $\mathbb{G} = X$. Then by Corollary 8.3 there is a surjective d-morphism $\rho : X \to W$ from X to $\mathcal{F}$, so by Corollary 4.3, φ is not valid in X. □

The same argument shows that KD4t.U is the $\mathcal{L}^{\langle dt \rangle}_{[d]\forall}$-logic of X, because KD4t.U has the finite model property over KD4-frames [6, Section 4.9]. Of course the connectedness axiom C is not relevant here, as X is totally disconnected.

In conclusion we extend Theorem 8.4 to a *strong completeness* result whose proof and ramifications are discussed on page 19 in the Appendix.

Theorem 8.5 *If Γ is any countable KD4t-consistent set of formulas, and x_0 is any point of any zero-dimensional dense-in-itself metric space X, then Γ is satisfiable at x_0 in X.* □

Appendix

This Appendix provides the proofs of Lemma 7.4 and Theorems 8.2 and 8.5.

Proof of Lemma 7.4

If $\text{cl}_X \mathbb{G} \subseteq \mathbb{G}$, then simply let $\mathbb{G}_0$ be any non-empty clopen proper subset of $\mathbb{G}$, and $\mathbb{G}_1 = \mathbb{G} \setminus \mathbb{G}_0$.

From now on, assume that $\text{cl}_X \mathbb{G} \setminus \mathbb{G} \neq \emptyset$. By Theorem 7.3, there are disjoint non-empty $\mathbb{I}_0, \mathbb{I}_1 \subseteq \mathbb{G}$ such that $\langle d_X \rangle \mathbb{I}_i = \text{cl}_X(\mathbb{G}) \setminus \mathbb{G}$ for each $i = 0, 1$. Both $\mathbb{I}_i$ are infinite (otherwise, $\langle d_X \rangle \mathbb{I}_i = \emptyset \neq \text{cl}_X \mathbb{G} \setminus \mathbb{G}$).

[4] Actually it holds for any infinite totally disconnected space, a weaker property than zero-dimensionality.

For each $x \in \mathbb{I}_0$, let $m(x) = \min(d(x, \mathbb{I}_0 \setminus \{x\}), d(x, \mathbb{I}_1))$. By assumption, $\mathbb{G} \cap \langle d_X \rangle \mathbb{I}_i = \emptyset$ for each i. So $m(x) > 0$. For each such x, *using zero-dimensionality*, choose a clopen neighbourhood $B(x)$ of x with

$$B(x) \subseteq N_{m(x)/6}(x). \tag{8.2}$$

Note that $B(x) \cap \mathbb{I}_1 = \emptyset$. Now define $\mathbb{G}_0 = \bigcup_{x \in \mathbb{I}_0} B(x)$ and $\mathbb{G}_1 = \mathbb{G} \setminus \mathbb{G}_0$. Plainly, $\mathbb{G}_0$ is open and $\mathbb{I}_0 \subseteq \mathbb{G}_0$, and $\mathbb{G}_0 \cap \mathbb{I}_1 = \emptyset$ so $\mathbb{I}_1 \subseteq \mathbb{G}_1$.

Claim. $\mathbb{G}_1$ is open (in X).
Proof of claim. Let $a \in \mathbb{G}_1$ be arbitrary, and let $s = d(a, \mathbb{I}_0)$. Again, $s > 0$. Fix $x \in \mathbb{I}_0$ with $d(a, x) < 2s$. Since $a \notin \mathbb{G}_0 \supseteq B(x)$, and $B(x)$ is clopen, we can choose an open neighbourhood N of a disjoint from $B(x)$. We can further suppose that $N \subseteq \mathbb{G}$ and $N \subseteq N_{s/2}(a)$.

The claim will be proved if we show that $N \subseteq \mathbb{G}_1$, which we do by showing that $N \cap B(y) = \emptyset$ for each $y \in \mathbb{I}_0 \setminus \{x\}$. Take such a y, and let $d(a, y) = t$, say (we have $s \leq t$). Then

$$m(y) \leq d(y, \mathbb{I}_0 \setminus \{y\}) \leq d(y, x) \leq d(y, a) + d(a, x) < t + 2s \leq 3t.$$

So $m(y)/6 \leq t/2$. By (8.2), $B(y) \subseteq N_{t/2}(y)$. So $B(y) \cap N_{t/2}(a) = \emptyset$. But $N \subseteq N_{s/2}(a) \subseteq N_{t/2}(a)$, so $B(y) \cap N = \emptyset$ as required. This proves the claim.

So we have partitioned $\mathbb{G}$ into two non-empty open sets $\mathbb{G}_i$ with $\mathbb{I}_i \subseteq \mathbb{G}_i$ $(i = 0, 1)$. It remains to check that $\mathrm{cl}_X(\mathbb{G}) \setminus \mathbb{G} = \mathrm{cl}_X(\mathbb{G}_i) \setminus \mathbb{G}_i$ for each i:

$$\begin{aligned}
\mathrm{cl}_X(\mathbb{G}) \setminus \mathbb{G} &= \langle d_X \rangle \mathbb{I}_i && \text{by choice of } \mathbb{I}_i \\
&= \langle d_X \rangle \mathbb{I}_i \setminus \mathbb{G} && \text{since } \langle d_X \rangle \mathbb{I}_i \text{ is disjoint from } \mathbb{G} \\
&\subseteq \mathrm{cl}_X(\mathbb{G}_i) \setminus \mathbb{G} && \text{since } \mathbb{I}_i \subseteq \mathbb{G}_i \text{ so } \langle d_X \rangle \mathbb{I}_i \subseteq \langle d_X \rangle \mathbb{G}_i \subseteq \mathrm{cl}_X \mathbb{G}_i \\
&\subseteq \mathrm{cl}_X(\mathbb{G}_i) \setminus \mathbb{G}_i && \text{since } \mathbb{G}_i \subseteq \mathbb{G} \\
&= \mathrm{cl}_X(\mathbb{G}_i) \setminus (\mathbb{G}_i \cup \mathbb{G}_{1-i}) && \text{since } \mathbb{G}_{1-i} \text{ is open and disjoint from } \mathbb{G}_i, \\
& && \quad \text{so also disjoint from } \mathrm{cl}_X \mathbb{G}_i \\
&= \mathrm{cl}_X(\mathbb{G}_i) \setminus \mathbb{G} && \text{since } \mathbb{G}_0 \cup \mathbb{G}_1 = \mathbb{G} \\
&\subseteq \mathrm{cl}_X(\mathbb{G}) \setminus \mathbb{G} && \text{since } \mathbb{G}_i \subseteq \mathbb{G}.
\end{aligned}$$

This proves the Lemma. □

Proof of Theorem 8.2

We construct a full d-morphism from $\mathbb{G}$ onto the rooted frame $\mathcal{F} = (W, R)$. To describe the structure of $\mathcal{F}$, define $R^\circ = R \cap R^{-1}$ and $R^\bullet = R \setminus R^{-1}$. Then $R^\circ(w) = \{v \in W : wRvRw\}$ and $R^\bullet(w) = \{v \in W : wRv \text{ and not } vRw\}$. The set $\{w\} \cup R^\circ(w)$ is the *cluster* of w. These clusters partition W. If w_0 is a root of $\mathcal{F}$, then $W = \{w_0\} \cup R^\circ(w_0) \cup R^\bullet(w_0)$, and there are two kinds of structure that $\mathcal{F}$ could have:

1. **w_0 is reflexive.** Then $w_0 \in R^\circ(w_0)$ and W is the disjoint union of $R^\circ(w_0)$ and $R^\bullet(w_0)$, with $R^\circ(w_0) \times W \subseteq R$. In this case it is possible that $R^\bullet(w_0) = \emptyset$, so then $\mathcal{F}$ consists of the single cluster $R^\circ(w_0)$. Either way, the relation R is universal on $R^\circ(w_0)$, whch has at least one element.

2. w_0 **is irreflexive.** Then $R^\circ(w_0) = \emptyset$ and W is the disjoint union of $\{w_0\}$ and $R^\bullet(w_0)$, with $\{w_0\} \times R^\bullet(w_0) \subseteq R$. If R is serial, then $R^\bullet(w_0) \neq \emptyset$.

The proof proceeds by induction on the size of W. We make the induction hypothesis that the result holds for all frames smaller than $\mathcal{F}$, and consider the two cases for w_0 just described.

Case 1: w_0 **is reflexive.** Then $W = R^\circ(w_0) \cup R^\bullet(w_0)$ as above. By Theorem 7.2, $\mathbb{G}$ can be partitioned into non-empty sets $\mathbb{B}_v$ ($v \in R^\circ(w_0)$) and non-empty open sets $\mathbb{G}_u$ ($u \in R^\bullet(w_0)$) such that, for each $u \in R^\bullet(w_0)$ and $v \in R^\circ(w_0)$, we have

$$\mathrm{cl}_X(\mathbb{G}_u) \setminus \mathbb{G}_u = \langle d_X \rangle \mathbb{B}_v = \mathrm{cl}_X(\mathbb{G}) \setminus \bigcup_{w \in R^\bullet(w_0)} \mathbb{G}_w = D, \text{ say.} \tag{8.3}$$

Each $\mathcal{F}^*(u)$ for $u \in R^\bullet(w_0)$ is a finite rooted KD4-frame smaller than $\mathcal{F}$, since it does not contain w_0, so by the inductive hypothesis, there is a surjective full d-morphism ρ_u from $\mathbb{G}_u$ to $\mathcal{F}^*(u)$. Define $\rho : \mathbb{G} \to W$ by

$$\rho(x) = \begin{cases} \rho_u(x), & \text{if } x \in \mathbb{G}_u \text{ for some (unique) } u \in R^\bullet(w_0), \\ v, & \text{if } x \in \mathbb{B}_v \text{ for some (unique) } v \in R^\circ(w_0). \end{cases}$$

That is, $\rho = \bigcup_{u \in R^\bullet(w_0)} \rho_u \; \cup \bigcup_{v \in R^\circ(w_0)} \mathbb{B}_v \times \{v\}$. We will show that ρ is a surjective full d-morphism from $\mathbb{G}$ to $\mathcal{F}$. The following claim will help.

Claim. $D \subseteq \langle d_X \rangle \rho^{-1}(w)$ for every $w \in W$, where D is defined in (8.3).
Proof of claim. There are two cases. The first is when $w \in R^\bullet(w_0)$. Now (8.3) gives $D = \mathrm{cl}_X\, \mathbb{G}_w \setminus \mathbb{G}_w$. As $\rho_w : \mathbb{G}_w \to \mathcal{F}^*(w)$ is full, $\mathrm{cl}_X\, \mathbb{G}_w \setminus \mathbb{G}_w \subseteq \langle d_X \rangle \rho_w^{-1}(w) \subseteq \langle d_X \rangle \rho^{-1}(w)$ (since $\rho_w \subseteq \rho$).

The second case is when $w \notin R^\bullet(w_0)$. Since $w \in W = R^\circ(w_0) \cup R^\bullet(w_0)$, we have $w \in R^\circ(w_0)$. Hence $\rho^{-1}(w) = \mathbb{B}_w$. By (8.3), $D = \langle d_X \rangle \mathbb{B}_w = \langle d_X \rangle \rho^{-1}(w)$. This proves the claim.

We now check that ρ is a d-morphism from $\mathbb{G}$ to $\mathcal{F}$. So let $w \in W$. We require

$$\mathbb{G} \cap \langle d_X \rangle \rho^{-1}(w) = \rho^{-1}(R^{-1}(w)). \tag{8.4}$$

Since $\mathbb{G}$ is partitioned into the $\mathbb{G}_u$'s and $\mathbb{B}_v$'s, it is enough to prove, for all $u \in R^\bullet(w_0)$ and $v \in R^\circ(w_0)$, that the following equations hold.

$$\mathbb{G}_u \cap \langle d_X \rangle \rho^{-1}(w) = \mathbb{G}_u \cap \rho^{-1}(R^{-1}(w)), \tag{8.5}$$

$$\mathbb{B}_v \cap \langle d_X \rangle \rho^{-1}(w) = \mathbb{B}_v \cap \rho^{-1}(R^{-1}(w)). \tag{8.6}$$

(i) **Proof of** (8.5). Since $\mathbb{G}_u$ is open and $\rho \restriction \mathbb{G}_u = \rho_u$, a d-morphism from $\mathbb{G}_u$ to the generated subframe $\mathcal{F}^*(u)$ of $\mathcal{F}$, Lemma 8.1(1) implies that $\rho \restriction \mathbb{G}_u$ is a d-morphism from $\mathbb{G}_u$ to $\mathcal{F}$. Then Lemma 8.1(2) yields (8.5).

(ii) **Proof of** (8.6). $\mathbb{B}_v \subseteq D$ by definition of D (8.3), so $\mathbb{B}_v \subseteq \langle d_X \rangle \rho^{-1}(w)$ by the claim. Since $v \in R^\circ(w_0)$, we have vRw_0Rw (as w_0 is a reflexive root), hence vRw and $\mathbb{B}_v = \rho^{-1}(v) \subseteq \rho^{-1}(R^{-1}(w))$. Thus $\mathbb{B}_v \cap \langle d_X \rangle \rho^{-1}(w) = \mathbb{B}_v = \mathbb{B}_v \cap \rho^{-1}(R^{-1}(w))$.

So ρ is indeed a d-morphism from $\mathbb{G}$ to $\mathcal{F}$. Now $\mathrm{cl}_X\,\mathbb{G}\setminus\mathbb{G}\subseteq D$ by (8.3), so by the claim, $\mathrm{cl}_X\,\mathbb{G}\setminus\mathbb{G}\subseteq\langle d_X\rangle\rho^{-1}(w)$ for every $w\in W$, showing that ρ is full.

We also need that ρ is surjective. But if $u\in R^\bullet(w_0)$ then $u=\rho(x)$ for some $x\in\mathbb{G}_u$ as $\rho_u:\mathbb{G}_u\to R^*(u)$ is surjective, and if $v\in R^\circ(w_0)$ then $v=\rho(x)$ for all $x\in\mathbb{B}_v$ by definition.

Case 2: w_0 **is irreflexive.** Then $W=\{w_0\}\cup R^\bullet(w_0)$. By Theorem 7.3 with $\mathbb{U}=\mathbb{G}$, there are disjoint non-empty $\mathbb{I},\mathbb{I}'\subseteq\mathbb{G}$ with $\langle d_X\rangle\mathbb{I}=\langle d_X\rangle\mathbb{I}'=\mathrm{cl}_X(\mathbb{G})\setminus\mathbb{G}$.

Let $\mathbb{G}'=\mathbb{G}\setminus\mathbb{I}$. Plainly, $\mathbb{G}'$ is non-empty (it contains $\mathbb{I}'$) and open (since $\langle d_X\rangle\mathbb{I}$ is disjoint from $\mathbb{G}$, we have $\mathbb{G}'=\mathbb{G}\setminus\mathbb{I}=\mathbb{G}\setminus(\mathbb{I}\cup\langle d_X\rangle\mathbb{I})=\mathbb{G}\setminus\mathrm{cl}_X\,\mathbb{I}$).

Claim. $\mathrm{cl}_X(\mathbb{G}')\setminus\mathbb{G}'=(\mathrm{cl}_X(\mathbb{G})\setminus\mathbb{G})\cup\mathbb{I}$.

Proof of claim. Plainly $\mathrm{cl}_X(\mathbb{G}')\setminus\mathbb{G}'=\mathrm{cl}_X(\mathbb{G}')\setminus(\mathbb{G}\setminus\mathbb{I})\subseteq(\mathrm{cl}_X(\mathbb{G}')\setminus\mathbb{G})\cup\mathbb{I}\subseteq(\mathrm{cl}_X(\mathbb{G})\setminus\mathbb{G})\cup\mathbb{I}$. Conversely, $\mathrm{cl}_X(\mathbb{G})\setminus\mathbb{G}=\langle d_X\rangle\mathbb{I}'\subseteq\langle d_X\rangle\mathbb{G}'\subseteq\mathrm{cl}_X\,\mathbb{G}'$, and

$$\begin{aligned}
\mathbb{I}&\subseteq\mathbb{G}\cap\mathrm{cl}_X\,\mathbb{G} && \text{clear}\\
&=\mathbb{G}\cap\langle d_X\rangle\mathbb{G} && \text{since } \mathrm{cl}_X\,\mathbb{G}=\langle d_X\rangle\mathbb{G} \text{ by Lemma 7.1(3)}\\
&=\mathbb{G}\cap(\langle d_X\rangle\mathbb{I}\cup\langle d_X\rangle\mathbb{G}') && \text{since } \mathbb{G}=\mathbb{I}\cup\mathbb{G}' \text{ and } \langle d_X\rangle \text{ is additive}\\
&=\mathbb{G}\cap\langle d_X\rangle\mathbb{G}' && \text{since } \mathbb{G}\cap\langle d_X\rangle\mathbb{I}=\emptyset\\
&\subseteq\langle d_X\rangle\mathbb{G}'\subseteq\mathrm{cl}_X\,\mathbb{G}' && \text{clear.}
\end{aligned}$$

So $(\mathrm{cl}_X(\mathbb{G})\setminus\mathbb{G})\cup\mathbb{I}\subseteq\mathrm{cl}_X\,\mathbb{G}'$. The converse inclusion $(\mathrm{cl}_X(\mathbb{G})\setminus\mathbb{G})\cup\mathbb{I}\subseteq\mathrm{cl}_X(\mathbb{G}')\setminus\mathbb{G}'$ now follows, since both $\mathrm{cl}_X(\mathbb{G})\setminus\mathbb{G}$ and $\mathbb{I}$ are disjoint from $\mathbb{G}'$. This proves the claim.

By Theorem 7.5, $\mathbb{G}'$ can be partitioned into non-empty open sets $\mathbb{G}_u$ ($u\in R^\bullet(w_0)$) with $\mathrm{cl}_X(\mathbb{G}')\setminus\mathbb{G}'=\mathrm{cl}_X(\mathbb{G}_u)\setminus\mathbb{G}_u$ for each such u. Here we need $R^\bullet(w_0)\neq\emptyset$, which holds by seriality of R and irreflexivity of w_0.

For each $u\in R^\bullet(w_0)$, the frame $\mathcal{F}^*(u)$ is a finite rooted KD4-frame smaller than $\mathcal{F}$. Inductively, there is a surjective full d-morphism ρ_u from $\mathbb{G}_u$ to $\mathcal{F}^*(u)$. Define $\rho:\mathbb{G}\to W$ by

$$\rho(x)=\begin{cases}\rho_u(x), & \text{if } x\in\mathbb{G}_u \text{ for some (unique) } u\in R^\bullet(w_0),\\ w_0, & \text{if } x\in\mathbb{I}.\end{cases}$$

Then ρ is surjective, similarly to Case 1. It helps below to note that

$$(\mathrm{cl}_X(\mathbb{G})\setminus\mathbb{G})\cup\mathbb{I}\subseteq\langle d_X\rangle\rho^{-1}(w)\quad\text{for all } w\in R^\bullet(w_0). \tag{8.7}$$

For, by the claim, for each $w\in R^\bullet(w_0)$ we have $(\mathrm{cl}_X(\mathbb{G})\setminus\mathbb{G})\cup\mathbb{I}=\mathrm{cl}_X(\mathbb{G}')\setminus\mathbb{G}'=\mathrm{cl}_X(\mathbb{G}_w)\setminus\mathbb{G}_w$. But $\mathrm{cl}_X(\mathbb{G}_w)\setminus\mathbb{G}_w\subseteq\langle d_X\rangle\rho^{-1}(w)$, by fullness of ρ_w and the fact that w is in $\mathcal{F}^*(w)$.

We now check that ρ is a d-morphism from $\mathbb{G}$ to $\mathcal{F}$. So let $w\in W$. We require (8.4) to hold. Since $\mathbb{G}$ is partitioned as $\mathbb{I}\cup\mathbb{G}'$, it is enough to prove the two equations

$$\mathbb{I}\cap\langle d_X\rangle\rho^{-1}(w)=\mathbb{I}\cap\rho^{-1}(R^{-1}(w)), \tag{8.8}$$

$$\mathbb{G}'\cap\langle d_X\rangle\rho^{-1}(w)=\mathbb{G}'\cap\rho^{-1}(R^{-1}(w)). \tag{8.9}$$

(i) **Proof of** (8.8).
 (a) If $w \in R^\bullet(w_0)$ then $\mathbb{I} \cap \langle d_X \rangle \rho^{-1}(w) = \mathbb{I}$ by (8.7), and $\mathbb{I} \cap \rho^{-1}(R^{-1}(w)) = \mathbb{I}$ since $\mathbb{I} \subseteq \rho^{-1}(w_0)$ and $w_0 \in R^{-1}(w)$.
 (b) Suppose instead that $w \notin R^\bullet(w_0)$. As R is transitive, $W = R^\bullet(w_0) \cup \{w_0\}$, so $w = w_0$ and $\rho^{-1}(w) = \mathbb{I}$. Thus $\mathbb{I} \cap \langle d_X \rangle \rho^{-1}(w) = \mathbb{I} \cap \langle d_X \rangle \mathbb{I} = \emptyset$. Also, $R^{-1}(w) = \emptyset$, so $\mathbb{I} \cap \rho^{-1}(R^{-1}(w)) = \emptyset$ as well.

(ii) **Proof of** (8.9). For each $w \in R^\bullet(w_0)$, $\mathbb{G}_w$ is non-empty and open, and $\rho \restriction \mathbb{G}_w = \rho_w$ is a d-morphism from $\mathbb{G}_w$ to $\mathcal{F}$. Since $\mathbb{G}' = \bigcup_{w \in R^\bullet(w_0)} \mathbb{G}_w$, Lemma 8.1(3) tells us that $\rho \restriction \mathbb{G}'$ is a d-morphism from $\mathbb{G}'$ to $\mathcal{F}$. Now (8.9) follows by Lemma 8.1(2).

So ρ is a d-morphism from $\mathbb{G}$ onto $\mathcal{F}$. It remains to show that it is full, i.e. that $\mathrm{cl}_X(\mathbb{G}) \setminus \mathbb{G} \subseteq \langle d_X \rangle \rho^{-1}(w)$ for all $w \in W$. For $w \in R^\bullet(w_0)$, the result follows from (8.7). But for $w = w_0$, we have $\mathrm{cl}_X(\mathbb{G}) \setminus \mathbb{G} = \langle d_X \rangle \mathbb{I} = \langle d_X \rangle \rho^{-1}(w_0)$.

This completes the induction and proves the Theorem. □

Proof of Theorem 8.5

Space restrictions prevent us from giving full details of the proof, but we can outline the main construction involved. First, a refinement of the proof of Theorem 8.4 shows that any non-theorem of KD4*t* is falsifiable at *any* x_0 in X. Hence any finite subset of Γ, being KD4*t*-consistent, is satisfiable at x_0 in some model.

We can assume without loss of generality that Γ is maximally KD4*t*-consistent. As Γ is countable, we can express it as $\bigcup_{n<\omega} \Gamma_n$, where each Γ_n is finite and $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots$. Then for each $n < \omega$, Γ_n is satisfiable at x_0, so there is an assignment $g_n : \mathsf{Var} \to \wp X$ such that $(X, g_n), x_0 \models \bigwedge \Gamma_n$. Because X is zero-dimensional and the Γ_n are finite, we can choose, by induction on n, a sequence $(C_n : n < \omega)$ of clopen sets with $X = C_0 \supseteq C_1 \supseteq \cdots$ and with the following properties holding for each $n > 0$, where we write $D_n = C_n \setminus C_{n+1}$:

C1 $x_0 \in C_n \subseteq N_{1/n}(x_0)$,

C2 for each formula $\langle d \rangle \varphi \in \Gamma_n$, there is $x \in D_n$ with $(X, g_n), x \models \varphi$,

C3 for each $\neg \langle d \rangle \varphi \in \Gamma_n$, we have $(X, g_n), x \not\models \varphi$ for each $x \in C_n \setminus \{x_0\}$.

Then by C1, $\bigcap_{n<\omega} C_n = \{x_0\}$. The D_n are clopen and pairwise disjoint and partition $X \setminus \{x_0\}$, and $C_m \setminus \{x_0\} = \bigcup_{m \le n < \omega} D_n$ for each $m < \omega$.

Now an assignment $g : \mathsf{Var} \to \wp X$ can be defined by declaring $x_0 \in g(p)$ iff $p \in \Gamma$, and if $x \neq x_0$, then $x \in g(p)$ iff $x \in g_n(p)$ for the unique n such that $x \in D_n$. Since each D_n is open, an induction on φ (cf. Theorem 3.4) shows that for each formula φ and each $x \in D_n$ we have

$$(X, g), x \models \varphi \text{ iff } (X, g_n), x \models \varphi.$$

Using this, a further inductive argument involving C2 and C3 shows that for each formula φ we have $\varphi \in \Gamma$ iff $(X, g), x_0 \models \varphi$. Hence Γ is satisfiable in X at x_0 as required.

By contrast, strong completeness fails for the Kripke frame semantics. If

$$\Sigma = \{p_0, q, [d]^*(p_{2n} \to \langle d\rangle(p_{2n+1} \wedge \neg q)), [d]^*(p_{2n+1} \to \langle d\rangle(p_{2n+2} \wedge q)) : n < \omega\},$$

then the set $\Sigma \cup \{\neg\langle dt\rangle\{q, \neg q\}\}$, discussed in [6, Section 4.4], is KD4t-consistent because each of its finite subsets is satisfiable in a KD4-frame. But $\Sigma \cup \{\neg\langle dt\rangle\{q, \neg q\}\}$ is not itself satisfiable at any point of a transitive frame. For if Σ is satisfied at some such point w, then there is an endless R-path in the frame starting from w along which q and $\neg q$ are each satisfied infinitely often in the model in question, ensuring that $\neg\langle dt\rangle\{q, \neg q\}$ is false at w.

Strong completeness for topological semantics does not hold in general if the language is enriched by the universal modality. In [6, Section 10.5] it is shown that there exists a countable set of $\mathcal{L}_{[d]\forall}$-formulas that is finitely satisfiable, but not satisfiable, in any dense-in-itself metric space that is compact and locally connected.

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