On the Priorean temporal logic with 'around now' over the real line

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Abstract

We consider the temporal language with the Priorean operators G and H expressing that a formula is true at all future times and all past times, plus an operator $\Box$ expressing that a formula is true throughout some open interval containing the evaluation time (i.e., it is true 'around now'). We show that the logic of time based on the real numbers in this language is finitely axiomatisable, answering an implicit question of Shehtman (1993). We also show that the logic has PSPACE-complete complexity, but is not Kripke complete and has no strongly complete axiomatisation.

Keywords Weak completeness, finite axiomatisation, filtration, lexicographic sum, Kripke-incompleteness.

1 Introduction

Modal formulas can be given semantics in models based on topological spaces. In a topological model, the formula $\Box\varphi$ is true at a point if φ is true throughout some open neighbourhood of that point. So the set of points satisfying $\Box\varphi$ is the interior of the set of points satisfying φ. Topological semantics predates Kripke semantics and was first considered by McKinsey and Tarski [20], who proved that the logic of any separable dense-in-itself metric space, such as the rationals ($\mathbb{Q}$) and reals ($\mathbb{R}$) with the usual metric, is S4. Interest in this theorem is undergoing a renaissance and several new proofs have recently appeared [21, 22, 2, 1, 17, 10], either for $\mathbb{R}$ alone or for the general case. The assumption of separability was removed in [24]. The theorem was extended by Kremer [13, 12] to a strong completeness result for any dense-in-itself metric space (for countable languages).

Additional connectives have also been considered. Shehtman added the universal modality $\forall$: a formula $\forall\varphi$ is true at an arbitrary point of a topological model if φ is true at every point. He showed in [29] that the logic of any connected separable

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dense-in-itself metric space, such as $\mathbb{R}$, is S4UC, with S4 axioms for $\Box$, the usual axioms U for $\forall$, and a connectedness axiom C: $\forall(\Box p \vee \Box\neg p) \to \forall p \vee \forall\neg p$.

Kudinov added the difference operator $[\neq]$: a formula $[\neq]\varphi$ is true at a point if φ is true at every *other* point. The difference operator is more expressive than $\forall$. In the language with $\Box, [\neq]$, Kudinov axiomatised the logic of all topological spaces, all dense-in-themselves topological spaces, and any zero-dimensional dense-in-itself metric space [14]. He also axiomatised the logic of $\mathbb{R}^n$ for $n \geq 2$ (unpublished), but proved [15] that the logic of $\mathbb{R}$ is not finitely axiomatisable, and not even axiomatisable by formulas using finitely many variables in total.

In [28], Shehtman shifted attention to temporal logic by adding the Priorean temporal connectives G and H to the original $\Box$. This language is given semantics in *ordered topological models*. An ordered topological model is a topological model whose topology is the interval topology arising from an irreflexive linear order $<$ on the set of points. Examples include models based on $\mathbb{Q}$ and $\mathbb{R}$ with their usual orderings and topologies. Such models can be viewed temporally. We can regard the points as times and the order $<$ as the earlier-later relation, so that $x < y$ denotes that x is in the past of y and y in the future of x. A formula $G\varphi$ is true at a point or time x in such a model iff φ is true at all future times — all y satisfying $x < y$. A formula $H\varphi$ is true at x if φ is true at all past times $y < x$. Together, the connectives G, H are even more expressive than $[\neq]$. In the temporal context, $\Box\varphi$ can still be read topologically, but it also has a reasonable temporal reading as 'φ is true around now', and this view was promulgated by Scott. In [28], Shehtman gave a finite axiomatisation of the logic of $\mathbb{Q}$ in this language, observed that the logic of $\mathbb{R}$ in the same language is decidable and hence recursively axiomatisable, and implicitly posed [28, p.256] the problem of axiomatising it explicitly. Although the area of topological semantics of modal logic has recently attracted a good deal of attention, this problem has remained open.

Although it has no topological $\Box$-modality, the very expressive temporal language with U and S (Until and Since) is worth mentioning here. A formula $U(\varphi, \psi)$ is true at a time point x if there is a point $y > x$ at which φ is true and such that ψ is true at every z with $x < z < y$ — informally, ψ is true until φ becomes true. The meaning of S is obtained by swapping $<$ with $>$. The connectives U and S were introduced by Kamp [11] and they can easily express all the connectives we have considered so far. Indeed, over $\mathbb{R}$, they can express every connective whose meaning is definable in first-order logic [11]. Reynolds gave a finite axiomatisation of the logic of $\mathbb{R}$ with U, S in [25], and showed the logic to be PSPACE-complete in [26].

In the current paper we consider Shehtman's temporal language with G, H, and $\Box$, interpreted over $\mathbb{R}$. We answer Shehtman's implicit question [28] by showing that the logic of $\mathbb{R}$ in this language is finitely axiomatisable. Given Kudinov's result, this is perhaps surprising, but given Reynolds's, it is less so. It suggests that G, H, and $\Box$ are in some sense closer to Until and Since over $\mathbb{R}$ than to $[\neq]$ and $\Box$. We obtain only 'weak completeness', and we show that no strong completeness result can be proven. We also show that the logic is not Kripke compete. As we said, Shehtman observed in [28] that it is decidable, and we show here that it is PSPACE-complete.

Our axiom system is similar to the one for $\mathbb{Q}$ given by Shehtman in [28] — the only difference is that we include an additional connectedness axiom $F(p \wedge Fq) \wedge F(\neg p \wedge Fq) \to F(\Diamond p \wedge \Diamond\neg p \wedge Fq)$, where $F\varphi$ abbreviates $\neg G \neg\varphi$. Our completeness proof starts in the same way as well, by a certain filtration of the canonical model. We then apply selective filtration and a closure technique designed to give a well behaved finite Kripke model, which we employ as a template to construct a model over $\mathbb{R}$, using lexicographic sums.

Layout of paper. Section 2 contains the basic definitions, and section 3 the system of axioms and inference rules for the logic of $\mathbb{R}$ in the language with G, H, and $\Box$. In section 4 we prove that the logic has no strong axiomatisation and is not Kripke complete, but is (decidable and) PSPACE-complete (decidability was known to Shehtman). Section 5 outlines the coming completeness proof. Section 6 builds the well behaved finite Kripke model referred to above, and section 7 constructs from it a model over $\mathbb{R}$. We conclude in section 8 with some open problems.

Throughout, we use $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ to denote the ordered sets of natural numbers, integers, rationals, and real numbers (respectively).

2 Generalities

Here, we lay down the syntax and semantics of our logic, and define some basic terms.

2.1 Syntax

Let PV be a fixed countably infinite set of propositional atoms. We write $p, q, r, \ldots$ for atoms. We define the language $\mathcal{L}$ to consist of the following formulas:

1. $\top$ is a formula.
2. Every $p \in PV$ is a formula.
3. If φ, ψ are formulas then so are $\neg\varphi$, $\varphi \wedge \psi$, $G\varphi$, $H\varphi$, and $\Box\varphi$.

The *mirror image* of a formula φ is the formula obtained by replacing every G in φ by H, and every H in φ by G. As abbreviations we let $\bot = \neg\top$, $\varphi \vee \psi = \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \to \psi = \neg(\varphi \wedge \neg\psi)$, $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$, $F\varphi = \neg G \neg\varphi$, $P\varphi = \neg H \neg\varphi$, and $\Diamond\varphi = \neg\Box\neg\varphi$.

2.2 Semantics over $\mathbb{R}$

We define semantics for $\mathcal{L}$-formulas over $\mathbb{R}$ as follows. Let $h : PV \to \wp(\mathbb{R})$ be an assignment to atoms (where $\wp$ denotes power-set). The pair $(\mathbb{R}, h)$ is then called a *model over* $\mathbb{R}$. For each $x \in \mathbb{R}$ and formula φ we define $(\mathbb{R}, h), x \models \varphi$ by induction:

1. $(\mathbb{R}, h), x \models \top$,
2. $(\mathbb{R}, h), x \models p$ iff $x \in h(p)$, for $p \in PV$,
3. $(\mathbb{R}, h), x \models \neg\varphi$ iff $(\mathbb{R}, h), x \not\models \varphi$,
4. $(\mathbb{R}, h), x \models \varphi \wedge \psi$ iff $(\mathbb{R}, h), x \models \varphi$ and $(\mathbb{R}, h), x \models \psi$,

5. $(\mathbb{R}, h), x \models G\varphi$ iff $(\mathbb{R}, h), y \models \varphi$ for all $y \in \mathbb{R}$ with $y > x$,
6. $(\mathbb{R}, h), x \models H\varphi$ iff $(\mathbb{R}, h), y \models \varphi$ for all $y \in \mathbb{R}$ with $y < x$,
7. $(\mathbb{R}, h), x \models \Box\varphi$ iff there exist $y, z \in \mathbb{R}$ with $y < x < z$ and $(\mathbb{R}, h), t \models \varphi$ for all $t \in \mathbb{R}$ with $y < t < z$.

A model $(\mathbb{R}, h)$ over $\mathbb{R}$ is said to *satisfy* a formula φ if there is some $x \in \mathbb{R}$ with $(\mathbb{R}, h), x \models \varphi$. We say that φ is *satisfiable over* $\mathbb{R}$ if some model over $\mathbb{R}$ satisfies it, and φ is *valid over* $\mathbb{R}$ if $\neg\varphi$ is not satisfiable over $\mathbb{R}$. A set Σ of $\mathcal{L}$-formulas is said to be *satisfiable over* $\mathbb{R}$ if there exist an assignment $h : PV \to \wp(\mathbb{R})$ and $x \in \mathbb{R}$ with $(\mathbb{R}, h), x \models \varphi$ for every $\varphi \in \Sigma$.

The *$\mathcal{L}$-logic of* $\mathbb{R}$ is the set of all $\mathcal{L}$-formulas that are valid over $\mathbb{R}$.

2.3 Kripke semantics

Formulas have an alternative Kripke semantics. A *binary relation* on a set W is a subset R of $W \times W$. For $w, u \in W$, we may write any of Rwu, $w\,R\,u$, $R(w, u)$ to indicate that $(w, u) \in R$.

A *Kripke frame* (for $\mathcal{L}$) is a triple $(W, \sqsubset, R)$, where W is a non-empty set and $\sqsubset, R$ are binary relations on W, associated with $G, \Box$, respectively. Occasionally we consider frames of the form (W, R) as well.

The choice of the symbol $\sqsubset$ for the 'temporal' relation may be controversial, so we will spend a little space justifying it. After all, in [28], Shehtman used the symbol S. We find $\sqsubset$ convenient because it gives rise to readily understood symbols $\sqsupset, \sqsubseteq, \sqsupseteq$ for various derived relations, and also because the temporal relations in the main proof will always be transitive, a property that is suggested by the symbol $\sqsubset$. However, we stress at the outset that *in spite of what the notation may suggest, $\sqsubset$ will not necessarily be irreflexive. That is, we may have $w \sqsubset w$ for some elements $w \in W$.* The reader needs to guard against this possibly misleading aspect of the symbols $\sqsubset, \sqsupset$ throughout. The symbol $<$ suggests irreflexivity even more strongly than $\sqsubset$, so we avoid it.

Given an assignment $h : PV \to \wp(W)$, the tuple $\mathcal{M} = (W, \sqsubset, R, h)$ is called a *Kripke model* (for $\mathcal{L}$). For $w \in W$, we define $\mathcal{M}, w \models \varphi$ by induction on formulas φ:

1. $\mathcal{M}, w \models \top$,
2. $\mathcal{M}, w \models p$ iff $w \in h(p)$, for $p \in PV$,
3. $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$,
4. $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$,
5. $\mathcal{M}, w \models G\varphi$ iff $\mathcal{M}, u \models \varphi$ for all $u \in W$ with $w \sqsubset u$,
6. $\mathcal{M}, w \models H\varphi$ iff $\mathcal{M}, u \models \varphi$ for all $u \in W$ with $u \sqsubset w$,
7. $\mathcal{M}, w \models \Box\varphi$ iff $\mathcal{M}, u \models \varphi$ for all $u \in W$ with Rwu.

Let $\mathcal{F} = (W, \sqsubset, R)$ be a Kripke frame, and $\mathcal{M} = (W, \sqsubset, R, h)$ a Kripke model. We say that a formula φ is *satisfiable (or satisfied) in $\mathcal{M}$,* and that $\mathcal{M}$ *satisfies φ,* if there is $w \in W$ with $\mathcal{M}, w \models \varphi$, and *satisfiable in $\mathcal{F}$* if there are $h : PV \to \wp(W)$ and $w \in W$ with $(W, \sqsubset, R, h), w \models \varphi$. A formula φ is said to be *valid in $\mathcal{F}$* (respectively, $\mathcal{M}$) if $\neg\varphi$ is not satisfiable in $\mathcal{F}$ (respectively, $\mathcal{M}$).

2.4 General definitions

For a map $f : X \to Y$, and $X' \subseteq X$, we write $f(X')$ for $\{f(x) : x \in X'\}$. We write $\operatorname{dom} f$ for X and $\operatorname{rng} f$ for $f(X)$.

DEFINITION 2.1. Let W be a set, and R a binary relation on it.

1. For $w, u \in W$, we write any of Rwu, $w\,R\,u$, $R(w, u)$ to indicate that $(w, u) \in R$.
2. We let $R^\bullet$ denote the binary relation on W defined by $R^\bullet wu$ iff $Rwu \wedge \neg Ruw$.
3. For $w \in W$ we write $R(w) = \{u \in W : Rwu\}$.
4. A subset $X \subseteq W$ is said to be *R-generated* if $R(x) \subseteq X$ for every $x \in X$.
5. For $X \subseteq W$, we write $R \restriction X$ for the binary relation $R \cap (X \times X)$ on X.
6. We say that R is *prelinear* if for all $x, y \in W$ we have $Rxy \vee x = y \vee Ryx$. (Note that more than one disjunct may hold. The term *connex* is also used in the literature, but here R will usually be transitive, in which case 'prelinear' seems more evocative.)

DEFINITION 2.2. Let $\mathcal{M} = (W, \sqsubset, R, h)$ be a Kripke model.

1. Let $u, w \in W$. We write $w \sqsubseteq u$ to abbreviate $w \sqsubset u \vee w = u$, and $w \sqsupset\sqsubset u$ to abbreviate $w \sqsubset u \wedge u \sqsubset w$. We take $u \sqsupset w$ as synonymous for $w \sqsubset u$, and similarly for $\sqsupseteq$.
2. An element $w \in W$ is said to be *$\sqsubset$-reflexive* if $w \sqsubset w$, and *$\sqsubset$-irreflexive,* otherwise.
3. A *$\sqsubset$-cluster* of/in $\mathcal{M}$ is a $\subseteq$-maximal non-empty subset $C \subseteq W$ such that $w \sqsubset u$ for all $w, u \in C$. This usage of 'cluster' is slightly different from that in (e.g.) [28]. Plainly, every member of a cluster is $\sqsubset$-reflexive.
4. Let $\mathcal{N} = (W', \sqsubset', R', h')$ be a Kripke model. We say that $\mathcal{N}$ is a *submodel of $\mathcal{M}$,* and write $\mathcal{N} \subseteq \mathcal{M}$, if $W' \subseteq W$, $\sqsubset' = \sqsubset \restriction W'$, $R' = R \restriction W'$, and $h'(p) = W' \cap h(p)$ for every atom $p \in PV$.
5. Let $\mathcal{N} \subseteq \mathcal{M}$ be as above. We say that $\mathcal{N}$ is an *R-generated submodel (of $\mathcal{M}$)* if W' is an R-generated subset of W, a *$\sqsubset$-generated submodel* if W' is a $\sqsubset$-generated subset of W, and a *$\sqsupset$-generated submodel* if W' is a $\sqsupset$-generated subset of W.
6. We say that $\mathcal{N}$ is a *generated submodel of* $\mathcal{M}$ if it is an R-generated, a $\sqsubset$-generated, and a $\sqsupset$-generated submodel of $\mathcal{M}$.

LEMMA 2.3. *Let $\mathcal{M}, \mathcal{N}$ be Kripke models as above, and suppose that $\mathcal{N}$ is a generated submodel of $\mathcal{M}$.*

1. *If $\sqsubset$ is transitive, then any $\sqsubset$-reflexive point $w \in W$ lies in a unique $\sqsubset$-cluster, namely, $\{u \in W : w \sqsupset\sqsubset u\}$.*
2. *$\mathcal{N}, w \models \varphi$ iff $\mathcal{M}, w \models \varphi$ for every $\mathcal{L}$-formula φ and every $w \in W'$.*
3. *Every $\mathcal{L}$-formula that is valid in the frame $(W, \sqsubset, R)$ of $\mathcal{M}$ is valid in the frame $(W', \sqsubset', R')$ of $\mathcal{N}$.*

Proof. We leave the proof of part 1 as an exercise. Part 2 is well known and easily proved by induction on φ. Part 3 follows from part 2. $\square$

2.5 Linear orders

A *linear order* is a structure $(I, <)$, where I is a non-empty set and $<$ is an irreflexive transitive binary relation on I that is prelinear according to definition 2.1(6): $(I, <) \models \forall xy(x < y \vee x = y \vee x > y)$. Since $<$ is irreflexive and transitive, exactly one disjunct holds, and we say in this case that $<$ is *linear.* See, e.g., [27] for information about linear orders. We often write $(I, <)$ simply as I. As usual, we let $x \leq y$ abbreviate $x < y \vee x = y$. A subset $U \subseteq I$ is *unbounded (in I)* if for all $x \in I$ there are $y, z \in U$ with $y < x < z$. An *interval of I* is a non-empty convex subset of I. We will often regard an interval as a linear order in its own right, under the ordering induced from I. We use standard notation for intervals specifiable by endpoints: if $x, y \in I$ and $x \leq y$ then $(x, y) = \{z \in I : x < z < y\}$, $[x, y) = \{z \in I : x \leq z < y\}$, $[x, y] = \{z \in I : x \leq z \leq y\}$, $(-\infty, x) = \{z \in I : z < x\}$, $[x, \infty) = \{z \in I : z \geq x\}$, etc. An *open interval* is one containing no least or greatest element.

3 Axioms

We now present a Hilbert system that, as we will show, axiomatises the $\mathcal{L}$-logic of $\mathbb{R}$. It is based on a system of Shehtman [28, §2] that was shown to axiomatise the $\mathcal{L}$-logic of $\mathbb{Q}$. The only difference is that we have added a 'connectedness' axiom, axiom 5.

3.1 The system

The axioms are as follows. We assume familiarity with Sahlqvist formulas in temporal logic: see, e.g., [3]. The axioms 2–4 are Sahlqvist formulas and their first-order correspondents are reproduced below. (The normality axioms can be turned into Sahlqvist formulas by replacing q by $\neg q$; their correspondents are equivalent to $\top$, and are omitted.) Each correspondent is true in a Kripke frame iff the axiom is valid in the frame. Moreover, the correspondents are true in the frame of the canonical model of the logic axiomatised by the system.

1. all propositional tautologies
2. axioms for dense linear time without endpoints:

$G(p \to q) \to (Gp \to Gq)$	normality
$Gp \to GGp$	transitivity: $\forall xyz(x \sqsubset y \wedge y \sqsubset z \to x \sqsubset z)$
$p \to GPp$	$\forall xy(x \sqsubset y \to y \sqsupset x)$
$GGp \to Gp$	density: $\forall xy(x \sqsubset y \to \exists z(x \sqsubset z \wedge z \sqsubset y))$
$FPp \to p \vee Fp \vee Pp$	$\forall xyz(x \sqsubset y \wedge y \sqsupset z \to x = z \vee x \sqsubset z \vee x \sqsupset z)$

3. S4 axioms for $\Box$:

$\Box(p \to q) \to (\Box p \to \Box q)$	normality
$\Box p \to p$	reflexivity: $\forall x R(x, x)$
$\Box p \to \Box\Box p$	transitivity: $\forall xyz(R(x, y) \wedge R(y, z) \to R(x, z))$

4. Shehtman's 'special axioms':

(a) $Hp \wedge p \wedge Gp \to \Box p$ $\forall xy(Rxy \to x \sqsupset y \vee x = y \vee x \sqsubset y)$
(b) $Gp \to G\Box p$ $\forall xyz(x \sqsubset y \wedge Ryz \to x \sqsubset z)$
(c) $Gp \wedge \Box p \to \Box Gp$ $\forall xyz(Rxy \wedge y \sqsubset z \to x \sqsubset z \vee Rxz)$
(d) $\Box p \to Fp$ $\forall x \exists y(x \sqsubset y \wedge Rxy)$

5. $F(p \wedge Fq) \wedge F(\neg p \wedge Fq) \to F(\Diamond p \wedge \Diamond \neg p \wedge Fq)$ (connectedness)
6. all mirror images of the above axioms (swap G with H, and F with P; also swap $\sqsubset$ with $\sqsupset$ in the correspondents).

The rules of inference are the standard ones:

1. modus ponens: $\dfrac{\varphi, \ \varphi \to \psi}{\psi}$
2. generalisation: $\dfrac{\varphi}{G\varphi}, \quad \dfrac{\varphi}{H\varphi}, \quad \dfrac{\varphi}{\Box\varphi}$
3. substitution: $\dfrac{\varphi(p)}{\varphi(\psi/p)}$

Some mirror images, such as $Hp \to HHp$, are redundant and can be omitted. We have not investigated the exact extent to which this can be done.

As usual, the *logic axiomatised by this system* is the smallest set L of $\mathcal{L}$-formulas that contains all the axioms listed above and is closed under the rules of inference. We say that an $\mathcal{L}$-formula φ is *provable in the system* if $\varphi \in \mathsf{L}$, and *consistent* if $\neg\varphi \notin \mathsf{L}$. A set Γ of $\mathcal{L}$-formulas is *consistent* if $\gamma_0 \wedge \ldots \wedge \gamma_n$ is consistent for every $n \in \mathbb{N}$ and $\gamma_0, \ldots, \gamma_n \in \Gamma$, and *maximal consistent* if it is consistent but has no proper consistent extension.

We aim to show that L is the $\mathcal{L}$-logic of $\mathbb{R}$. The inclusion '$\subseteq$' (soundness) is straightforward:

THEOREM 3.1. *The system is sound over $\mathbb{R}$.*

Proof (sketch). All axioms other than axiom 5 are shown to be valid over any dense flow of time without endpoints in [28, lemma 2.2(2)]. Axiom 5 is valid over $\mathbb{R}$ because every interval of $\mathbb{R}$ is connected. Indeed, assume that for some model $(\mathbb{R}, h)$ and $t \in \mathbb{R}$ we have

$$(\mathbb{R}, h), t \models F(p \wedge Fq) \wedge F(\neg p \wedge Fq).$$

Let $v_1, v_2 > t$ satisfy $(\mathbb{R}, h), v_1 \models p \wedge Fq$ and $(\mathbb{R}, h), v_2 \models \neg p \wedge Fq$. We can find $u > \max(v_1, v_2)$ with $u \in h(q)$. Assume wlog. that $v_1 < v_2$. Let

$$s = \sup\{x \in \mathbb{R} : \forall y(v_1 \leq y < x \to y \in h(p))\}.$$

Then $t < v_1 \leq s \leq v_2 < u$. Hence, $(\mathbb{R}, h), s \models Fq$. By definition of s, we have $(\mathbb{R}, h), s \models \Diamond p \wedge \Diamond \neg p$. We deduce that $(\mathbb{R}, h), t \models F(\Diamond p \wedge \Diamond \neg p \wedge Fq)$ as required.

The inference rules obviously preserve validity. $\square$

3.2 Simple theorems of the system

The first three lemmas do not use the connectedness axiom 5 or its mirror image.

LEMMA 3.2. *$F\top$ and $P\top$ are provable in the system.*

Proof. As $\top$ is a tautology, it is provable, and we get $\Box\top$ by $\Box$-generalisation. By axiom 4d, we prove $F\top$. We prove $P\top$ similarly. $\Box$

LEMMA 3.3. *$G\neg p \wedge HFp \to \Diamond p$ is provable in the system.*

Proof. We can prove $G\neg p \wedge \Box\neg p \to \Box G\neg p$ by axiom 4c. By the mirror image of axiom 4d we have $\Box G\neg p \to PG\neg p$. Using propositional tautologies we deduce $G\neg p \wedge \Box\neg p \to \neg HFp$, and then the result. $\Box$

The connectedness axiom (5) has an important consequence: the well known Prior axiom

$$Fq \wedge FG\neg q \to F(G\neg q \wedge HFq). \tag{3.1}$$

We will prove this using the following lemma.

LEMMA 3.4. *$\Diamond Gq \to Gq$ and $\Diamond Hq \to Hq$ are provable in the system.*

Proof. The following are provable:

1.	$\Diamond Gq \to GP\Diamond Gq$	by axiom $p \to GPp$
2.	$P\Diamond Gq \to PGq$	Gq-instance of contrapositive of axiom 4b ($Hp \to H\Box p$)
3.	$GP\Diamond Gq \to GPGq$	from previous by G-generalisation and normality
4.	$PGq \to q$	contrapositive of temporal axiom $p \to HFp$
5.	$GPGq \to Gq$	from previous by G-generalisation and normality

The result now follows from lines 1, 3, and 5 by propositional tautologies. The proof of $\Diamond Hq \to Hq$ is a mirror image. $\Box$

Now we will use the connectedness axiom for the first time.

COROLLARY 3.5. *The Prior axiom $Fq \wedge FG\neg q \to F(G\neg q \wedge HFq)$ and its mirror image are provable in the system.*

Proof. We give a more informal proof along the lines of the preceding lemma. Assume $Fq \wedge FG\neg q$. Using the density axiom, this yields $FFq \wedge FG\neg q$. Taking $p = G\neg q$ and $q = \top$ in the connectedness axiom (axiom 5) gives $F(G\neg q \wedge F\top) \wedge F(\neg G\neg q \wedge F\top) \to F(\Diamond G\neg q \wedge \Diamond\neg G\neg q \wedge F\top)$. By lemma 3.2, $F\top$ is equivalent to $\top$, so this reduces to

$$FG\neg q \wedge FFq \to F(\Diamond G\neg q \wedge \Diamond Fq).$$

So we obtain $F(\Diamond G\neg q \wedge \Diamond Fq)$. Now by standard temporal logic, we can prove $Fq \to HF(Fq)$ and $HF(Fq) \to HFq$. This gives us $F(\Diamond G\neg q \wedge \Diamond HFq)$. By lemma 3.4, we obtain $F(G\neg q \wedge HFq)$ as required. The mirror image can be derived similarly. $\Box$

REMARK 3.6. In connection with the consequent of the Prior axiom, we point out that if $\mathcal{M}$ is a Kripke model, $w \in \mathcal{M}$, φ an $\mathcal{L}$-formula, and $\mathcal{M}, w \models G\neg\varphi \wedge HF\varphi$, then w is $\sqsubset$-irreflexive. For if $w \sqsubset w$ then as $\mathcal{M}, w \models HF\varphi$ we have $\mathcal{M}, w \models F\varphi$, contradicting $\mathcal{M}, w \models G\neg\varphi$.

4 Some facts about the $\mathcal{L}$-logic of $\mathbb{R}$

Here we prove some fairly straightforward results about the $\mathcal{L}$-logic of $\mathbb{R}$.

4.1 Complexity

We will see below that the $\mathcal{L}$-logic of $\mathbb{R}$ does not have the finite model property. Since this property is often used to show decidability, it may be surprising that the logic is decidable [28, p. 256]. We begin by establishing its complexity.

THEOREM 4.1. *The problem of deciding whether an $\mathcal{L}$-formula is valid over $\mathbb{R}$ is PSPACE-complete.*

Proof (sketch). We assume knowledge of temporal logic with Until and Since (U, S) as in [26], where it is proved that the problem of determining satisfiability over $\mathbb{R}$ of a formula written with U, S is PSPACE-complete. Given an $\mathcal{L}$-formula φ, introduce a new propositional atom q_ψ for each subformula ψ of φ, and define the formula $\widehat{\psi}$ as follows, where $\forall\psi$ abbreviates $\psi \wedge \neg U(\neg\psi, \top) \wedge \neg S(\neg\psi, \top)$:

- $\widehat{\top} = \forall q_\top$
- $\widehat{p} = \forall(p \leftrightarrow q_p)$ for $p \in PV$
- $\widehat{\neg\psi} = \forall(q_{\neg\psi} \leftrightarrow \neg q_\psi)$
- $\widehat{\psi \wedge \chi} = \forall(q_{\psi\wedge\chi} \leftrightarrow q_\psi \wedge q_\chi)$
- $\widehat{F\psi} = \forall(q_{F\psi} \leftrightarrow U(q_\psi, \top))$
- $\widehat{P\psi} = \forall(q_{P\psi} \leftrightarrow S(q_\psi, \top))$
- $\widehat{\Box\psi} = \forall(q_{\Box\psi} \leftrightarrow q_\psi \wedge U(\top, q_\psi) \wedge S(\top, q_\psi))$.

Let φ^* be the conjunction of all $\widehat{\psi}$ for subformulas ψ of φ, together with q_φ. It can be checked that φ is satisfiable over $\mathbb{R}$ iff φ^* is, and that φ^* can be constructed from φ in polynomial time. Given φ, we may construct φ^* and then decide its satisfiability over $\mathbb{R}$ in PSPACE [26]. The combined procedure can be done in polynomial space, so the satisfiability, and hence the validity, of φ can be decided in PSPACE. The logic of $\mathbb{R}$ with $\Box$ alone is S4 [20], which is already PSPACE-hard [16]. $\Box$

4.2 Strong completeness

A Hilbert system (of axioms and rules) is said to be *sound* if all satisfiable formulas are consistent, and *strongly complete* if any consistent set of formulas using in all only countably many atoms is satisfiable.

THEOREM 4.2. *There is no sound and strongly complete Hilbert system for the $\mathcal{L}$-logic of $\mathbb{R}$.*

Proof. Let Σ be the following set of formulas written with atoms p, q, r:

$$\begin{array}{l} \Box p \\ F(r \wedge G\neg r) \\ G(r \vee Fr \to \Box p \vee \Box q) \\ F(\neg p \wedge F(\neg q \wedge F(\neg p \wedge F(\ \cdots \wedge Fr \underbrace{)) \cdots)}_{n \text{ brackets}} \text{ for each integer } n \geq 1 \end{array}$$

It is easy to see that any finite subset of Σ is satisfiable over $\mathbb{R}$. However, Σ itself is not satisfiable over $\mathbb{R}$. For suppose that Σ is satisfied at 0 and $r \wedge G\neg r$ is true at 1, say. By the first and third formulas, each $x \in [0,1]$ belongs to some open interval $I_x \subseteq \mathbb{R}$ with $I_x \subseteq h(p)$ or $I_x \subseteq h(q)$. By the Heine–Borel theorem, $[0,1]$ is compact, so there are $n \in \mathbb{N}$ and $x_0 < \cdots < x_{n-1}$ in $[0,1]$ such that $[0,1] \subseteq \bigcup_{i<n} I_{x_i}$. By the final set of formulas, there are $0 < y_0 < y_1 < \cdots < y_n < 1$ with $y_j \notin h(p)$ if j is even and $y_j \notin h(q)$ if j is odd (each $j \leq n$). Now by the pigeonhole principle and convexity of the I_x, there are $i, j < n$ with $y_j, y_{j+1} \in I_{x_i}$. But $I_{x_i} \subseteq h(p)$ or $I_{x_i} \subseteq h(q)$, a contradiction.

If the $\mathcal{L}$-logic of $\mathbb{R}$ had a sound and strongly complete Hilbert system, then since Σ is finitely satisfiable, it would be consistent and so satisfiable, contradicting the above. □

4.3 Kripke completeness

Finally, we consider Kripke completeness. I would like to thank Nick Bezhanishvili for helpful discussions on this material. First, a minor lemma, used only here.

LEMMA 4.3. *Let $\mathcal{F} = (W, \sqsubset, R)$ be a Kripke frame that validates the axioms of §3.1. Suppose that $w, u, x \in W$ satisfy $w \sqsubset u$, $w \sqsubset x$, and $\neg(x \sqsubset u)$. Then there is $y \in W$ satisfying $w \sqsubset y$, $\neg(y \sqsubset u)$, and Ryu.*

Proof. Let $w, u, x \in W$ be as stated. Let $h : PV \to W$ be an assignment with $h(p) = \{u\}$, and let $\mathcal{N} = (W, \sqsubset, R, h)$. Since $\neg(x \sqsubset u)$, we have $\mathcal{N}, x \models G\neg p$. As $w \sqsubset u, x$, this gives $\mathcal{N}, w \models Fp \wedge FG\neg p$. As all axioms are valid in $\mathcal{F}$ and frame validity is preserved by the inference rules, by corollary 3.5 the Prior axiom (3.1) is valid in $\mathcal{F}$, so $\mathcal{N}, w \models F(G\neg p \wedge HFp)$. Hence there is $y \in W$ with $w \sqsubset y$ and

$$\mathcal{N}, y \models G\neg p \wedge HFp. \tag{4.1}$$

Hence, $\neg(y \sqsubset u)$. By (4.1) and lemma 3.3, $\mathcal{N}, y \models \Diamond p$. So Ryu. □

We will now consider the following formula θ, where a, b are atoms:

$$\theta = H\neg a \wedge H\neg b \wedge \neg a \wedge \neg b \wedge \Diamond a \wedge \Diamond b \wedge G\neg(\Diamond a \wedge \Diamond b) \wedge FG\neg a. \tag{4.2}$$

LEMMA 4.4. *The formula θ is satisfiable over $\mathbb{R}$, but is not satisfiable in any Kripke model whose frame validates the axioms of §3.1.*

Proof. Let $h : PV \to \wp(\mathbb{R})$ be an assignment satisfying $h(a) = \{1/2^n : n \in \mathbb{N}\}$ and $h(b) = \{2/3^n : n \in \mathbb{N}\}$. Evidently, $(\mathbb{R}, h), 0 \models \theta$. So θ is satisfiable over $\mathbb{R}$.

Let $\mathcal{M} = (W, \sqsubset, R, h)$ be a Kripke model whose frame $\mathcal{F} = (W, \sqsubset, R)$ validates the axioms of §3.1. Let $w \in W$ and assume for contradiction that $\mathcal{M}, w \models \theta$.

Recall from definition 2.1 that $R(w) = \{u \in W : Rwu\}$. Plainly, $\mathcal{M}, w \models \Diamond a \wedge \Diamond b$, so there are $u, v \in R(w)$ with $\mathcal{M}, u \models a$ and $\mathcal{M}, v \models b$. Now the correspondents of the axioms in §3.1, where given, are all true in $\mathcal{F}$. By (the correspondent of) axiom 4a, $u \sqsupset w \vee u \sqsubseteq w$. As $\mathcal{M}, w \models H\neg a \wedge \neg a$, we cannot have $u \sqsubseteq w$. So $u \sqsupset w$. Since also Rwv, by axiom 4b we obtain $u \sqsupset v$. Similarly, $v \sqsupset u$. By the S4 axioms, $\sqsubset$ is transitive, so $\{u, v\}$ is contained in a $\sqsubset$-cluster.

As $\mathcal{M}, w \models FG\neg a$, we can choose $x \in W$ with $w \sqsubset x$ and $\mathcal{M}, x \models G\neg a$. Since $\mathcal{M}, u \models a$, we have $\neg(x \sqsubset u)$. So by lemma 4.3, there is $y \in W$ with $w \sqsubset y$, $\neg(y \sqsubset u)$, and Ryu. But also, $u \sqsubset v$, so by axiom 4c we obtain $Ryv \vee y \sqsubset v$.

If Ryv, then both $u, v \in R(y)$, so $\mathcal{M}, y \models \Diamond a \wedge \Diamond b$. Since $w \sqsubset y$, we obtain $\mathcal{M}, w \models F(\Diamond a \wedge \Diamond b)$, contradicting that $\mathcal{M}, w \models \theta$. If instead $y \sqsubset v$, then since $v \sqsubset u$ we have $y \sqsubset u$, contradicting that $\neg(y \sqsubset u)$. Either way, our assumption that $\mathcal{M}, w \models \theta$ has led to a contradiction. □

Readers wondering whether θ could be simplified to a formula with only one atom should bear in mind that $H\neg a \wedge \neg a \wedge \Diamond a \wedge G\neg\Diamond a$ is not satisfiable over $\mathbb{R}$, and its negation is provable (the proof uses axiom 4a and the S4 reflexivity axiom for $\Box$). Separately, lemma 4.4 fails if the final conjunct $FG\neg a$ is omitted from θ. We will return to this example in §5 and §7.6.

Recall that a modal logic L is said to be *Kripke complete* (respectively, to have the *finite model property*) if there exists a class $\mathcal{K}$ of (resp. finite) Kripke frames such that L is the set of all modal formulas that are valid in every frame in $\mathcal{K}$.

THEOREM 4.5. *The $\mathcal{L}$-logic of $\mathbb{R}$ is not Kripke complete and does not have the finite model property.*

Proof. If the $\mathcal{L}$-logic of $\mathbb{R}$ were the logic of a class $\mathcal{K}$ of Kripke frames of the form $(W, \sqsubset, R)$, then as the formula θ in (4.2) is satisfiable over $\mathbb{R}$, it would be satisfiable over a frame in $\mathcal{K}$. By lemma 4.4, such a frame could not validate the axioms of §3.1. So by theorem 3.1, it could not validate the logic of $\mathbb{R}$ either. □

As N. Bezhanishvili has observed, the $\mathcal{L}$-logic of $\mathbb{R}$ is a 'naturally occurring' example of a non-Kripke complete logic.

5 Towards completeness

The main aim of the paper is to show that the Hilbert system given in §3.1 is sound and complete for the $\mathcal{L}$-logic of $\mathbb{R}$. In this section, we take some first steps in that direction.

5.1 A problem

We begin by observing that certain naïve approaches will not succeed. For example, we might try to prove that every consistent formula is satisfiable in some finite Kripke model whose frame $(W, \sqsubset, R)$ has the following special form. Ordered by $\sqsubset$, which is transitive and prelinear, it falls into a sequence

$$C_0 \sqsubset u_0 \sqsubset C_1 \sqsubset u_1 \sqsubset \cdots \sqsubset u_{n-1} \sqsubset C_n,$$

where $n \geq 0$, the C_i are $\sqsubset$-clusters, the u_i are $\sqsubset$-irreflexive, the relation $\sqsubset$ is defined between sets and points as in definition 5.1(1), and $R(u_i) = C_i \cup \{u_i\} \cup C_{i+1}$ for each $i < n$. (Recall that $R(u_i) = \{w \in W : Ru_i w\}$.) We could also require that every cluster is connected as an R-frame — see definition 5.2 for the meaning of this. It would then be not so hard to construct a model over $\mathbb{R}$ satisfying the formula.

This idea is unlikely to work — by theorem 4.5, the logic of $\mathbb{R}$ is not characterised by any class of finite frames at all. And in fact, the formula θ of (4.2) — which is consistent since it is satisfiable over $\mathbb{R}$ — is not satisfiable in any Kripke model of this form. For, θ being true at a world w would force w to be irreflexive — say $w = u_i$ — and a and b to be true at some worlds in the cluster C_{i+1} immediately following u_i in the order $\sqsubset$. This cluster could not be $\sqsubset$-final in the model because of the conjunct $FG\neg a$ of θ. So $i + 1 < n$. But now, $\Diamond a \wedge \Diamond b$ would be true at u_{i+1}, contradicting the truth of $G\neg(\Diamond a \wedge \Diamond b)$ at u_i.

5.2 Ψ-linked models

So it seems that we are forced to work with Kripke models that may contain adjacent $\sqsubset$-clusters with no intervening irreflexive point. (Since the $\mathcal{L}$-logic of $\mathbb{R}$ is not Kripke complete, it seems that we cannot work with frames and have to use models.) We will focus our attention on 'nice' models in which any two such clusters contain 'similar' points. We will show that any consistent formula is satisfiable in a nice Kripke model, and that any such model can be transformed into a model over $\mathbb{R}$. To define 'nice', we need the following somewhat disparate preliminary definitions.

DEFINITION 5.1. Let $\mathcal{M} = (W, \sqsubset, R, h)$ be a Kripke model.

1. For $X, Y \subseteq W$, we write $X \sqsubset Y$ if $x \sqsubset y$ for every $x \in X$ and $y \in Y$. We abbreviate $\{x\} \sqsubset Y$ to $x \sqsubset Y$, etc.
2. We say that an ordered pair (C, D) of $\sqsubset$-clusters in $\mathcal{M}$ is
   - *consecutive (in $\mathcal{M}$)* if $C \neq D$ and $\{u \in W : u \sqsubset\text{-reflexive}, C \sqsubset u \sqsubset D\} = C \cup D$,
   - *adjacent (in $\mathcal{M}$)* if $C \neq D$ and $\{u \in W : C \sqsubset u \sqsubset D\} = C \cup D$.

   In each case, $C \sqsubset D$. Consecutive clusters have no $\sqsubset$-reflexive points between them, but may have $\sqsubset$-irreflexive ones. Adjacent clusters have nothing between them.

DEFINITION 5.2. A frame (W, R) is said to be *connected* if there do not exist non-empty disjoint R-generated subsets $X, Y \subseteq W$ with $W = X \cup Y$.

DEFINITION 5.3. Let Ψ be a set of $\mathcal{L}$-formulas.

1. We write $\mathcal{B}\Psi$ for the set of formulas in Ψ of the form $\Box\theta$, $G\theta$, or $H\theta$ (the 'box-formulas').
2. Let $\mathcal{M} = (W, \sqsubset, R, h)$ be a Kripke model. We define an equivalence relation $\equiv_\Psi^\mathcal{M}$ on W by: $c \equiv_\Psi^\mathcal{M} d$ iff for every $\psi \in \Psi$, we have $\mathcal{M}, c \models \psi$ iff $\mathcal{M}, d \models \psi$.

We can now give our definition of 'nice' model:

DEFINITION 5.4. Let Ψ be a set of $\mathcal{L}$-formulas. We say that a Kripke model $\mathcal{M} = (W, \sqsubset, R, h)$ is Ψ-*linked* if:

1. W is finite.
2. The frame $(W, \sqsubset, R)$ validates all axioms of §3.1 (including mirror images), except possibly the connectedness axiom 5 and its mirror image.
3. The relation $\sqsubset$ is prelinear (see definition 2.1(6)).
4. For every $\sqsubset$-cluster $C \subseteq W$, the frame $(C, R \restriction C)$ is connected.
5. For every pair (C, D) of adjacent $\sqsubset$-clusters in $\mathcal{M}$, there are $c \in C$ and $d \in D$ with $c \equiv_{\mathcal{B}\Psi}^\mathcal{M} d$.

In a Ψ-linked model $\mathcal{M}$, adjacent clusters are 'linked' by 'similar' (formally, $\equiv_{\mathcal{B}\Psi}^\mathcal{M}$-equivalent) points. Agreement of these points on formulas $\Box\theta \in \Psi$ is critical for our later work — in lemma 7.25 in particular. Their agreement on formulas $G\theta$ and $H\theta$ is more a convenience that allows a simple definition of 'nice' model. Condition 4 is weaker than saying that the frame of $\mathcal{M}$ validates the connectedness axiom 5. For example, if $\mathcal{M}$ consists of two $\sqsubset$-reflexive points c, d, with $c \sqsubset d$, $\neg Rcd$, $\neg Rdc$, $\mathcal{M}, c \models p$, and $\mathcal{M}, d \models \neg p$, then $\mathcal{M}$ is $\emptyset$-linked but does not validate the connectedness axiom.

5.3 Structure of Ψ-linked models

By definition 5.4, the frame of a Ψ-linked model validates the Shehtman axioms, and these impose considerable structure on Kripke frames validating them. The following lemma sheds a lot of light on this structure. It can be obtained from [28, lemmas 3.5–3.6], but we give a proof to make the paper more self-contained.

LEMMA 5.5 (λ, ρ-lemma)**.** *Let $\mathcal{F} = (W, \sqsubset, R)$ be a Kripke frame that validates the axioms of §3.1 except possibly the connectedness axiom 5 and its mirror image, and let $w \in W$.*

1. *Every $\sqsubset$-cluster in $\mathcal{F}$ is an R-generated subset of W.*
2. *If w is $\sqsubset$-reflexive then $R(w)$ is a subset of a $\sqsubset$-cluster.*
3. *If w is $\sqsubset$-irreflexive, then there are disjoint $\sqsubset$-clusters $\lambda(w)$, $\rho(w)$ such that $R(w) = \lambda(w) \cup \{w\} \cup \rho(w)$. For every $t \in W$ we have $t \sqsubset w$ iff $t \sqsubset \lambda(w)$, and $w \sqsubset t$ iff $\rho(w) \sqsubset t$.*

Proof. Since $\mathcal{F}$ validates all axioms in §3.1 with first-order correspondents, those correspondents are true in $\mathcal{F}$. To prove part 1 of the lemma, let $C \subseteq W$ be a $\sqsubset$-cluster and let $w \in C$. By lemma 2.3, $C = \{u \in W : u \sqsupset\sqsubset w\}$ (recall from definition 2.2 that $u \sqsupset\sqsubset w$ denotes that $u \sqsubset w \wedge w \sqsubset u$). By axiom 4b and its mirror image, C is R-generated.

For part 2, suppose that $w \sqsubset w$. By lemma 2.3, w is contained in a $\sqsubset$-cluster $C = \{u \in W : u \sqsupset\sqsubset w\}$. By part 1, C is R-generated, so $R(w) \subseteq C$.

For part 3, suppose that $\neg(w \sqsubset w)$ and let

$$\begin{array}{rcl} \lambda(w) & = & R(w) \cap \{u \in W : u \sqsubset w\}, \\ \rho(w) & = & R(w) \cap \{u \in W : w \sqsubset u\}. \end{array}$$

These sets are non-empty by axiom 4d and its mirror image, and disjoint as w is irreflexive and $\sqsubset$ transitive. By axiom 4a we have $R(w) = \lambda(w) \cup \{w\} \cup \rho(w)$.

We show that $\lambda(w)$ is a $\sqsubset$-cluster. If $t \in W$ and $u \in \lambda(w)$, then axiom 4b and $\sqsubset$-transitivity yield

$$t \sqsubset w \text{ iff } t \sqsubset u. \tag{5.1}$$

Recalling that $\lambda(w) \neq \emptyset$, pick arbitrary $u \in \lambda(w)$. Taking $t = u$ in (5.1), we see that u is $\sqsubset$-reflexive, so by lemma 2.3 again, the set $U = \{v \in W : v \sqsupset\sqsubset u\}$ is a $\sqsubset$-cluster.

We show that $\lambda(w) = U$. By (5.1), $t \sqsubset v$ for all $t, v \in \lambda(w)$, and it follows that $\lambda(w) \subseteq U$. To show that $U \subseteq \lambda(w)$, let $t \in U$ be arbitrary. Since $u \sqsubset w$, by $\sqsubset$-transitivity we have $t \sqsubset w$. Also, $Rwu \wedge u \sqsubset t$, so by axiom 4c, $w \sqsubset t \vee Rwt$. If $w \sqsubset t$, then as $t \sqsubset w$ and $\sqsubset$ is transitive, we would have $w \sqsubset w$, contradicting irreflexivity of w. So $\neg(w \sqsubset t)$, and hence Rwt. We already have $t \sqsubset w$, so $t \in \lambda(w)$. As $t \in U$ was arbitrary, $U \subseteq \lambda(w)$ as required.

The last claim in the lemma (for λ) is immediate from (5.1). The case of ρ is handled similarly. $\square$

It follows from the lemma that for $\sqsubset$-irreflexive w we have $u \sqsubset w \sqsubset v$ for every $u \in \lambda(w)$ and $v \in \rho(w)$, by taking $t = u$ and $t = v$. That is, $\lambda(w) \sqsubset w \sqsubset \rho(w)$. We can think of $\lambda(w)$ as the set of points lying infinitesimally near to w in the past, and $\rho(w)$ as the set of points lying infinitesimally near to w in the future.

We can now elucidate the structure of a Ψ-linked model $\mathcal{M} = (W, \sqsubset, R, h)$. Recall that $x \sqsupset\sqsubset y$ means $x \sqsubset y \wedge y \sqsubset x$. For $x, y \in W$ define $x \sim y$ iff $x = y \vee x \sqsupset\sqsubset y$. As $\sqsubset$ is transitive, $\sim$ is an equivalence relation on W. Each equivalence class is either a singleton consisting of an $\sqsubset$-irreflexive point, or (by lemma 2.3) a $\sqsubset$-cluster. As $\sqsubset$ is prelinear, we can enumerate the clusters without repetitions as $C_0, \ldots, C_k$, with $C_0 \sqsubset C_1 \sqsubset \cdots \sqsubset C_k$. Each remaining $\sim$-class, if any, is a singleton $\{u\}$ with u $\sqsubset$-irreflexive. The frame $\mathcal{F} = (W, \sqsubset, R)$ validates the axioms required by lemma 5.5, so for each such u the sets $\lambda(u), \rho(u)$ are defined and are $\sqsubset$-clusters. Say, $\lambda(u) = C_i$. By the lemma, $C_i \sqsubset u$, and $u \sqsubset w$ iff $\rho(u) \sqsubset w$ for each $w \in W$. It follows that $i < k$ and $\rho(u) = C_{i+1}$. If u, v are irreflexive and $u \sqsubset v$, then by the lemma, $u \sqsubset \rho(u) \sqsubset \lambda(v) \sqsubset v$ (possibly $\rho(u) = \lambda(v)$), so for each $i < k$ there is at most one irreflexive point lying between C_i and C_{i+1}.

We conclude that for some finite $k \geq 0$, the frame $(W, \sqsubset)$ is the union of distinct $\sqsubset$-clusters $C_0 \sqsubset C_1 \sqsubset \cdots \sqsubset C_k$ and $\sqsubset$-irreflexive points u_i ($i \in I$) for some $I \subseteq$

$\{0, \ldots, k-1\}$, with $C_i = \lambda(u_i) \sqsubset u_i \sqsubset \rho(u_i) = C_{i+1}$ for each $i \in I$. We never get two adjacent irreflexive points. This is as in the suggested form of frames in §5.1, except that we may get two adjacent $\sqsubset$-clusters with no intervening irreflexive point. In this case, they are linked by containing $\equiv^{\mathcal{M}}_{\mathcal{B}\Psi}$-equivalent points.

5.4 Problem resolved

We saw in §5.1 that the formula θ of (4.2) is not satisfiable in any model whose frame is as described there. Let us now show that θ is satisfiable in a linked model.

PROPOSITION 5.6. *The formula*

$$\theta = H\neg a \wedge H\neg b \wedge \neg a \wedge \neg b \wedge \Diamond a \wedge \Diamond b \wedge G\neg(\Diamond a \wedge \Diamond b) \wedge FG\neg a$$

of (4.2) *is satisfied in a* Ψ*-linked model, where* Ψ *is the set of subformulas of* θ*.*

Proof. The formula θ is true at world u_0 in the Kripke model $\mathcal{M} = (W, R, \sqsubset, h)$ shown in figure 1. In the figure, the black dots are $\sqsubset$-irreflexive and the white dots

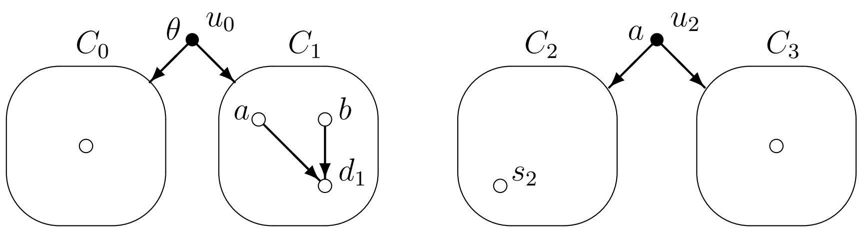

Figure 1: Kripke model $\mathcal{M}$ satisfying θ

are $\sqsubset$-reflexive. The relation $\sqsubset$ is given by the left-to-right ordering, except within each $\sqsubset$-cluster C_i $(i \leq 3)$ where of course all points are $\sqsubset$-related. The relation R is given by the reflexive closure of the arrows. The atoms a, b are true only where shown at u_2 and the two upper points in C_1.

Let Ψ be the set of subformulas of θ. Recalling that $\Diamond$ abbreviates $\neg\Box\neg$ and F abbreviates $\neg G\neg$, the set $\mathcal{B}\Psi$ of 'box-formulas' in Ψ is

$$\mathcal{B}\Psi = \{H\neg a, H\neg b, \Box\neg a, \Box\neg b, G\neg(\Diamond a \wedge \Diamond b), GFa, G\neg a\}.$$

We claim that $\mathcal{M}$ is Ψ-linked. It is finite, and its frame validates all axioms in §3.1 (including mirror images), except possibly the connectedness axiom 5 and its mirror image. Obviously, the relation $\sqsubset$ is prelinear and the frame $(C, R \restriction C)$ is connected for each $\sqsubset$-cluster C. The only adjacent $\sqsubset$-clusters in $\mathcal{M}$ are C_1 and C_2, and the lower dots d_1, s_2 in them are $\equiv^{\mathcal{M}}_{\mathcal{B}\Psi}$-equivalent — $\Box\neg a$, $\Box\neg b$, and $G\neg(\Diamond a \wedge \Diamond b)$ are true at both of them, and $H\neg a$, $H\neg b$, GFa, and $G\neg a$ are false. So $\mathcal{M}$ is indeed Ψ-linked. $\Box$

We will return to this example in §7.6.

5.5 Strategy of completeness proof

Our approach to proving completeness will now be as follows.

Step 1. We show that any formula φ_0 that is consistent with the system of §3.1 is satisfiable in a Ψ-linked Kripke model, where Ψ is the set of subformulas of $P\varphi_0$. This is done in §6.

Step 2. Given any Ψ-linked model, where Ψ is a finite set of formulas closed under subformulas, we construct a model over $\mathbb{R}$ that satisfies the same formulas from Ψ. This is done in §7.

These two steps are of roughly equal length and can be read in either order. Readers may prefer to read §7 first, as Ψ-linked models may be better motivated that way. Completeness of the system follows immediately by putting the two steps together — this will be done in theorem 8.1.

6 Consistent formulas have linked models

In this section we prove the following.

THEOREM 6.1. *Let φ_0 be an $\mathcal{L}$-formula that is consistent with the system of §3.1. Let Ψ be a finite set of $\mathcal{L}$-formulas containing $P\varphi_0$ and closed under taking subformulas. Then φ_0 is satisfiable in a Ψ-linked Kripke model.*

To prove it, we will successively construct five Kripke models $\mathcal{M}_0, \dots, \mathcal{M}_4$ satisfying φ_0 and getting closer to our goal. The first three are exactly as in Shehtman's axiomatisation of the logic of $F, P, \Box$ over $\mathbb{Q}$ in [28]. The last, $\mathcal{M}_4$, will be a Ψ-linked model satisfying φ_0. Each $\mathcal{M}_i$ will be written $(W_i, \sqsubset_i, R_i, h_i)$, but sometimes we drop the index i from these entries. Also, we sometimes identify (notationally) $\mathcal{M}_i$ with its domain W_i.

0. $\mathcal{M}_0$ is the canonical model.
1. $\mathcal{M}_1$ is a generated submodel of $\mathcal{M}_0$ satisfying φ_0, in which the relation $\sqsubset$ is prelinear.
2. $\mathcal{M}_2$ is got by filtrating all $\sqsubset$-clusters of $\mathcal{M}_1$, which consequently become finite.
3. $\mathcal{M}_3$ is a finite R-generated submodel of $\mathcal{M}_2$ got by selective filtration for $\sqsubset$. We use prelinearity and the Prior axiom to obtain it.
4. $\mathcal{M}_4$ is obtained from $\mathcal{M}_3$ by adding some extra points to arrange that any two adjacent clusters contain $\equiv^{\mathcal{M}_4}_{\mathcal{B}\Psi}$-equivalent points. We use the Prior axiom and induction on the number of '$\equiv^{\mathcal{M}_2}_{\mathcal{B}\Psi}$-types' of points in an interval.

As we go, we will indicate why we apparently cannot jump from our latest model directly to a model of φ_0 over $\mathbb{R}$.

Now to the details. Let φ_0 be an $\mathcal{L}$-formula consistent with the system defined in §3.1. We will explain in turn how each model $\mathcal{M}_0, \dots, \mathcal{M}_4$ is constructed.

6.1 Model $\mathcal{M}_0$

This is just the canonical model of the system given in §3.1, over the set PV of atoms. So W_0 is the set of all maximal consistent sets of $\mathcal{L}$-formulas. We write $\Gamma, \Delta, \Xi, \Theta, \ldots$ for arbitrary members of W_0. The relations and assignment are defined by:

- $\Gamma \sqsubset_0 \Delta$ iff $\varphi \in \Delta$ for every formula $G\varphi \in \Gamma$ (this is equivalent to each of the three statements $\varphi \in \Delta \Rightarrow F\varphi \in \Gamma$, $H\varphi \in \Delta \Rightarrow \varphi \in \Gamma$, and $\varphi \in \Gamma \Rightarrow P\varphi \in \Delta$),
- $\Gamma\, R_0\, \Delta$ iff $\varphi \in \Delta$ for every formula $\Box\varphi \in \Gamma$ (equivalently, $\varphi \in \Delta \Rightarrow \Diamond\varphi \in \Gamma$),
- $h_0(p) = \{\Gamma \in W_0 : p \in \Gamma\}$ for each atom $p \in PV$.

We assume familiarity with basic facts about canonical models — see, e.g., [5, 3] for details. The most important ones are that $\mathcal{M}_0, \Gamma \models \varphi$ iff $\varphi \in \Gamma$, for each $\Gamma \in \mathcal{M}_0$ and each $\mathcal{L}$-formula φ, all substitution instances of axioms from §3.1 are valid in $\mathcal{M}_0$, and the first-order correspondents listed in §3.1 are true in the canonical frame $(W_0, \sqsubset_0, R_0)$. For example, $\sqsubset_0$ is transitive and R_0 is reflexive and transitive. All substitution instances of the connectedness axiom 5 are valid in $\mathcal{M}_0$, but we do not know that this axiom is valid in the canonical frame.

6.2 Model $\mathcal{M}_1$

Since φ_0 is consistent, we can take $\Gamma_0 \in W_0$ containing φ_0, and then $\mathcal{M}_0, \Gamma_0 \models \varphi_0$. So $\mathcal{M}_0$ satisfies φ_0. However, $\mathcal{M}_0$ is a little unwieldy for us. Much of it is irrelevant: it has smaller and more manageable submodels satisfying φ_0, and in particular, ones in which $\sqsubset$ is prelinear. So our first step is to restrict to such a submodel.

We define $\mathcal{M}_1 = (W_1, \sqsubset_1, R_1, h_1)$ to be the submodel of $\mathcal{M}_0$ with domain $W_1 = \{\Delta \in W_0 : \Delta \sqsubseteq_0 \Gamma_0 \vee \Gamma_0 \sqsubset_0 \Delta\}$. Let us establish its basic properties.

LEMMA 6.2. *$\mathcal{M}_1$ is a generated submodel of $\mathcal{M}_0$ in which $\sqsubset_1$ is prelinear. The model $\mathcal{M}_1$ satisfies φ_0.*

Proof. Suppose $\Delta \in W_1$, $\Theta \in W_0$, and $\Delta \sqsubset_0 \Theta$. We show that $\Theta \in W_1$. If $\Gamma_0 \sqsubseteq_0 \Delta$ then by transitivity, $\Gamma_0 \sqsubset_0 \Theta$ and so $\Theta \in W_1$. If not, then $\Delta \sqsubset_0 \Gamma_0, \Theta$, and we obtain $\Theta \in W_1$ by the correspondent of the mirror image $PFp \to p \vee Fp \vee Pp$ of the linearity axiom. So $\mathcal{M}_1$ is a $\sqsubset_0$-generated submodel of $\mathcal{M}_0$. By symmetry, $\mathcal{M}_1$ is also a $\sqsupset_0$-generated submodel of $\mathcal{M}_0$. A similar argument, left to the reader, shows that $\sqsubset_1$ is prelinear as in definition 2.1(6).

To show that $\mathcal{M}_1$ is an R_0-generated submodel as well, note that if $\Delta \in W_1$, $\Theta \in W_0$, and $R_0\Delta\Theta$, then by the correspondent of axiom 4a we have $\Theta \sqsubseteq_0 \Delta$ or $\Delta \sqsubset_0 \Theta$. As W_1 is $\sqsubset_0$- and $\sqsupset_0$-generated, $\Theta \in W_1$.

Hence, $\mathcal{M}_1$ is a generated submodel of $\mathcal{M}_0$. By lemma 2.3, $\mathcal{M}_1, \Gamma_0 \models \varphi_0$ and φ_0 is satisfied in $\mathcal{M}_1$. □

As $\mathcal{M}_1$ is a generated submodel of $\mathcal{M}_0$, lemma 2.3 tells us that all substitution instances of axioms from §3.1 are valid in $\mathcal{M}_1$ and all axioms other than the connectedness axiom 5 (and its mirror image) are valid in its frame. Hence, the first-order correspondents of these axioms are true in the frame of $\mathcal{M}_1$. The following is now immediate from lemma 5.5 (see also [28, lemma 3.6]).

LEMMA 6.3. *Every $\sqsubset_1$-cluster in $\mathcal{M}_1$ is an R_1-generated subset of W_1.*

To end, we establish the perhaps surprising result that $\mathcal{M}_1$ contains initial and final $\sqsubset_1$-clusters (possibly equal).

LEMMA 6.4. *There are $\sqsubset_1$-clusters $C_\infty, C_{-\infty} \subseteq W_1$ such that $C_{-\infty} \sqsubset_1 \Gamma \sqsubset_1 C_\infty$ for every $\Gamma \in W_1$.*

Proof. Let $\Lambda_0 = \{P\varphi : \varphi \text{ is satisfied in } \mathcal{M}_1\}$. Then Λ_0 is consistent. For suppose that $n > 0$ is finite and $\varphi_0, \ldots, \varphi_{n-1}$ are satisfied in $\mathcal{M}_1$ at $\Delta_0, \ldots, \Delta_{n-1}$, respectively. Choose $i < n$ such that $|\{j < n : \Delta_j \sqsubseteq_1 \Delta_i\}|$ is maximal. It follows by prelinearity and transitivity of $\sqsubset_1$ that for every $j < n$ we have $\Delta_j \sqsubseteq_1 \Delta_i$. By lemma 3.2, $F\top \in \Delta_i$, so there is $\Gamma \in \mathcal{M}_1$ with $\Delta_i \sqsubset_1 \Gamma$. Then $\Delta_0, \ldots, \Delta_{n-1} \sqsubset_1 \Gamma$, so $P\varphi_0, \ldots, P\varphi_{n-1} \in \Gamma$, proving consistency of Λ_0.

Let $\Lambda \supseteq \Lambda_0$ be maximal consistent. Then $\Lambda \in W_0$. Let $\Gamma \in W_1$ be arbitrary. Then $\varphi \in \Gamma \Rightarrow P\varphi \in \Lambda_0 \subseteq \Lambda$, so $\Gamma \sqsubset_0 \Lambda$. This means that $\Lambda \in W_1$ and that Λ is a $\sqsubset_1$-greatest point in $\mathcal{M}_1$. Hence also, $\Lambda \sqsubset_1 \Lambda$, so Λ lies in a $\sqsubset_1$-cluster C_∞, say, of $\mathcal{M}_1$. Since C_∞ is a cluster, transitivity of $\sqsubset_1$ yields that $\Gamma \sqsubset_1 C_\infty$ as required.

The cluster $C_{-\infty}$ is obtained by a mirror image argument. □

6.3 Model $\mathcal{M}_2$

There is little hope of obtaining directly from $\mathcal{M}_1$ a model over $\mathbb{R}$ satisfying φ_0. For one thing, if we could do it, it would be likely that any consistent set of formulas could be shown satisfiable over $\mathbb{R}$, contradicting theorem 4.2. The more practical problem is that we do not know enough about $\sqsubset_1$ and R_1. We can glean some information about them from lemma 5.5. Essentially, $\mathcal{M}_1$ consists of $\sqsubset_1$-irreflexive points and $\sqsubset_1$-clusters. To make a real model, we would like to 'represent' each cluster C over $\mathbb{R}$, using methods originating in [20] (see §7). Unfortunately, it is not clear that the frame $(C, R_1 \restriction C)$ is connected. This makes the task hard — and in the Ψ-linked model that we aim to build, clusters must be connected.

The purpose of our next model $\mathcal{M}_2$ is to make all $\sqsubset$-clusters connected. We achieve this by making them finite, using a certain filtration of $\mathcal{M}_1$ due to Shehtman [28, §3]. The connectedness axiom can then be used to prove that every $\sqsubset_2$-cluster in $\mathcal{M}_2$ is connected. We will also show that $\mathcal{M}_2$ contains certain well-behaved submodels, which will be used to construct the final model $\mathcal{M}_4$. So our study of $\mathcal{M}_2$ will be quite elaborate.

6.3.1 Definition of $\mathcal{M}_2$

We will need a finite set of formulas to define the filtration. It will be the set Ψ of theorem 6.1 — a finite set of formulas containing $P\varphi_0$ and closed under taking subformulas. Our filtration equivalence relation is now defined as in [28, §3]. Recall again that $\Gamma \sqsupset\sqsubset \Delta$ means that $\Gamma \sqsubset \Delta$ and $\Delta \sqsubset \Gamma$. Let $\sim$ be the binary relation on W_1 defined by:

$$\Gamma \sim \Delta \iff \Gamma = \Delta \vee \big((\Gamma \sqsupset\sqsubset_1 \Delta) \wedge (\Gamma \cap \Psi = \Delta \cap \Psi)\big). \tag{6.1}$$

As $\sqsubset_1$ is transitive, $\sim$ is an equivalence relation on W_1.

DEFINITION 6.5. For $\Gamma \in W_1$ we write $\Gamma/{\sim}$ for the equivalence class $\{\Delta \in W_1 : \Gamma \sim \Delta\}$. For $X \subseteq W_1$, we write $X/{\sim}$ for the set $\{\Gamma/{\sim} : \Gamma \in X\}$ of equivalence classes having a non-empty intersection with X.

Generally, but not always, X will be $\sim$-closed (i.e., a union of $\sim$-classes).

The domain W_2 of $\mathcal{M}_2$ is now defined to be the set $W_1/{\sim}$ of $\sim$-equivalence classes in $\mathcal{M}_1$. The relations on $\mathcal{M}_2$ are: $\sqsubset_2$ is induced existentially from $\sqsubset_1$, and R_2 is the transitive closure of the relation induced existentially from R_1. Formally:

$$\begin{aligned}
&\sqsubset_2 = \{(\Gamma/{\sim}, \Delta/{\sim}) : \Gamma, \Delta \in W_1,\ \Gamma \sqsubset_1 \Delta\},\\
&R_2^0 = \{(\Gamma/{\sim}, \Delta/{\sim}) : \Gamma, \Delta \in W_1,\ \Gamma\, R_1\, \Delta\},\\
&R_2 \text{ is the transitive closure of } R_2^0.
\end{aligned}$$

We set $h_2(p) = \{\Gamma/{\sim} : \Gamma \in W_1,\ p \in \Gamma\}$ for each atom $p \in PV$. This defines the model $\mathcal{M}_2$.

In case of doubt, we remark that the expressions above are of the form $\{f(x) : x \in X\}$ for some set X and some class function f defined on X, and thus are well-defined sets by the axiom of replacement of ZF. For example, for $\sqsubset_2$ we can take $X = \sqsubset_1$ and $f : (\Gamma, \Delta) \mapsto (\Gamma/{\sim}, \Delta/{\sim})$.

6.3.2 Filtration lemma for $\mathcal{M}_2$

Filtration is designed to preserve truth of formulas, so let us confirm this first. To begin, it is worth knowing that $\sqsubset_2$ is closely related to $\sqsubset_1$:

LEMMA 6.6. *Let $\Gamma, \Delta \in W_1$. Then $\Gamma \sqsubset_1 \Delta$ iff $\Gamma/{\sim} \sqsubset_2 \Delta/{\sim}$.*

Proof. If $\Gamma \sqsubset_1 \Delta$ then $\Gamma/{\sim} \sqsubset_2 \Delta/{\sim}$ by definition of $\sqsubset_2$. Conversely, if $\Gamma/{\sim} \sqsubset_2 \Delta/{\sim}$ then by definition of $\sqsubset_2$ there are $\Gamma', \Delta' \in W_1$ with $\Gamma \sim \Gamma'$, $\Delta \sim \Delta'$, and $\Gamma' \sqsubset_1 \Delta'$. The definition of $\sim$ gives $\Gamma \sqsubseteq_1 \Gamma'$ and $\Delta' \sqsubseteq_1 \Delta$, so $\Gamma \sqsubset_1 \Delta$ by transitivity of $\sqsubset_1$. $\square$

In [28], a 'filtration lemma' is proved: all formulas in Ψ are preserved from $\mathcal{M}_1$ to $\mathcal{M}_2$. Because of the simple definition of $\sqsubset_2$, we can actually go a little further:

LEMMA 6.7 (filtration)**.** *Let ψ be an $\mathcal{L}$-formula formed from formulas in Ψ by using only the boolean and temporal operators (without using $\Box$ or $\Diamond$). Then for all $\Gamma \in W_1$ we have $\mathcal{M}_2, \Gamma/{\sim} \models \psi$ iff $\mathcal{M}_1, \Gamma \models \psi$ (iff $\psi \in \Gamma$). Hence, φ_0 is satisfied in $\mathcal{M}_2$.*

Proof. By induction on ψ. If $\psi \in \Psi$, the result is proved in [28, lemmas 3.2–3.3]. The boolean cases are easy and left to the reader. Assuming the result for ψ, we prove it for $G\psi$. Let $\Gamma \in W_1$ be given. Then the following are equivalent:

$\mathcal{M}_2, \Gamma/{\sim} \models G\psi$	
$\mathcal{M}_2, \Delta/{\sim} \models \psi$ for all $\Delta \in W_1$ with $\Gamma/{\sim} \sqsubset_2 \Delta/{\sim}$	by semantics of G,
$\mathcal{M}_1, \Delta \models \psi$ for all $\Delta \in W_1$ with $\Gamma \sqsubset_1 \Delta$	by ind. hyp. & lemma 6.6,
$\mathcal{M}_1, \Gamma \models G\psi$	by semantics of G.

The case $H\psi$ is similar.

We know there is some $\Gamma_0 \in W_1$ containing φ_0. As $\varphi_0 \in \Psi$, the above yields $\mathcal{M}_2, \Gamma_0/{\sim} \models \varphi_0$, so φ_0 is satisfied in $\mathcal{M}_2$. $\square$

6.3.3 Structure of $\mathcal{M}_2$

We will need some simple facts about the form of $\mathcal{M}_2$. Happily, by [28, lemma 3.3], the frame of $\mathcal{M}_2$ validates all axioms of the system of §3.1 other than axiom 5 (connectedness) and its mirror image. Hence, their correspondents are true in the frame $(W_2, \sqsubset_2, R_2)$, and lemma 5.5 applies.

Some limited instances of (connectedness and) the Prior axiom are also valid in $\mathcal{M}_2$:

LEMMA 6.8. *Let β be an $\mathcal{L}$-formula formed from formulas in Ψ by using only the boolean and temporal operators (without using $\Box$ or $\Diamond$). Then the β-instance*

$$\pi = F\beta \wedge FG\neg\beta \to F(G\neg\beta \wedge HF\beta)$$

of the Prior axiom is valid in $\mathcal{M}_2$.

Proof. Take any $w \in W_2$ and $\Gamma \in w$. By corollary 3.5, the Prior axiom is provable, so Γ contains all substitution instances of it. So $\pi \in \Gamma$. This formula is made from formulas in Ψ by using only the boolean and temporal operators, so by the filtration lemma 6.7 we obtain $\mathcal{M}_2, w \models \pi$ as required. $\Box$

However, in lemmas 6.12 and 6.20 we will need arbitrary instances of these axioms, and we do not know that they are valid in $\mathcal{M}_2$. They are of course valid in $\mathcal{M}_1$, so we will work in $\mathcal{M}_1$ in these lemmas.

LEMMA 6.9. 1. *The relation $\sqsubset_2$ is transitive and prelinear.*

2. *If $X \subseteq W_1$ is a $\sqsubset_1$-cluster in $\mathcal{M}_1$ then $X/{\sim}$ is a $\sqsubset_2$-cluster in $\mathcal{M}_2$.*
3. *If $C \subseteq W_2$ is a $\sqsubset_2$-cluster in $\mathcal{M}_2$ then $\bigcup C$ is a $\sqsubset_1$-cluster in $\mathcal{M}_1$.*
4. *Every $\sqsubset_2$-cluster in $\mathcal{M}_2$ is a finite R_2-generated subset of W_2.*

Proof. The relation $\sqsubset_1$ is transitive (since $\mathcal{M}_1 \subseteq \mathcal{M}_0$ and $\sqsubset_0$ is transitive), and prelinear (by lemma 6.2), and it follows from lemma 6.6 that $\sqsubset_2$ is as well. This proves part 1.

For parts 2–3, let $X \subseteq W_1$ and $C \subseteq W_2$ be sets. Plainly, $X \subseteq \bigcup(X/{\sim})$ and $C = (\bigcup C)/{\sim}$. Call X a $\sqsubset_1$-*precluster* if $\Gamma \sqsubset_1 \Delta$ for every $\Gamma, \Delta \in X$, and C a $\sqsubset_2$-*precluster* if $c \sqsubset_2 d$ for every $c, d \in C$. By lemma 6.6, parts 2 and 3 hold if we replace 'cluster' by 'precluster'. A cluster is just a maximal precluster, and by Zorn's lemma, every precluster extends to a cluster.

Suppose that X is a $\sqsubset_1$-cluster. Then $X/{\sim}$ is a $\sqsubset_2$-precluster. Extend it to a $\sqsubset_2$-cluster D. Then $X \subseteq \bigcup(X/{\sim}) \subseteq \bigcup D$, and since $\bigcup D$ is a $\sqsubset_1$-precluster we have $X = \bigcup D$. So $X/{\sim} = (\bigcup D)/{\sim} = D$, a $\sqsubset_2$-cluster.

Suppose that C is a $\sqsubset_2$-cluster. Then $\bigcup C$ is a $\sqsubset_1$-precluster. Extend it to a $\sqsubset_1$-cluster Y. Then $C = (\bigcup C)/{\sim} \subseteq Y/{\sim}$, and since $Y/{\sim}$ is a $\sqsubset_2$-precluster we have $C = Y/{\sim}$. So $Y \subseteq \bigcup(Y/{\sim}) = \bigcup C \subseteq Y$. Consequently, $\bigcup C = Y$ is a $\sqsubset_1$-cluster.

For part 4, let C be a $\sqsubset_2$-cluster. Then C has the form $Y/{\sim}$ for a $\sqsubset_1$-cluster $Y = \bigcup C$, and by definition of $\sim$, for each $\Gamma, \Delta \in Y$ we have $\Gamma \sim \Delta$ iff $\Gamma \cap \Psi = \Delta \cap \Psi$. Hence, the map $f : C \to \wp(\Psi)$ given by $f(\Gamma/{\sim}) = \Gamma \cap \Psi$ is well defined and one-one, so $|C| \leq |\wp(\Psi)|$ and C is finite (this is [28, lemma 3.4]). That C is R_2-generated follows from lemma 5.5. $\Box$

Lemma 6.4 also extends to $\mathcal{M}_2$:

LEMMA 6.10. *The sets $C_\infty/{\sim}, C_{-\infty}/{\sim}$ are $\sqsubset_2$-clusters in $\mathcal{M}_2$, and for every $w \in W_2$ we have $C_{-\infty}/{\sim} \sqsubset_2 w \sqsubset_2 C_\infty/{\sim}$.*

Proof. Immediate from lemmas 6.4, 6.6, and 6.9. □

6.3.4 Connectedness of $\sqsubset$-clusters in $\mathcal{M}_2$

By lemma 6.9, all $\sqsubset_2$-clusters in $\mathcal{M}_2$ are finite, and using the connectedness axiom, we can prove that they are connected as R_2-frames. To do this, we first write down formulas to define individual elements within a $\sqsubset_2$-cluster.

DEFINITION 6.11. For $w \in \mathcal{M}_2$ let $\chi_w = \bigwedge(\Psi \cap \Gamma) \wedge \neg \bigvee(\Psi \setminus \Gamma)$ for arbitrary $\Gamma \in w$. (By convention, $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$.)

By definition of $\sim$ in (6.1), this definition is independent of the choice of Γ, and obviously $\chi_w \in \Gamma$. The set $\{\chi_w : w \in \mathcal{M}_2\}$ is finite, because Ψ is finite.

LEMMA 6.12. *Let C be a $\sqsubset_2$-cluster in $\mathcal{M}_2$. Then $(C, R_2 \restriction C)$ is a connected frame.*

Proof. Suppose on the contrary that C is a $\sqsubset_2$-cluster in $\mathcal{M}_2$ that is the union of disjoint non-empty R_2-generated sets X, Y. Let $\alpha = \bigvee_{w \in X} \chi_w$. Then α defines X within C: for any $\Gamma \in \bigcup C$ we have

$$\alpha \in \Gamma \iff \Gamma/{\sim} \in X. \tag{6.2}$$

Choose any $\Gamma \in \bigcup C$ and let

$$\Delta_0 = \{\Diamond\alpha, \Diamond\neg\alpha\} \cup \{F\gamma, P\gamma : \gamma \in \Gamma\}.$$

We show that Δ_0 is consistent. Since Γ is closed under conjunction, it suffices to take arbitrary $\gamma \in \Gamma$ and show that $\delta = \Diamond\alpha \wedge \Diamond\neg\alpha \wedge F\gamma \wedge P\gamma$ is consistent. Choose any $\Gamma_X \in \bigcup X$ and $\Gamma_Y \in \bigcup Y$. By lemma 6.9, $\bigcup C$ is a $\sqsubset_1$-cluster in $\mathcal{M}_1$. So $\Gamma_X \sqsupset\sqsubset_1 \Gamma$ and similarly for Γ_Y. Now $\alpha \wedge F\gamma \in \Gamma_X$ and $\neg\alpha \wedge F\gamma \in \Gamma_Y$. So $F(\alpha \wedge F\gamma), F(\neg\alpha \wedge F\gamma) \in \Gamma$. By the connectedness axiom 5, $F(\Diamond\alpha \wedge \Diamond\neg\alpha \wedge F\gamma) \in \Gamma$. By temporal axioms, $GP\gamma \in \Gamma$ as well, so $F\delta \in \Gamma$. If δ is inconsistent then $\neg\delta$ and hence $G\neg\delta$ are provable, so $G\neg\delta \in \Gamma$, contradicting its consistency. So δ is consistent, as required.

So we may take $\Delta \in \mathcal{M}_0$ with $\Delta \supseteq \Delta_0$. By definition of Δ_0 we have $\Delta \sqsupset\sqsubset_0 \Gamma$, so $\Delta \in \mathcal{M}_1$ and $\Delta \in \bigcup C$. As $\Diamond\alpha, \Diamond\neg\alpha \in \Delta$, we may find $\Delta_X, \Delta_Y \in R_0(\Delta)$ with $\alpha \in \Delta_X$ and $\neg\alpha \in \Delta_Y$. Then $\Delta_X, \Delta_Y \in R_1(\Delta)$ as $\mathcal{M}_1$ is a generated submodel of $\mathcal{M}_0$. By lemma 6.3, $\bigcup C$ is an R_1-generated subset of $\mathcal{M}_1$, so $\Delta_X, \Delta_Y \in \bigcup C$ as well.

Let $w = \Delta/{\sim}$, $w_X = \Delta_X/{\sim}$, and $w_Y = \Delta_Y/{\sim}$. By (6.2), $w_X \in X$ and $w_Y \in Y$. By definition of R_2, we have $w_X, w_Y \in R_2(w)$. Since $w \in X \cup Y$, this contradicts that X and Y are disjoint and R_2-generated. □

6.3.5 Submodels of $\mathcal{M}_2$

It may appear that $\mathcal{M}_2$ could be our final Kripke model. Each $\sqsubset_2$-cluster C in $\mathcal{M}_2$ is finite and connected as an R_2-frame, so classical work (see §7) will give us a model over $\mathbb{R}$ respecting truth within C. We can represent an $\sqsubset_2$-irreflexive point of $\mathcal{M}_2$ by a single point of $\mathbb{R}$. Could we not string these together somehow, to make a model over $\mathbb{R}$ satisfying φ_0?

To do so would require the following. Let J be the linear order obtained from $(W_2, \sqsubset_2)$ by replacing each $\sqsubset_2$-cluster by a copy of $\mathbb{R}$. Then we would need $J \cong \mathbb{R}$.

There is no reason to suppose that $J \cong \mathbb{R}$. For example, $\mathcal{M}_2$ may have uncountably many $\sqsubset_2$-clusters. (Indeed, the argument of theorem 4.2 can be used to create an example in which $J \ncong \mathbb{R}$.) Even a *finite* submodel of $\mathcal{M}_2$ may not work: if we form J for it as we did for $\mathcal{M}_2$, we may still not have $J \cong \mathbb{R}$, because the submodel may have consecutive $\sqsubset$-clusters that are actually adjacent, with no intervening point. Consideration of the formula θ of (4.2) shows that this can indeed happen. Replacing the two adjacent clusters by copies of $\mathbb{R}$ gives a linear order isomorphic to $((-1, 0) \cup (0, 1), <)$. This is not Dedekind complete — it has a 'gap' at 0. It follows that $J \ncong \mathbb{R}$ in this case.

In Ψ-linked models, any two adjacent clusters contain 'similar' points satisfying exactly the same formulas $\Box\psi, G\psi, H\psi \in \Psi$. They allow us to 'fill the gap'. So we will now show that $\mathcal{M}_2$ contains abundant finite submodels with this property. This is in a sense the heart of the paper. It will lead us shortly to our final model $\mathcal{M}_4$.

First we introduce formulas that will help us to pick out 'similar' points. Recall (definition 5.3) that $\mathcal{B}\Psi$ is the set of all formulas in Ψ of the form $\Box\psi$, $G\psi$, or $H\psi$ (the 'box-formulas').

DEFINITION 6.13. 1. For a subset $B \subseteq \mathcal{B}\Psi$, let $\beta_B = (\bigwedge B) \wedge \neg \bigvee(\mathcal{B}\Psi \setminus B)$.

2. For a Kripke model $\mathcal{M}$ and $w \in \mathcal{M}$, we write $\tau_{\mathcal{M}}(w)$ (or if no ambiguity is likely, $\tau(w)$) for the set $\{\psi \in \mathcal{B}\Psi : \mathcal{M}, w \models \psi\}$.

We think of $\tau_{\mathcal{M}}(w)$ as the 'type' of w in $\mathcal{M}$. The following is proved by standard boolean manipulations.

LEMMA 6.14.

1. *For every Kripke model $\mathcal{M}$ and $w \in \mathcal{M}$, the set $\tau_{\mathcal{M}}(w)$ is the unique subset $B \subseteq \mathcal{B}\Psi$ with $\mathcal{M}, w \models \beta_B$.*
2. *For every $\Gamma \in W_1$, the set $\tau_{\mathcal{M}_1}(\Gamma)$ is the unique subset $B \subseteq \mathcal{B}\Psi$ such that $\beta_B \in \Gamma$. Indeed, $\tau_{\mathcal{M}_1}(\Gamma) = \Gamma \cap \mathcal{B}\Psi$.*
3. *For every Kripke model $\mathcal{M}$ and $c, d \in \mathcal{M}$, we have $c \equiv^{\mathcal{M}}_{\mathcal{B}\Psi} d$ (so c and d are 'similar') iff $\tau_{\mathcal{M}}(c) = \tau_{\mathcal{M}}(d)$.*

So to show that two clusters C, D contain similar points, it suffices to find points $c \in C$ and $d \in D$ of the same type — satisfying the same formula β_B for some $B \subseteq \mathcal{B}\Psi$. We will use the Prior axiom to do this. However, because this axiom delivers an $\sqsubset$-irreflexive point after which a formula is false but at which the formula may actually be true (see remark 3.6), whereas we want a point in a cluster, we will actually use $\Box\beta_B$ rather than β_B.

Not every world in a model satisfies some $\Box\beta_B$, so (roughly) we now pick out the worlds that do. We will call such worlds *links,* because they will 'link' adjacent clusters.

We would like to define $w \in W_2$ to be a link if $\mathcal{M}_2, w \models \Box\beta_B$ for some B. Unfortunately, there is a second complication. We would like to use the Prior axiom to find a point satisfying some $\Box\beta_B$. But we do not know that the $\Box\beta_B$-instance of the Prior axiom is valid in $\mathcal{M}_2$. Moreover, the point should also lie in a certain temporal range, and to achieve this we will need to work in $\mathcal{M}_1$, where $\sqsubset_1$ is defined by formulas. Now the Prior axiom is valid in $\mathcal{M}_1$, and it will deliver points of $\mathcal{M}_1$ in the right range and satisfying $\Box\beta_B$, but their 'representatives' (their $\sim$-classes) in $\mathcal{M}_2$ may not satisfy $\Box\beta_B$ as the filtration lemma 6.7 does not apply to $\Box\beta$. Consequently, with this definition of 'link' we cannot guarantee that adjacent clusters will contain 'similar' points. So in the formal definition of 'link', we work directly in $\mathcal{M}_1$:

DEFINITION 6.15. An element $w \in W_2$ is said to be a *link* if $\Box\beta_B \in \bigcup w$ for some $B \subseteq \mathcal{B}\Psi$ — that is, there are $\Gamma \in w$ and $B \subseteq \mathcal{B}\Psi$ with $\Box\beta_B \in \Gamma$.

We first show that links are rather common, and that they work well enough to deliver 'similar' points.

LEMMA 6.16. *Every $\sqsubset_2$-cluster in $\mathcal{M}_2$ contains a link.*

Proof. Let C be a $\sqsubset_2$-cluster in $\mathcal{M}_2$. By lemma 6.9(4), C is finite and R_2-generated. Let $c \in C$ be such that $|R_2(c)|$ is least possible. We will show that c is a link.

Let $B = \tau(c)$. First we show that $\mathcal{M}_2, c \models \Box\beta_B$. So take arbitrary $d \in R_2(c)$. As C is R_2-generated, $d \in C$. By transitivity of R_2 we have $R_2(d) \subseteq R_2(c)$, so by choice of c we have $R_2(d) = R_2(c)$. Moreover, since $c \sqsupset\sqsubset_2 d$, by transitivity of $\sqsubset_2$ we see that for all $w \in W_2$ we have $w \sqsubset_2 c$ iff $w \sqsubset_2 d$, and $w \sqsupset_2 c$ iff $w \sqsupset_2 d$. So c and d 'see' the same elements of W_2 via R_2, $\sqsubset_2$, and $\sqsupset_2$. It plainly follows that $c \equiv^{\mathcal{M}_2}_{\mathcal{B}\Psi} d$, so $B = \tau(c) = \tau(d)$ and hence $\mathcal{M}_2, d \models \beta_B$ as well. As d was arbitrary, $\mathcal{M}_2, c \models \Box\beta_B$ as required.

Now let $\Gamma \in c$ be arbitrary; we will show that $\Box\beta_B \in \Gamma$, so that c is a link by definition. Let $\Delta \in R_1(\Gamma)$ be arbitrary, and let $d = \Delta/{\sim} \in W_2$. By definition of R_2 we have $d \in R_2(c)$, so by the above, $\mathcal{M}_2, d \models \beta_B$. As β_B is a boolean combination of formulas in Ψ, the filtration lemma 6.7 yields $\beta_B \in \Delta$. As Δ was arbitrary, we obtain $\Box\beta_B \in \Gamma$. $\Box$

LEMMA 6.17. *Let $B \subseteq \mathcal{B}\Psi$ and $w \in \mathcal{M}_2$. Then $\Box\beta_B \in \bigcup w$ iff w is a link and $\tau(w) = B$.*

Proof. If $\Box\beta_B \in \Gamma \in w$, then w is plainly a link. By the S4 reflexivity axiom, $\beta_B \in \Gamma$ as well. By the filtration lemma 6.7, $\mathcal{M}_2, w \models \beta_B$, so by lemma 6.14, $\tau(w) = B$.

Conversely, suppose that w is a link. By definition, $\Box\beta_{B'} \in \bigcup w$ for some $B' \subseteq \mathcal{B}\Psi$. By the first part, $\tau(w) = B'$. So if additionally $\tau(w) = B$, we have $B' = B$ and hence $\Box\beta_B \in \bigcup w$. $\Box$

We will need to count the types of links in an interval:

DEFINITION 6.18. For $\sqsubset_2$-clusters $C, D \subseteq \mathcal{M}_2$ with $C \sqsubset_2 D$ (possibly, $C = D$), let

$$\sharp(C, D) = |\{\tau_{\mathcal{M}_2}(w) : w \in W_2 \text{ a link, } C \sqsubset_2 w \wedge w \sqsubset_2 D\}|.$$

The value is plainly finite, because $\mathcal{B}\Psi$ is finite. It is the number of types of links in the interval (C, D) of $\mathcal{M}_2$.

COROLLARY 6.19. *Let $C, D \subseteq \mathcal{M}_2$ be $\sqsubset_2$-clusters with $C \sqsubset_2 D$. Then we have $\sharp(C, D) > 0$.*

Proof. By lemma 6.16, there is a link $w \in C$. Plainly, $C \sqsubset_2 w \sqsubset_2 D$. So $\tau(w)$ contributes one to the total for $\sharp(C, D)$, which is therefore nonzero. $\square$

Suppose that (C, D) is a pair of adjacent $\sqsubset_2$-clusters in $\mathcal{M}_2$. Using lemma 6.16, take links $c \in C$ and $d \in D$. Imagine that $\sharp(C, D) = 1$. As $\tau(c)$ and $\tau(d)$ both contribute one to $\sharp(C, D)$, we must have $\tau(c) = \tau(d)$, and it follows that $c \equiv^{\mathcal{M}_2}_{\mathcal{B}\Psi} d$. We have found similar points in C, D, as required for a Ψ-linked model. This suggests trying to find similar points in more general situations by induction on $\sharp(C, D)$, and that is what we will do in lemma 6.22 below. For the induction step, we will need the following important technical lemma. Part 3 of the lemma shows that the value of $\sharp$ drops, facilitating the induction.

LEMMA 6.20. *Let C be a $\sqsubset_2$-cluster in $\mathcal{M}_2$, and let $w \in \mathcal{M}_2$ with $C \sqsubset_2 w$ and $w \notin C$. Then there is a $\sqsubset_2$-irreflexive $u \in \mathcal{M}_2$ such that:*

1. *$C \sqsubset_2 u \sqsubseteq_2 w$,*
2. *there are $c \in C$ and $d \in \lambda(u)$ with $c \equiv^{\mathcal{M}_2}_{\mathcal{B}\Psi} d$,*
3. *if w is $\sqsubset_2$-irreflexive and $u \sqsubset_2 w$, then $\sharp(\rho(u), \lambda(w)) < \sharp(C, \lambda(w))$.*

(See lemma 5.5 for λ, ρ.) The mirror image also holds.

Proof. By lemma 6.16, there is a link $c \in C$. Let $B = \tau(c)$. By lemma 6.17, there is $\Gamma \in c$ with $\Box\beta_B \in \Gamma$.

Now pick any $\Delta \in w$. Since $C \sqsubset_2 w \notin C$, by lemma 6.6 we have $\Gamma \sqsubset_1 \Delta$ and $\neg(\Delta \sqsubset_1 \Gamma)$. By the latter, there is a formula γ with

$$\gamma \in \Gamma \text{ and } G\neg\gamma \in \Delta. \tag{6.3}$$

If w is $\sqsubset_2$-irreflexive, then Δ is $\sqsubset_1$-irreflexive, and in that case, as the reader may confirm, we can suppose that $H\gamma \in \Delta$ as well.

So $\gamma \wedge \Box\beta_B \in \Gamma$ and $G\neg(\gamma \wedge \Box\beta_B) \in \Delta$. As Γ is in a $\sqsubset_1$-cluster ($\bigcup C$), we have $\Gamma \sqsubset_1 \{\Gamma, \Delta\}$, so $F(\gamma \wedge \Box\beta_B) \wedge FG\neg(\gamma \wedge \Box\beta_B) \in \Gamma$. Now by corollary 3.5, the Prior axiom is provable, so its instance $F(G\neg(\gamma \wedge \Box\beta_B) \wedge HF(\gamma \wedge \Box\beta_B))$ is in Γ. This lets us take $\Theta \sqsupset_1 \Gamma$ in $\mathcal{M}_1$ with

$$G\neg(\gamma \wedge \Box\beta_B) \wedge HF(\gamma \wedge \Box\beta_B) \in \Theta. \tag{6.4}$$

It follows from (6.4) that Θ is $\sqsubset_1$-irreflexive. Let $u = \Theta/\!\sim$. By lemma 6.6, u is $\sqsubset_2$-irreflexive. Since $\Gamma \sqsubset_1 \Theta$, we have $C \sqsubset_2 u$. (So by lemma 5.5, $C \sqsubset_2 \lambda(u)$; note that $C = \lambda(u)$ is possible.) Since $HF\gamma \in \Theta$ but $G\neg\gamma \in \Delta$, we see that $\Delta \not\sqsubseteq_1 \Theta$. By prelinearity, $\Theta \sqsubseteq_1 \Delta$, so $u \sqsubseteq_2 w$. This proves part 1 of the lemma.

It follows from (6.4) and lemma 3.3 that $\Diamond(\gamma \wedge \Box\beta_B) \in \Theta$. So we may choose $\Theta' \in R_1(\Theta)$ containing $\gamma \wedge \Box\beta_B$. There are two possibilities, the first being the reason why we need to use $\Box\beta_B$ rather than just β_B. If $\Theta' = \Theta$, choose arbitrary $\Xi \in R_1(\Theta)$ with $\Xi \sqsubset_1 \Theta$ (there is such a Ξ by the mirror image of axiom 4d). By transitivity of R_1 we have $\Box\beta_B \in \Xi$. Alternatively, if $\Theta' \neq \Theta$, then by (6.4) we have $\neg(\Theta \sqsubseteq_1 \Theta')$, and prelinearity of $\sqsubset_1$ gives $\Theta' \sqsubset_1 \Theta$. In that case let $\Xi = \Theta'$. Again we have $\Box\beta_B \in \Xi$.

Let $d = \Xi/\sim$. Then $d \in \lambda(u)$. By lemma 6.17, $\tau(d) = B = \tau(c)$. By lemma 6.14, $c \equiv^{\mathcal{M}_2}_{\mathcal{B}\Psi} d$. This proves part 2 of the lemma.

For part 3, suppose that w is $\sqsubset_2$-irreflexive and $u \sqsubset_2 w$, so that $\rho(u) \sqsubset_2 \lambda(w)$ by lemma 5.5 (possibly $\rho(u) = \lambda(w)$). We have $H\gamma \in \Delta$ in this case, and by (6.4), $G\neg(\gamma \wedge \Box\beta_B) \in \Theta$. It follows that $\Box\beta_B \notin \Xi'$ for every $\Xi' \in W_1$ with $\Theta \sqsubset_1 \Xi' \sqsubset_1 \Delta$. As $\Theta/\sim = u$ and $\Delta/\sim = w$, by lemmas 6.6 and 6.17 there is no link v with $\tau(v) = B$ and $u \sqsubset_2 v \sqsubset_2 w$ — equivalently, with $\rho(u) \sqsubset_2 v \sqsubset_2 \lambda(w)$ (see lemma 5.5). So

$$\begin{aligned}&\{\tau(v) : v \in W_2 \text{ a link}, \rho(u) \sqsubset_2 v \sqsubset_2 \lambda(w)\}\\ &\qquad\qquad \subseteq \{\tau(v) : v \in W_2 \text{ a link}, C \sqsubset_2 v \sqsubset_2 \lambda(w)\} \setminus \{B\}.\end{aligned}$$

Since there certainly is a link v of type B with $C \sqsubset_2 v \sqsubset_2 \lambda(w)$ — for example, $v = c$ — we see that $\sharp(\rho(u), \lambda(w)) < \sharp(C, \lambda(w))$. $\Box$

We remark that γ in (6.3) may be very complex and the instance of the Prior axiom used to obtain (6.4) may not be valid in $\mathcal{M}_2$. That is why we work in $\mathcal{M}_1$.

DEFINITION 6.21. Let $\mathcal{M} = (W, \sqsubset, R, h)$ be a submodel of $\mathcal{M}_2$.

1. We say that $\mathcal{M}$ is *good* if it is finite, R_2-generated, and every $\sqsubset$-cluster in $\mathcal{M}$ is a $\sqsubset_2$-cluster in $\mathcal{M}_2$.
2. We say that $\mathcal{M}$ is *perfect* if it is good, and for every pair (C, D) of adjacent $\sqsubset$-clusters in $\mathcal{M}$, there are $c \in C$ and $d \in D$ with $c \equiv^{\mathcal{M}_2}_{\mathcal{B}\Psi} d$.

Note that we use $\equiv^{\mathcal{M}_2}_{\mathcal{B}\Psi}$ and not $\equiv^{\mathcal{M}}_{\mathcal{B}\Psi}$ here. We will convert to $\equiv^{\mathcal{M}}_{\mathcal{B}\Psi}$ in §6.4 but we will need a little more machinery for that.

LEMMA 6.22. *Every good submodel of $\mathcal{M}_2$ extends to a perfect submodel of $\mathcal{M}_2$.*

Proof. Let us say that a *defect* in a good submodel $\mathcal{M} = (W, \sqsubset, R, h)$ of $\mathcal{M}_2$ is a pair (C, D) of adjacent $\sqsubset$-clusters in $\mathcal{M}$ such that there do not exist $c \in C$, $d \in D$ with $c \equiv^{\mathcal{M}_2}_{\mathcal{B}\Psi} d$. Again, we use $\mathcal{M}_2$ here, not $\mathcal{M}$. Let

$$d(\mathcal{M}) = \sum \big\{\sharp(C, D) : (C, D) \text{ a defect of } \mathcal{M}\big\}.$$

Then $d(\mathcal{M}) \geq 0$, and $d(\mathcal{M})$ is finite because $\mathcal{M}$ is.

Now let $\mathcal{M} = (W, \sqsubset, R, h)$ be an arbitrary good submodel of $\mathcal{M}_2$. Among all good submodels $\mathcal{M}^* = (W^*, \sqsubset^*, R^*, h^*)$ with $\mathcal{M} \subseteq \mathcal{M}^* \subseteq \mathcal{M}_2$, choose one with $d(\mathcal{M}^*)$ as small as possible. We will show that $d(\mathcal{M}^*) = 0$. Since by corollary 6.19, $\sharp(C, D) > 0$ for every defect (C, D), it follows that $\mathcal{M}^*$ has no defects and is therefore perfect.

Assume for contradiction that $d(\mathcal{M}^*) > 0$. Pick any defect (C, D) in $\mathcal{M}^*$ and any $w \in D$. As $\mathcal{M}^* \subseteq \mathcal{M}_2$, we have $C \sqsubset_2 w$ and $\neg(w \sqsubset_2 C)$. Let $u \in \mathcal{M}_2$ be as provided by lemma 6.20, and let $\mathcal{N}$ be the submodel of $\mathcal{M}_2$ consisting of $\mathcal{M}^*$ together with $R_2(u)$. We let $\sqsubset$ denote $\sqsubset_2 \restriction \mathcal{N}$. So

$$\{v \in \mathcal{N} : C \sqsubset v \sqsubset D\} = C \cup \lambda(u) \cup \{u\} \cup \rho(u) \cup D, \tag{6.5}$$

shown left to right in $\sqsubset$-order. (Possibly, $\lambda(u) = C$ or $\rho(u) = D$ or both.) Plainly, $\mathcal{N}$ is good and $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{M}_2$. So by choice of $\mathcal{M}^*$ we have $d(\mathcal{N}) \geq d(\mathcal{M}^*)$.

Now outside the range C–D shown in (6.5), all defects and their $\sharp$-values are the same in $\mathcal{M}^*$ and $\mathcal{N}$. (Remember that $\sharp$ is evaluated with respect to $\mathcal{M}_2$.) So let us consider the remaining potential defects in $\mathcal{N}$. From inspection of (6.5), these are $(C, \lambda(u))$ and $(\rho(u), D)$. (Note that $(\lambda(u), \rho(u))$ is not a defect since it is not a pair of adjacent clusters: u is $\sqsubset$-irreflexive and $\lambda(u) \sqsubset u \sqsubset \rho(u)$.)

The pair $(C, \lambda(u))$ is not a defect, because lemma 6.20 provides that C and $\lambda(u)$ contain $\equiv^{\mathcal{M}_2}_{\mathcal{B}\Psi}$-equivalent points (possibly even $C = \lambda(u)$).

So because $d(\mathcal{N}) \geq d(\mathcal{M}^*)$, we see that $(\rho(u), D)$ must be a defect in $\mathcal{N}$ and moreover that $\sharp(\rho(u), D) \geq \sharp(C, D)$.

But since $C \sqsubset_2 \rho(u)$, we have $\rho(u) \sqsubset_2 t \sqsubset_2 D \Rightarrow C \sqsubset_2 t \sqsubset_2 D$ for all $t \in W_2$, so every link type contributing to $\sharp(\rho(u), D)$ also contributes to $\sharp(C, D)$. Hence, $\sharp(\rho(u), D) \leq \sharp(C, D)$. So in fact, $\sharp(\rho(u), D) = \sharp(C, D)$ and $d(\mathcal{N}) = d(\mathcal{M}^*)$.

Now u is irreflexive, so $u \sqsubset_2 D$ and $\neg(D \sqsubset_2 u)$. Applying the mirror image of lemma 6.20, we obtain irreflexive $v \in \mathcal{M}_2$ with $u \sqsubseteq_2 v \sqsubset_2 D$, where $\rho(v)$ and D contain $\equiv^{\mathcal{M}_2}_{\mathcal{B}\Psi}$-equivalent points. But $\rho(u)$ and D do not contain $\equiv^{\mathcal{M}_2}_{\mathcal{B}\Psi}$-equivalent points, since $(\rho(u), D)$ is a defect in $\mathcal{N}$. It follows that $u \neq v$. We conclude that $u \sqsubset_2 v \sqsubset_2 D$.

Let $\mathcal{N}'$ be the submodel of $\mathcal{M}_2$ consisting of $\mathcal{N}$ together with $R_2(v)$. We let $\sqsubset$ denote $\sqsubset_2 \restriction \mathcal{N}'$. So

$$\{v \in \mathcal{N}' : \rho(u) \sqsubset v \sqsubset D\} = \rho(u) \cup \lambda(v) \cup \{v\} \cup \rho(v) \cup D, \tag{6.6}$$

again shown left to right in $\sqsubset$-order. (Possibly, $\rho(u) = \lambda(v)$ or $\rho(v) = D$ or both.) Outside the range $\rho(u)$–D shown in (6.6), $\mathcal{N}'$ has the same defects as $\mathcal{N}$. Within this range, the possible defects in $\mathcal{N}'$ are $(\rho(u), \lambda(v))$ and $(\rho(v), D)$. Lemma 6.20 provides that $(\rho(v), D)$ is not a defect in $\mathcal{N}'$ and that $\sharp(\rho(u), \lambda(v)) < \sharp(\rho(u), D)$. It follows that $d(\mathcal{N}') < d(\mathcal{N}) = d(\mathcal{M}^*)$. Since again, $\mathcal{N}'$ is good and $\mathcal{M} \subseteq \mathcal{N}' \subseteq \mathcal{M}_2$, this contradicts the minimality of $d(\mathcal{M}^*)$, and completes the proof. $\square$

6.4 Models $\mathcal{M}_3$ and $\mathcal{M}_4$

Our final model $\mathcal{M}_4$ will be a perfect submodel of $\mathcal{M}_2$ obtained by lemma 6.22 from a good submodel $\mathcal{M}_3$ satisfying φ_0 that we have to construct first. So we do a selective filtration of $\mathcal{M}_2$ to deliver our first verifiably finite model: an R_2-generated submodel $\mathcal{M}_3$ of $\mathcal{M}_2$ satisfying φ_0. We select the points of $\mathcal{M}_2$ to include in $\mathcal{M}_3$ in three steps.

1. Select the whole of $C_\infty/{\sim}$ and $C_{-\infty}/{\sim}$, which by lemma 6.10 are finite $\sqsubset_2$-clusters and R_2-generated subsets of W_2, and add them to $\mathcal{M}_3$.
2. Now consider in turn each $\psi \in \Psi$ (if any) such that $F\neg\psi \wedge FG\psi$ is satisfied in $\mathcal{M}_2$. Here, Ψ is as in theorem 6.1. Choose any $w \in W_2$ with $\mathcal{M}_2, w \models F\neg\psi \wedge FG\psi$. By lemma 6.8, $\mathcal{M}_2, w \models F(G\psi \wedge HF\neg\psi)$, so choose $u \in W_2$ with $\mathcal{M}_2, u \models G\psi \wedge HF\neg\psi$ (and with $u \sqsupset_2 w$). Select the whole finite set $R_2(u)$ and add it to $\mathcal{M}_3$.
3. Also do the mirror image of step 2.

Clearly, $\mathcal{M}_3$ is a non-empty good submodel of $\mathcal{M}_2$. So by lemma 6.22, we may choose a perfect model $\mathcal{M}_4$ with $\mathcal{M}_3 \subseteq \mathcal{M}_4 \subseteq \mathcal{M}_2$. We have arrived at our final model. We show first that truth of formulas in Ψ is preserved between it and $\mathcal{M}_2$.

LEMMA 6.23. *For every $w \in W_4$ and $\psi \in \Psi$ we have $\mathcal{M}_4, w \models \psi$ iff $\mathcal{M}_2, w \models \psi$.*

Proof. The proof is by induction on ψ. The argument is fairly standard for selective filtration. If ψ is atomic, it is true because $\mathcal{M}_4$ is a submodel of $\mathcal{M}_2$, and the boolean cases are straightforward (note here and below that Ψ is closed under subformulas, so the inductive hypothesis applies to subformulas of ψ). The case $\Box\psi$ is also straightforward, because $\mathcal{M}_4$ is an R_2-generated submodel of $\mathcal{M}_2$. The main cases are formulas in Ψ of the form $G\psi$ and $H\psi$. Then $\psi \in \Psi$; inductively assume the result for ψ.

If $\mathcal{M}_2, w \models G\psi$, take arbitrary $u \sqsupset_4 w$ in $\mathcal{M}_4$. Then $u \sqsupset_2 w$ because $\mathcal{M}_4 \subseteq \mathcal{M}_2$. So $\mathcal{M}_2, u \models \psi$. Inductively, $\mathcal{M}_4, u \models \psi$. Since u was arbitrary, $\mathcal{M}_4, w \models G\psi$.

Assume now that $\mathcal{M}_2, w \models \neg G\psi$ — i.e., $\mathcal{M}_2, w \models F\neg\psi$. It suffices to find $v \in W_3$ with $w \sqsubset_2 v$ and $\mathcal{M}_2, v \models \neg\psi$. For then, we have $v \in W_4$ (since $W_3 \subseteq W_4$), $w \sqsubset_4 v$ (since $\mathcal{M}_4 \subseteq \mathcal{M}_2$), and $\mathcal{M}_4, v \models \neg\psi$ (by the inductive hypothesis). So v is a witness to $\mathcal{M}_4, w \models \neg G\psi$.

There are two cases. The first is when $\mathcal{M}_2, w \models GF\neg\psi$. By lemma 6.10, for every $u \in C_\infty/{\sim}$ we have $u \sqsupset_2 w$, so $\mathcal{M}_2, u \models F\neg\psi$ and there is $v \in W_2$ with $v \sqsupset_2 u$ and $\mathcal{M}_2, v \models \neg\psi$. By lemma 6.10, $v \in C_\infty/{\sim}$ as well. So by construction, $v \in W_3$, and by transitivity, $w \sqsubset_2 v$ as required.

The second case is when $\mathcal{M}_2, w \models F\neg\psi \wedge FG\psi$. By definition of $\mathcal{M}_3$, there is $u \in W_2$ with $\mathcal{M}_2, u \models G\psi \wedge HF\neg\psi$ and $R_2(u) \subseteq W_3$. If $w \sqsupseteq_2 u$ then $\mathcal{M}_2, w \models G\psi$, a contradiction. So by prelinearity, $w \sqsubset_2 u$. By lemma 3.3, $G\psi \wedge HF\neg\psi \to \Diamond\neg\psi$ is provable in the system without using the connectedness axiom (or its mirror image). Since all axioms required for the proof are valid in the frame of $\mathcal{M}_2$, and the inference rules preserve frame validity, we see that $G\psi \wedge HF\neg\psi \to \Diamond\neg\psi$ is valid in $\mathcal{M}_2$ and so $\mathcal{M}_2, u \models \Diamond\neg\psi$. Choose $v \in R_2(u)$ with $\mathcal{M}_2, v \models \neg\psi$. Then $v \in W_3$ (since $R_2(u) \subseteq W_3$). As $w \sqsubset_2 u \, R_2 \, v$, by axiom 4b we have $w \sqsubset_2 v$, as required.

So in either case, $\mathcal{M}_4, w \models \neg G\psi$ as required. The case of $H\psi$ is handled similarly. This completes the induction. □

Our final lemma establishes the main theorem 6.1:

LEMMA 6.24. *The model $\mathcal{M}_4$ is Ψ-linked, and φ_0 is satisfied in $\mathcal{M}_4$.*

Proof. We refer to definition 5.4 for the meaning of 'Ψ-linked'. Plainly, W_4 is finite. We check that the frame $(W_4, \sqsubset_4, R_4)$ of $\mathcal{M}_4$ validates all axioms of §3.1 excluding the connectedness axiom 5 and its mirror image. All these axioms are valid in the frame of $\mathcal{M}_2$, so their first-order correspondents are true in this frame. The correspondents of all axioms except temporal density and axiom 4d are universal first-order sentences, and so remain true in the frame of $\mathcal{M}_4$ which is a substructure of the frame of $\mathcal{M}_2$. The correspondent of axiom 4d is preserved by R-generated subframes and so remains true in the frame of $\mathcal{M}_4$. The frame of $\mathcal{M}_4$ consists of $\sqsubset_4$-clusters sometimes interspersed by single irreflexive points. It is plain from this that $\sqsubset_4$ is dense, so the density axiom is valid in the frame.

Since $\mathcal{M}_4 \subseteq \mathcal{M}_2$, and $\sqsubset_2$ is prelinear (by lemma 6.9), $\sqsubset_4$ is prelinear too.

Let $C \subseteq W_4$ be a $\sqsubset_4$-cluster. Since $\mathcal{M}_4$ is good, C is also a $\sqsubset_2$-cluster of $\mathcal{M}_2$. By lemma 6.12, the frame $(C, R_2 \restriction C)$ is connected. But this frame is $(C, R_4 \restriction C)$, since $\mathcal{M}_4 \subseteq \mathcal{M}_2$. So $(C, R_4 \restriction C)$ is connected.

As $\mathcal{M}_4$ is perfect, for every pair (C, D) of adjacent $\sqsubset_4$-clusters there are $c \in C$ and $d \in D$ with $c \equiv^{\mathcal{M}_2}_{\mathcal{B}\Psi} d$. By lemma 6.23, $c \equiv^{\mathcal{M}_4}_{\mathcal{B}\Psi} d$ as well. So $\mathcal{M}_4$ is Ψ-linked.

By lemma 6.7, φ_0 is satisfied in $\mathcal{M}_2$. Pick any $w \in C_\infty/\sim$. Then $\mathcal{M}_2, w \models P\varphi_0$. Recall that $P\varphi_0 \in \Psi$. Then $w \in W_4$ and by lemma 6.23, $\mathcal{M}_4, w \models P\varphi_0$. It follows that φ_0 is satisfied in $\mathcal{M}_4$. □

7 From linked models to real models

In §§7.1–7.3 we provide a simple way to build maps defined on intervals of $\mathbb{R}$. As far as we know, the method essentially originates in [4]. We will use it in §7.4 to 'represent' any connected S4-frame over $\mathbb{R}$, and in §7.5 to 'represent' any Ψ-linked Kripke model over $\mathbb{R}$ in a way that respects the formulas in Ψ. An example of the construction for the formula θ of (4.2) will be given in §7.6.

Linear orders, intervals, and other related notions were introduced in §2.5. In particular, recall that we often write a linear order $(I, <)$ simply as I. In this section, we will write ordered pairs in the form $\langle i, j\rangle$ where they might be confused with intervals.

7.1 Lexicographic sums of linear orders

Let $(J, <)$ be a linear order, and for each $j \in J$ let I_j be an interval of $\mathbb{R}$. (More generally, I_j can be any linear order, but we are concerned only with the case of intervals of $\mathbb{R}$.) We write

$$I = \sum_{j \in J} I_j = \{\langle i, j\rangle : j \in J,\ i \in I_j\},$$

and define an order $<$ on I lexicographically by $\langle i, j\rangle < \langle i', j'\rangle$ iff $j < j'$ or ($j = j'$ and $i < i'$). Clearly, $(I, <)$ is a linear order. It can be thought of as the linear order obtained from $(J, <)$ by replacing each $j \in J$ by a copy of I_j. If $(J, <) = (\{0, 1, \ldots, n\}, <)$ for some $n \in \mathbb{N}$, we can write I explicitly as $I_0 + \cdots + I_n$. It can be thought of as a copy of I_0 followed by copies of $I_1, \ldots, I_n$ in order.

When $J = \{0, 1\}$, it is plain that if I_0 has a greatest element and I_1 has no least element, or if I_0 has no greatest element and I_1 has a least element, then $I_0 + I_1$ is order-isomorphic to an interval of $\mathbb{R}$. For example, $(0, 1] + (0, 1] \cong (0, 1) + [0, 1] \cong (0, 1]$. More generally:

PROPOSITION 7.1. *Suppose that one of the following holds.*

1. *J is finite, say $(J, <) = (\{0, 1, \ldots, n\}, <)$ for some $n \in \mathbb{N}$. I_0 has no least or greatest element. Each I_j for $j > 0$ has a least element but no greatest element.*
2. *$(J, <) = (\mathbb{Z}, <)$, each I_j (j odd) has no least or greatest element, and each I_j (j even) is a singleton.*
3. *$(J, <) = (\mathbb{R}, <)$, each I_j has a least and a greatest element, and I_j is a singleton for every irrational j.*

Then $(\sum_{j \in J} I_j, <) \cong (\mathbb{R}, <)$.

Proof (sketch). A linear order is isomorphic to $(\mathbb{R}, <)$ iff it is dense, has no endpoints, has a countable dense subset, and is Dedekind complete. It is well known and easy to check that each of the three sum-orders above has these properties. See [4, 27] for more information. $\square$

When J and the I_j meet one of the conditions in proposition 7.1, the linear order $(I, <)$ is isomorphic to $(\mathbb{R}, <)$, and we will generally identify the two. Sometimes we will identify $(I, <)$ with an open interval of $\mathbb{R}$. It will always be stated explicitly when these identifications are made.

7.2 Functions on linear orders

We continue to let $(J, <)$ be a linear order, I_j ($j \in J$) an interval of $\mathbb{R}$, and $I = \sum_{j \in J} I_j$. Let W be a non-empty set, and for each $j \in J$ let $f_j : I_j \to W$ be a map. We define a map

$$f = \sum_{j \in J} f_j \quad : \quad I \to W$$

$$\text{by} \qquad f(\langle i, j \rangle) \;=\; f_j(i).$$

In the case where $(J, <) = (\{0, 1, \ldots, n\}, <)$, we may write the sum explicitly as $f_0 + \cdots + f_n$. If I_j is a singleton $\{x\}$ and $f_j(x) = s$, say, we may write the map f_j simply as s.

This 'sum' notation for functions should not be confused with (for example) the pointwise sum of real-valued functions, and in fact, in our applications the set W will be the set of worlds of a Kripke frame and will have no '+' operation defined on it.

EXAMPLE 7.2. If $w_0, \ldots, w_n \in W$ then, modulo a renaming of the elements of its domain, $w_0 + \cdots + w_n$ is the map $f : \{0, \ldots, n\} \to W$ given by $f(i) = w_i$ for each $i \leq n$.

For $j \in J$ we define $\mathrm{dom}_f(f_j) = I_j \times \{j\} \subseteq I$. We will sometimes regard f_j as a map $f_j : \mathrm{dom}_f(f_j) \to W$, via $\langle i, j\rangle \mapsto f_j(i)$ for each $\langle i, j\rangle \in \mathrm{dom}_f(f_j)$. In effect, we identify f_j with $f \restriction \mathrm{dom}_f(f_j)$.

The notation $\mathrm{dom}_f(f_j)$ is more convenient than the plain $I_j \times \{j\}$ for two reasons. First, in several places we will not have explicit notation for the I_j, and the notation saves us from the need to introduce any. Second, we will frequently be identifying I with $\mathbb{R}$ via some tacit order isomorphism $\rho : I \to \mathbb{R}$, and we will carry over the notation $\mathrm{dom}_f(f_j)$ via the identification. Formally, when this identification is in operation,

1. we will identify f with the map $f \circ \rho^{-1} : \mathbb{R} \to W$,
2. we will redefine $\mathrm{dom}_f(f_j)$ to denote the interval $\rho(I_j \times \{j\})$ of $\mathbb{R}$,
3. we will sometimes identify f_j with the map $f \circ \rho^{-1} \restriction \rho(I_j \times \{j\})$ defined on this interval.

7.3 Shuffles

There is an important special case known as the shuffle. Reynolds [26] described a shuffle as a 'thorough mixture' of its ingredients. Keeping W as above, let $\mathcal{G}$ be a countable (possibly empty) set of maps of the form $g : K_g \to W$, where K_g is an interval of $\mathbb{R}$ with a least and a greatest element. Suppose also that $g_0 : K_{g_0} \to W$ is a map, where K_{g_0} is a singleton interval of $\mathbb{R}$. Choose any $\theta : \mathbb{R} \to \mathcal{G} \cup \{g_0\}$ such that $\theta(j) = g_0$ for every irrational j, and $\theta^{-1}(g)$ is a dense subset of $\mathbb{Q}$ for each $g \in \mathcal{G}$. This is not difficult to do. Then $\theta^{-1}(g)$ is dense in $\mathbb{R}$ for every $g \in \mathcal{G} \cup \{g_0\}$. Now define $I_j = K_{\theta(j)}$ for each $j \in \mathbb{R}$, so that $\theta(j) : I_j \to W$, and let

$$I = \sum_{j \in \mathbb{R}} I_j, \qquad \sigma = \sum_{j \in \mathbb{R}} \theta(j) \ : I \to W.$$

Then $\sigma(\langle i, j\rangle) = (\theta(j))(i) \in W$.

An element $x \in I$ is said to be a *σ-endpoint* if it is of the form $\langle i, j\rangle$, where $j \in \mathbb{R}$ and i is the least or greatest element of I_j.

LEMMA 7.3. *Let I, σ be as above, let $x, y, z \in I$ with $y < x < z$, and suppose that x is a σ-endpoint. Then $\sigma((y, z)) = \mathrm{rng}(\sigma)$.*

Proof. We show that $\mathrm{rng}\,\sigma \subseteq \sigma((y, z))$ (the converse inclusion is trivial). Fix arbitrary $s \in \mathrm{rng}\,\sigma$. Pick $g \in \mathcal{G} \cup \{g_0\}$ and $k \in K_g$ with $g(k) = s$. Suppose $x = \langle i, j\rangle$, say, and suppose that i is the least element of I_j (the case where it is the greatest element of I_j is similar). If $y = \langle i', j'\rangle$, say, then we must have $j' < j$. Now $\theta^{-1}(g)$ is dense in $\mathbb{R}$, so we may pick $j^* \in \theta^{-1}(g)$ with $j' < j^* < j$. Then $y < \langle k, j^*\rangle < z$ and $s = \sigma(\langle k, j^*\rangle) \in \sigma((y, z))$ as required. □

COROLLARY 7.4. *Let I, σ be as above. Then $\sigma^{-1}(s)$ is unbounded in I for each $s \in \mathrm{rng}(\sigma)$.*

Proof. This follows from the lemma, as the set of σ-endpoints is unbounded in I. □

By proposition 7.1(3), the linear order $(I, <)$ is isomorphic to $(\mathbb{R}, <)$, so by choosing a suitable isomorphism we can regard σ as a map $\sigma : \mathbb{R} \to W$. This map depends on the choices of the isomorphism and θ, but any choices will do for us and in fact all choices lead to the same result modulo an automorphism (order-preserving permutation) of $(\mathbb{R}, <)$. So we let

$$\mathrm{Shuffle}\big(\mathcal{G}\ ;\ g_0\big)$$

denote a map $\sigma : \mathbb{R} \to W$ as above, for arbitrary tacit choices of these items. The elements of $\mathcal{G} \cup \{g_0\}$ are called the *ingredients* of the shuffle.

EXAMPLE 7.5. If $a, b, c \in W$ then $\mathrm{Shuffle}(\{a, b\}; c)$ can be taken to be a map $\sigma : \mathbb{R} \to \{a, b, c\}$ such that $\sigma^{-1}(a)$, $\sigma^{-1}(b)$ are dense sets of rationals and $\sigma^{-1}(c) = \mathbb{R} \setminus \mathbb{Q}$.

7.4 S4 frames

We now use lexicographic sums to establish a relative of the McKinsey–Tarski theorem that the logic of $\mathbb{R}$ in the language with $\Box$ is S4 [20]. It will be needed in §7.5. A similar method is used in [10] to prove the McKinsey–Tarski theorem itself, and others. There is also very substantial similarity to the methods used in [20, 24].

DEFINITION 7.6. An *S*4*-frame* is a pair (W, R), where R is a reflexive and transitive binary relation on the non-empty set W.

DEFINITION 7.7. Let $\mathcal{F} = (W, R)$ be an S4-frame, $(I, <)$ a linear order, and $g : I \to W$ a map. We say that an element $x \in I$ is *g-fair (with respect to $\mathcal{F}$)* if there are $y, z \in I$ with $y < x < z$ and such that $g((y', z')) = R(g(x))$ for every $y', z' \in I$ with $y \leq y' < x < z' \leq z$.

To motivate the definition, consider a Kripke model $\mathcal{M} = (W, R, h)$ and a map $g : \mathbb{R} \to W$. Let $h' : PV \to \wp(\mathbb{R})$ be the assignment induced from $\mathcal{M}$ by g, via $h'(p) = g^{-1}(h(p))$ for $p \in PV$. The reader is invited to check that *if every $x \in \mathbb{R}$ is g-fair then g preserves all modal formulas:* $(\mathbb{R}, h'), x \models \varphi$ iff $\mathcal{M}, g(x) \models \varphi$ for every $x \in \mathbb{R}$ and every $\mathcal{L}$-formula φ not involving G or H. See the claim in the proof of lemma 7.25 below.

REMARK 7.8. Clearly, if $\mathcal{F}$ is an R-generated subframe of an S4-frame $\mathcal{G}$, a point $x \in I$ is g-fair with respect to $\mathcal{F}$ iff it is g-fair with respect to $\mathcal{G}$.

Where the meaning is clear from the context, we will usually say simply that x is g-fair.

REMARK 7.9. Fairness is clearly a 'local' property depending only on arbitrarily small neighbourhoods of the point in question. So if $g = \sum_{j \in J} f_j$, $j \in J$, and $x \in \mathrm{dom}_g(f_j)$ is in the interior of $\mathrm{dom}_g(f_j)$ (that is, x is not a least or greatest element of $\mathrm{dom}_g(f_j)$), then x is g-fair iff it is f_j-fair. (Recall here that we identify f_j with $g \restriction \mathrm{dom}_g(f_j)$.)

THEOREM 7.10. *Let $\mathcal{F} = (W, R)$ be a finite connected S4-frame (connected frames were defined in definition 5.2). Then there is a map $g : \mathbb{R} \to W$ satisfying:*

1. *every $x \in \mathbb{R}$ is g-fair (with respect to $\mathcal{F}$),*
2. *$g^{-1}(w)$ is unbounded in $\mathbb{R}$ for every $w \in W$.*

Proof. Recall from definition 2.1 that $R^\bullet wu$ means that $Rwu \wedge \neg Ruw$. As $\mathcal{F}$ is finite, we can define for each $w \in W$ a map $\nu_w : \mathbb{R} \to W$ by complete induction on $|R(w)|$:

$$\nu_w = \text{Shuffle}\big(\{w + \nu_u + w : u \in R^\bullet(w)\} \cup \{u : Rwu \wedge Ruw\} \ ; \ w\big).$$

This is well defined because (a) for each $u \in R^\bullet(w)$, since R is transitive we have $R(u) \subseteq R(w)$, and plainly $w \in R(w) \setminus R(u)$, so $|R(u)| < |R(w)|$ and hence ν_u is defined inductively, (b) the domain of each map $w + \nu_u + w$ and of each map u can be taken to be an interval of $\mathbb{R}$ with a least and a greatest point (for u it is a singleton interval), and (c) the domain of the map w is some singleton interval of $\mathbb{R}$.

Claim. For each $w \in W$:

1. $\text{rng}(\nu_w) = R(w)$.
2. Every $x \in \mathbb{R}$ is ν_w-fair.

Proof of claim. The proof is by complete induction on $|R(w)|$. Inductively assume the claim for all $u \in R^\bullet(w)$. For part 1, $\text{rng}(\nu_w)$ is clearly the union of the ranges of the ingredients of the shuffle defining ν_w. For any $u \in W$, the range of the map u is just $\{u\}$. So using the inductive hypothesis,

$$\begin{array}{rcl}\text{rng}(\nu_w) & = & \big(\bigcup_{u \in R^\bullet(w)} (\{w\} \cup \text{rng}(\nu_u) \cup \{w\})\big) \cup \{u : Rwu \wedge Ruw\} \cup \{w\} \\ & = & \big(\bigcup_{u \in R^\bullet(w)} R(u)\big) \cup \{u : Rwu \wedge Ruw\} \cup \{w\} \\ & = & R(w).\end{array}$$

This proves part 1. For part 2, take $x \in \mathbb{R}$. We show that x is ν_w-fair. If x is a ν_w-endpoint (see §7.3), suppose that $\nu_w(x) = u$, say. The definition of ν_w tells us that either $u = w$, or Rwu and Ruw. In both cases, by transitivity of R we have $R(u) = R(w)$. By lemma 7.3 and part 1, whenever $y < x < z$ we have

$$\nu_w((y, z)) = \text{rng}(\nu_w) = R(w) = R(u).$$

It follows that x is ν_w-fair. If x is not a ν_w-endpoint, then x is in the interior of the domain of ν_u for some $u \in R^\bullet(w)$. Inductively, x is ν_u-fair, and hence (see remark 7.9) it is ν_w-fair as well. This proves the claim.

Now $\mathcal{F}$ is connected and R is reflexive and transitive. It follows that $\mathcal{F}$ is path-connected in the sense that for each $u, v \in W$ there are $w_0, \ldots, w_n \in W$ (for some finite n) with $w_0 = u$, $w_n = v$, $R(w_i, w_{i+1})$ for each even $i < n$, and $R(w_{i+1}, w_i)$ for each odd $i < n$. Using this and the finiteness of $\mathcal{F}$, it is straightforward to find an infinite zigzag path through $\mathcal{F}$ that visits each point infinitely often. Formally, there are $a_i, b_i \in W$ $(i \in \mathbb{Z})$ satisfying:

- Ra_ib_{i-1} and Ra_ib_i for each $i \in \mathbb{Z}$,
- for each $w \in W$, the set $\{i \in \mathbb{Z} : a_i = w\}$ is unbounded in $\mathbb{Z}$.

We would like to define $g = \sum_{i\in\mathbb{Z}}(a_i + \nu_{a_i} + a_i + \nu_{b_i})$, but the notation may be easier to follow if we avoid nested sums. So for each $i \in \mathbb{Z}$, define functions as follows:

$$\begin{array}{rclcl} f_{4i} & = & f_{4i+2} & = & a_i, \\ & & f_{4i+1} & = & \nu_{a_i}, \\ & & f_{4i+3} & = & \nu_{b_i}. \end{array}$$

We now define our desired map $g : \mathbb{R} \to W$ by

$$g = \sum_{j\in\mathbb{Z}} f_j.$$

By proposition 7.1(2), $\mathrm{dom}(g)$ is order-isomorphic to $\mathbb{R}$. As usual, we identify the two, and identify each restriction $g \restriction \mathrm{dom}_g(f_j)$ of g with f_j.

Let $x \in \mathbb{R}$ be arbitrary. We show that it is g-fair. Fix the unique $j \in \mathbb{Z}$ such that $x \in \mathrm{dom}_g(f_j)$. There are four cases. If $j = 4i+1$ for some $i \in \mathbb{Z}$, then $f_j = \nu_{a_i}$. By the claim, x is ν_{a_i}-fair, and hence (remark 7.9) it is g-fair. The case where $j = 4i+3$ for some $i \in \mathbb{Z}$ is similar.

Suppose $j = 4i+2$ for some $i \in \mathbb{Z}$. So f_j is the map a_i, $g(x) = a_i$, and the maps $f_{j-1} = \nu_{a_i}$ and $f_{j+1} = \nu_{b_i}$ are shuffles. Take any $y \in \mathrm{dom}_g(f_{j-1})$ and $z \in \mathrm{dom}_g(f_{j+1})$, so that $y < x < z$. By applying corollary 7.4 and part 1 of the claim to ν_{a_i} and ν_{b_i}, we see that

$$\begin{aligned} g((y,x)) &= \mathrm{rng}(\nu_{a_i}) = R(a_i), \\ g((x,z)) &= \mathrm{rng}(\nu_{b_i}) = R(b_i). \end{aligned}$$

Consequently, $g((y,z)) = g((y,x)) \cup \{g(x)\} \cup g((x,z)) = R(a_i) \cup \{a_i\} \cup R(b_i)$. But $a_i \in R(a_i)$, and moreover, since Ra_ib_i, transitivity yields $R(b_i) \subseteq R(a_i)$. So this union is just $R(a_i)$ — i.e., $R(g(x))$. Thus, $g((y,z)) = R(g(x))$ for every $y \in \mathrm{dom}_g(f_{j-1})$ and $z \in \mathrm{dom}_g(f_{j+1})$. Since all sufficiently large $y < x$ lie in $\mathrm{dom}_g(f_{j-1})$ and all sufficiently small $z > x$ lie in $\mathrm{dom}_g(f_{j+1})$, it follows that x is g-fair as required.

A similar argument covers the case where $j = 4i$ for some $i \in \mathbb{Z}$. We simply note that the left and right neighbours of x are then $\nu_{b_{i-1}}$ and ν_{a_i}, respectively, and that Ra_ib_{i-1}. So in all cases, x is g-fair. This proves part 1 of the theorem.

For part 2, let $w \in W$ and $r \in \mathbb{R}$ be given. The set $\{i \in \mathbb{Z} : a_i = w\}$ is unbounded in $\mathbb{Z}$, so we can take $i, j \in \mathbb{Z}$ such that $a_i = w$, $j = 4i$ (so $f_j = a_i$), and so large that $r < x$ for the unique $x \in \mathrm{dom}_g(f_j)$. Then $g(x) = a_i = w$. So $g^{-1}(w)$ has no upper bound, and a symmetrical argument shows that it has no lower bound either. □

7.5 Representing Ψ-linked models

Here, we prove the main result of this section:

THEOREM 7.11. *Let Ψ be a finite set of $\mathcal{L}$-formulas closed under subformulas, let $\varphi \in \Psi$, and suppose that φ is satisfied in some Ψ-linked Kripke model. Then φ is satisfiable over $\mathbb{R}$.*

To prove it, fix φ, Ψ as in the statement of the theorem, and let $\mathcal{M} = (W, \sqsubset, R, h)$ be a Ψ-linked Kripke model in which φ is satisfied. We will use theorem 7.10 to define (in definition 7.18) a map $g : \mathbb{R} \to W$ that will induce a model over $\mathbb{R}$, and we will then prove that g preserves all formulas in Ψ. The theorem will follow from this.

To define g, we need to set out some terminology.

7.5.1 Clusters in the model

Since $\mathcal{M}$ is Ψ-linked (see definition 5.4), $\sqsubset$ is prelinear and lemma 5.5 applies to the frame of $\mathcal{M}$. Thus we may enumerate, without repetitions, the $\sqsubset$-clusters in $\mathcal{M}$ as $C_0, \ldots, C_k$ for some $k \geq 0$, with

$$C_0 \sqsubset C_1 \sqsubset \cdots \sqsubset C_k.$$

Let $i < k$. Then (C_i, C_{i+1}) is a pair of consecutive $\sqsubset$-clusters. As $\mathcal{M}$ is Ψ-linked, there are two possibilities.

1. C_i and C_{i+1} are not adjacent, so there is $u \in \mathcal{M}$ with $C_i \sqsubset u \sqsubset C_{i+1}$ and $u \notin C_i \cup C_{i+1}$. Then u is not in any $\sqsubset$-cluster, so is irreflexive. We must have $C_i = \lambda(u)$ and $C_{i+1} = \rho(u)$. It follows that u is unique. We define u_i to be this u, and we say that i is *open,* C_i is *right-open,* and C_{i+1} is *left-open.*
2. C_i and C_{i+1} are adjacent, so as $\mathcal{M}$ is Ψ-linked, we can select $d_i \in C_i$ and $s_{i+1} \in C_{i+1}$ with $d_i \equiv^{\mathcal{M}}_{\mathcal{B}\Psi} s_{i+1}$. In this case, we say that i is *closed,* C_i is *right-closed,* and C_{i+1} is *left-closed.*

We also say that C_0 is left-open and C_k right-open. (We do not define k itself as either open or closed.)

7.5.2 The maps f_i and f'_i

The map $g : \mathbb{R} \to W$ will be made from maps $f_0, \ldots, f_k$ and $f'_0, \ldots, f'_{k-1}$. Up to an order automorphism of $\mathbb{R}$, we will have $g = \sum_{i<k}(f_i + f'_i) + f_k$, but it may help the reader if we provide more specific notation for the domains of the components of this sum. So we choose elements

$$-\infty = l_0 < r_0 \leq l_1 < r_1 \leq l_2 < r_2 \leq \cdots < r_{k-1} \leq l_k < r_k = \infty \tag{7.1}$$

in $\mathbb{R} \cup \{\pm\infty\}$, such that $r_i = l_{i+1}$ iff i is open, for each $i < k$. We will use these elements to define

- surjective maps $f_i : (l_i, r_i) \to C_i$ for each $i \leq k$,
- 'filler' maps $f'_i : [r_i, l_{i+1}] \to W$ for each $i < k$.

Unfortunately, the definitions involve a number of cases, because the C_i come in four kinds: left- or right-open, and left- or right-closed.

DEFINITION 7.12. First we define the maps f_i for $i \leq k$. Fix such an i. We will actually define f_i by way of an auxiliary map, f^*_i. First note that because $\mathcal{M}$ is Ψ-linked, $(C_i, R \restriction C_i)$ is a finite connected S4-frame.

1. If C_i is left-open and right-open then, observing that (l_i, r_i) is order-isomorphic to $\mathbb{R}$, use theorem 7.10 to choose a map $f_i^* : (l_i, r_i) \to C_i$ satisfying the stated properties — to wit, every $x \in (l_i, r_i)$ is f_i^*-fair with respect to $(C_i, R \restriction C_i)$, and $(f_i^*)^{-1}(w)$ is unbounded in (l_i, r_i) for every $w \in C_i$.
2. If C_i is left-open and right-closed, first use theorem 7.10 to choose a map $f_i^* : (l_i, \infty) \to C_i$ with the stated properties. The properties ensure that there is $x > l_i$ with $f_i^*(x) = d_i$. By some scaling, we can suppose that $x = r_i$.
3. If C_i is left-closed and right-open, we use a mirror image argument to choose a map $f_i^* : (-\infty, r_i) \to C_i$ with $f_i^*(l_i) = s_i$.
4. If C_i is left-closed and right-closed, we combine the preceding two cases, with a little extra work to ensure surjectivity. Using theorem 7.10, choose a map $f_i^* : \mathbb{R} \to C_i$ with the stated properties. By the properties and the finiteness of C_i, there are $x < y$ in $\mathbb{R}$ with $f_i^*(x) = s_i$, $f_i^*(y) = d_i$, and $f_i^*((x, y)) = C_i$. By scaling, we can assume that $x = l_i$ and $y = r_i$.

We now define $f_i = f_i^* \restriction (l_i, r_i)$.

Before moving on, we collect some facts about the f_i.

LEMMA 7.13. *Let $i \leq k$ and $x \in (l_i, r_i)$. If C_i is left-open then $f_i((l_i, x)) = C_i$. If C_i is right-open then $f_i((x, r_i)) = C_i$.*

Proof. Suppose that C_i is left-open. Certainly, $f_i((l_i, x)) \subseteq \operatorname{rng}(f_i^*) = C_i$. To prove the converse inclusion, let $w \in C_i$ be given. Because C_i is left-open, $\operatorname{dom}(f_i^*)$ has the form (l_i, z) where either $z = r_i$ or $z = \infty$ (see definition 7.12(1,2)). By theorem 7.10, $(f_i^*)^{-1}(w)$ is unbounded in $\operatorname{dom}(f_i^*)$, so there is $y \in (l_i, x)$ with $f_i^*(y) = w$. By definition, $f_i = f_i^* \restriction (l_i, r_i)$, so $f_i(y) = w$ and hence $w \in f_i((l_i, x))$. Since w was arbitrary, $C_i \subseteq f_i((l_i, x))$, proving the first part. The second part is a mirror image. $\square$

COROLLARY 7.14. *For each $i \leq k$, the map $f_i : (l_i, r_i) \to C_i$ is surjective.*

Proof. This is immediate from lemma 7.13 except when C_i is left-closed and right-closed (case 4 of definition 7.12). But in that case, we arranged explicitly that $f_i((l_i, r_i)) = C_i$. $\square$

LEMMA 7.15. *Suppose that $i < k$ is closed. Then $f_i((y, r_i)) \subseteq R(d_i)$ for some $y \in (l_i, r_i)$, and $f_{i+1}((l_{i+1}, z)) \subseteq R(s_{i+1})$ for some $z \in (l_{i+1}, r_{i+1})$.*

Proof. Here, C_i is right-closed, C_{i+1} is left-closed, and d_i, s_{i+1} are defined. By theorem 7.10, r_i is f_i^*-fair, so there are $y, z \in \operatorname{dom}(f_i^*)$ with $l_i < y < r_i < z$ and $f_i^*((y, z)) = R(f_i^*(r_i)) = R(d_i)$. Then $f_i((y, r_i)) = f_i^*((y, r_i)) \subseteq f_i^*((y, z)) = R(d_i)$. This proves the first part of the lemma, and the second part is a mirror image. $\square$

DEFINITION 7.16. Next we define the maps $f_i' : [r_i, l_{i+1}] \to W$ for each $i < k$.

1. If i is open, then $r_i = l_{i+1}$ and u_i is defined. Define f_i' simply by $f_i'(r_i) = u_i$.

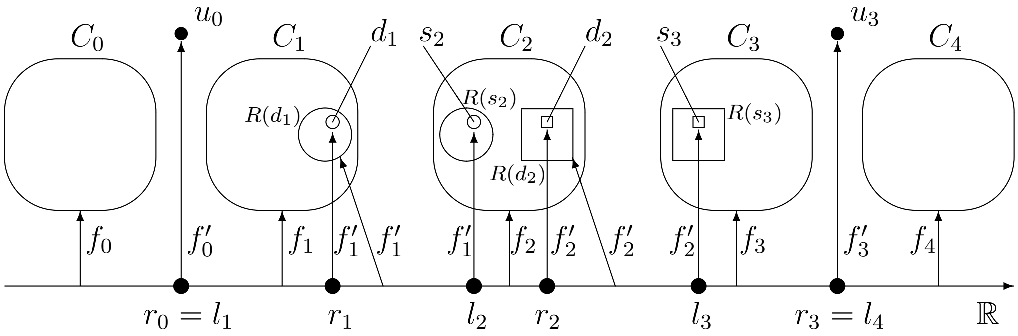

Figure 2: Example of parts of $g : \mathbb{R} \to \mathcal{M}$ when $k = 4$

2. If i is closed, then C_i is right-closed, C_{i+1} is left-closed, and d_i, s_{i+1} are defined. Plainly, $(R(d_i), R \restriction R(d_i))$ is a finite S4-frame. It is trivially connected, because if $R(d_i)$ is the union of disjoint R-generated subsets X, Y, then supposing without loss of generality that $d_i \in X$, we have $R(d_i) \subseteq X$ and so $Y = \emptyset$. Noting that (r_i, l_{i+1}) is order-isomorphic to $\mathbb{R}$ in this case, we may therefore choose a map $f'_i : (r_i, l_{i+1}) \to R(d_i)$ satisfying the conditions of theorem 7.10 — to wit, each $x \in (r_i, l_{i+1})$ is f'_i-fair with respect to $(R(d_i), R \restriction R(d_i))$, and $f'^{-1}_i(w)$ is unbounded in (r_i, l_{i+1}) for each $w \in R(d_i)$. We extend f'_i to the whole of $[r_i, l_{i+1}]$ by defining $f'_i(r_i) = d_i$ and $f'_i(l_{i+1}) = s_{i+1}$. (So in this case, $f'_i([r_i, l_{i+1})) \subseteq C_i$ but $f'_i(l_{i+1}) \in C_{i+1}$.)

LEMMA 7.17. *Suppose that $i < k$ is closed and let $x \in (r_i, l_{i+1})$. Then $f'_i((r_i, x)) = f'_i((x, l_{i+1})) = R(d_i)$.*

Proof. As in lemma 7.13, theorem 7.10's conditions imply that $(r_i, l_{i+1}) \cap f'^{-1}_i(w)$ is unbounded in (r_i, l_{i+1}) for each $w \in R(d_i)$, from which the lemma follows. $\square$

7.5.3 The map g

DEFINITION 7.18. We finally define $g = (\bigcup_{i \leq k} f_i) \cup (\bigcup_{i<k} f'_i) : \mathbb{R} \to W$. That is, for each $x \in \mathbb{R}$,

$$g(x) = \begin{cases} f_i(x), & \text{if } x \in (l_i, r_i) \text{ for some } i \leq k, \\ f'_i(x), & \text{if } x \in [r_i, l_{i+1}] \text{ for some } i < k. \end{cases}$$

This is plainly well defined.

EXAMPLE 7.19. An example of the construction of g is shown in figure 2. In the figure, C_0 and C_4 are left- and right-open, C_1 is left-open and right-closed, C_2 is left- and right-closed, and C_3 is left-closed and right-open. The small circles inside C_1 and C_2 are d_1 and s_2, respectively, and the big circles are $R(d_1)$ and $R(s_2)$, respectively. Similarly, the element d_2 is the small square inside C_2, and the large square is $R(d_2)$. The large square inside C_3 is $R(s_3)$ and s_3 is the small square inside it. We can see that $R(d_1)$ is used in a sense as intervening material for g between C_1 and C_2 via f'_1, and similarly with $R(d_2)$. For an example of how the construction produces a model of φ over $\mathbb{R}$, see §7.6.

7.5.4 Properties of g

We establish a few properties of g, useful below.

LEMMA 7.20. *For each $i \leq k$,*

$$Y_i = \{x \in \mathbb{R} : g(x) \sqsubset C_i\} = \begin{cases} (-\infty, l_{i+1}), & \text{if } i < k, \\ \mathbb{R}, & \text{if } i = k. \end{cases}$$

Proof. If $i = k$, the result is trivial since $W \sqsubset C_k$. Let $i < k$. If $x < l_{i+1}$, then $g(x)$ is $f_j(x)$ or $f'_j(x)$ for some $j \leq i$, so by inspection, $g(x) \in C_j$ for some $j \leq i$, or $g(x) = u_j$ for some $j < i$ (we have $j \neq i$ since $x \neq l_{i+1}$). It follows that $g(x) \sqsubset C_i$. If $x \geq l_{i+1}$ then $g(x) = u_j$ for some $j \geq i$ or $g(x) \in C_j$ for some $j > i$, so $\neg(g(x) \sqsubset C_i)$. □

LEMMA 7.21. *The map $g : \mathbb{R} \to W$ is surjective and order preserving: if $x < y$ in $\mathbb{R}$ then $g(x) \sqsubset g(y)$.*

Proof. Surjectivity is immediate from corollary 7.14 and the fact that $g(r_i) = u_i$ for every $i < k$ such that u_i is defined. We check that g is order preserving. Assume $x < y$. Suppose first that $g(y) \in C_i$ for some $i \leq k$. Then $y \in Y_i$. By lemma 7.20, Y_i is closed downwards under $<$, so $x \in Y_i$ as well. Hence $g(y) \in C_i \sqsupset g(x)$ as required. If on the other hand $g(y) = u_i$ for some $i < k$, then the definition of g yields $y = l_{i+1} > x$, so $x \in Y_i$ by lemma 7.20. Hence $g(x) \sqsubset C_i \sqsubset u_i = g(y)$ as required. □

LEMMA 7.22. *Each $x \in \mathbb{R} \setminus \{r_i, l_{i+1} : i < k\}$ is g-fair with respect to the frame of $\mathcal{M}$.*

Proof. Each such x is in the interior of $\mathrm{dom}(f_i)$ for some $i \leq k$ or the interior of $\mathrm{dom}(f'_i)$ for some $i < k$. As f_i and f'_i were defined using theorem 7.10, in the former case x is f_i-fair with respect to $(C_i, R \restriction C_i)$, and in the latter case x is f'_i-fair with respect to $(R(d_i), R \restriction R(d_i))$. These are R-generated subframes of the frame of $\mathcal{M}$. By remarks 7.8 and 7.9, x is g-fair with respect to the frame of $\mathcal{M}$. □

LEMMA 7.23. *Let $i \leq k$ and $x \in (l_i, r_i)$.*

1. *If C_i is left-open then $g((l_i, x)) = C_i$.*
2. *If C_i is right-open then $g((x, r_i)) = C_i$.*

Proof. Immediate from lemma 7.13, since $g \restriction (l_i, r_i) = f_i$. □

LEMMA 7.24. *Suppose that $i < k$ is closed.*

1. *$g((y, r_i)) \subseteq R(d_i)$ for some $y < r_i$.*
2. *$g((r_i, x)) = g((x, l_{i+1})) = R(d_i)$ for every $x \in (r_i, l_{i+1})$.*
3. *$g((l_{i+1}, z)) \subseteq R(s_{i+1})$ for some $z > l_{i+1}$.*

Proof. Since $g \restriction (l_i, r_i) = f_i$ and $g \restriction (l_{i+1}, r_{i+1}) = f_{i+1}$, parts 1 and 3 follow from lemma 7.15. Part 2 follows from lemma 7.17, since $g \restriction (r_i, l_{i+1}) = f'_i$. □

7.5.5 The model $\mathcal{R}$

We define an assignment h' into $\mathbb{R}$ by $h'(p) = g^{-1}(h(p))$, for each atom p. We let $\mathcal{R} = (\mathbb{R}, h')$. This is our intended model. We now prove a 'truth lemma' for it, from which theorem 7.11 will easily follow.

LEMMA 7.25. *For every $\psi \in \Psi$ and $x \in \mathbb{R}$ we have $\mathcal{R}, x \models \psi$ iff $\mathcal{M}, g(x) \models \psi$.*

Proof. By induction on ψ. The lemma for atomic ψ is immediate from the definition of h', and the boolean cases are easy. The main cases are $G\psi$, $H\psi$, and $\Box\psi$. Since Ψ is closed under subformulas, $\psi \in \Psi$ as well, so inductively assume the lemma for ψ:

$$\mathcal{R}, x \models \psi \iff \mathcal{M}, g(x) \models \psi \quad \text{for every } x \in \mathbb{R} \quad \text{(inductive hypothesis)} \qquad (7.2)$$

First we deal with the temporal operators. If $\mathcal{M}, g(x) \models G\psi$ then take arbitrary $y \in \mathbb{R}$ with $y > x$. By lemma 7.21, g is order preserving, so $g(x) \sqsubset g(y)$ and hence $\mathcal{M}, g(y) \models \psi$. Inductively, $\mathcal{R}, y \models \psi$, and as y was arbitrary, $\mathcal{R}, x \models G\psi$.

Conversely, suppose that $\mathcal{M}, g(x) \models \neg G\psi$. As W is finite and $\sqsubset$ transitive and prelinear, we may choose $w \in W$ such that $\mathcal{M}, w \models \neg\psi$ and $w \sqsupseteq u$ for all $u \in W$ with $\mathcal{M}, u \models \neg\psi$. Such a w is a '$\sqsubset$-maximal' witness to $\neg\psi$ in $\mathcal{M}$. It should be plain that $g(x) \sqsubset w$. We show that

$$\text{there is } y > x \text{ in } \mathbb{R} \text{ with } g(y) = w. \qquad (7.3)$$

To prove (7.3), there are two cases.

Case 1: $w = u_i$ for some $i < k$. We have $g(x) \sqsubset w = u_i$, so $g(x) \sqsubset \lambda(u_i) = C_i$. By lemma 7.20, $x \in Y_i = (-\infty, l_{i+1})$. So $x < l_{i+1}$, and by definition of g, $g(l_{i+1}) = u_i = w$, proving (7.3) in this case.

Case 2: $w \in C_i$ for some $i \leq k$. We show first that C_i *is right-open.* Suppose for contradiction that C_i is right-closed. Then $i < k$, the points d_i and s_{i+1} are defined, and w and d_i are in the same cluster, so $d_i \sqsubset w$ and hence $\mathcal{M}, d_i \models \neg G\psi$. But $d_i \equiv^{\mathcal{M}}_{\mathcal{B}\Psi} s_{i+1}$ and $G\psi \in \mathcal{B}\Psi$, so $\mathcal{M}, s_{i+1} \models \neg G\psi$ as well, and there is $u \in W$ with $s_{i+1} \sqsubset u$ and $\mathcal{M}, u \models \neg\psi$. By choice of w we have $u \sqsubseteq w$. So $s_{i+1} \sqsubset u \sqsubseteq w \sqsubset d_i$. By transitivity, $s_{i+1} \sqsubset d_i$. But this is a contradiction, because $d_i \in C_i$, $s_{i+1} \in C_{i+1}$, and C_{i+1} is a strictly $\sqsubset$-later cluster than C_i. So C_i is indeed right-open.

Because C_i is right-open, if $i < k$ then $r_i = l_{i+1}$. Also, $r_k = \infty$. So lemma 7.20 yields $Y_i = \{y \in \mathbb{R} : g(y) \sqsubset C_i\} = (-\infty, r_i)$. Since $g(x) \sqsubset w \in C_i$, we have $x \in Y_i$. As C_i is right-open, lemma 7.23(2) yields $C_i \subseteq g((x, r_i))$, so there is $y > x$ with $g(y) = w$, proving (7.3) in this case too.

So we may take y as in (7.3). Then $\mathcal{R}, y \models \neg\psi$ by the inductive hypothesis (7.2), so $\mathcal{R}, x \models \neg G\psi$ as required.

In spite of the temporal asymmetry of the definition of g, the case of $H\psi$ is so similar that we leave it to the reader.

Finally we consider $\Box\psi$, the most interesting case. The following claim will make light of all but one of the subsequent subcases.

Claim. If x is g-fair then $\mathcal{R}, x \models \Box\psi$ iff $\mathcal{M}, g(x) \models \Box\psi$.

Proof of claim. Assume that x is g-fair. First suppose that $\mathcal{M}, g(x) \models \Box\psi$. By g-fairness of x, we can choose $y, z \in \mathbb{R}$ with $y < x < z$ and $g((y,z)) \subseteq R(g(x))$. Take arbitrary $t \in (y,z)$. Then $g(t) \in R(g(x))$, so by Kripke semantics, $\mathcal{M}, g(t) \models \psi$. By the inductive hypothesis (7.2), $\mathcal{R}, t \models \psi$. As t was arbitrary, we obtain $\mathcal{R}, x \models \Box\psi$.

Conversely, if $\mathcal{R}, x \models \Box\psi$ then choose $y < x < z$ in $\mathbb{R}$ such that $\mathcal{R}, t \models \psi$ for all $t \in (y,z)$. Pick arbitrary $w \in R(g(x))$. By g-fairness of x we have $R(g(x)) \subseteq g((y,z))$, so there is $t \in (y,z)$ with $g(t) = w$. Since $t \in (y,z)$, we have $\mathcal{R}, t \models \psi$, and inductively (see (7.2)), $\mathcal{M}, w \models \psi$. Since w was arbitrary, $\mathcal{M}, g(x) \models \Box\psi$. This proves the claim.

There are now four subcases.

1. If $x \in \mathbb{R} \setminus \{r_i, l_{i+1} : i < k\}$ then by lemma 7.22, x is g-fair, and the claim gives $\mathcal{R}, x \models \Box\psi$ iff $\mathcal{M}, g(x) \models \Box\psi$ as required.

2. Suppose that $x = l_{i+1} = r_i$ for some $i < k$. We show that x is g-fair, and this is easy to do. As $r_i = l_{i+1}$, we see that C_i is right-open, C_{i+1} is left-open, and $g(x) = u_i$. Take any y, z with $l_i < y < x < z < r_{i+1}$. Recalling that $x = l_{i+1} = r_i$, by lemma 7.23 we have $g((y,x)) = C_i$ and $g((x,z)) = C_{i+1}$, so

$$g((y,z)) = g((y,x)) \cup \{g(x)\} \cup g((x,z)) = C_i \cup \{u_i\} \cup C_{i+1} = R(u_i).$$

   Hence, x is g-fair. By the claim, $\mathcal{R}, x \models \Box\psi$ iff $\mathcal{M}, g(x) \models \Box\psi$.

3. Suppose that $x = r_i < l_{i+1}$ for some $i < k$. Then i is closed and $g(x) = d_i$. By lemma 7.24(1), $g((y,x)) \subseteq R(d_i)$ for all large enough $y < x$. On the other side, if $z \in (x, l_{i+1})$ then by lemma 7.24(2), $g((x,z)) = R(d_i)$. Consequently, for all large enough y and small enough z with $y < x < z$, we have

$$g((y,z)) = g((y,x)) \cup \{g(x)\} \cup g((x,z)) = \overbrace{g((y,x))}^{\subseteq R(d_i)} \cup \{d_i\} \cup R(d_i) = R(d_i).$$

   So once again, x is g-fair, and the claim yields $\mathcal{R}, x \models \Box\psi$ iff $\mathcal{M}, g(x) \models \Box\psi$.

4. Finally suppose that $r_i < l_{i+1} = x$ for some $i < k$. This case is the culmination of our work. The claim does not apply, as x is not g-fair: indeed, $g((r_i, x))$ is *disjoint* from $R(g(x))$.

   Since $r_i < l_{i+1}$, we know that C_i is right-closed, C_{i+1} is left-closed, d_i and s_{i+1} are both defined, $g(x) = s_{i+1}$, and $d_i \equiv^{\mathcal{M}}_{\mathcal{B}\Psi} s_{i+1}$.

   Suppose on the one hand that $\mathcal{R}, x \models \Box\psi$. We show first that $\mathcal{M}, d_i \models \Box\psi$. Choose $y \in (r_i, x)$ with $\mathcal{R}, t \models \psi$ for all $t \in (y, x)$. Then $g((y,x)) = R(d_i)$ by lemma 7.24(2). Let $w \in R(d_i)$ be arbitrary. Choose $t \in (y,x)$ with $g(t) = w$. Inductively (see (7.2)), $\mathcal{M}, w \models \psi$. Since w was arbitrary, we obtain $\mathcal{M}, d_i \models \Box\psi$ as required. But $d_i \equiv^{\mathcal{M}}_{\mathcal{B}\Psi} s_{i+1} = g(x)$ and $\Box\psi \in \mathcal{B}\Psi$, so we have $\mathcal{M}, g(x) \models \Box\psi$ as well.

   Suppose on the other hand that $\mathcal{M}, g(x) \models \Box\psi$. As $g(x) = s_{i+1} \equiv^{\mathcal{M}}_{\mathcal{B}\Psi} d_i$ and $\Box\psi \in \mathcal{B}\Psi$, we have both $\mathcal{M}, s_{i+1} \models \Box\psi$ and $\mathcal{M}, d_i \models \Box\psi$. Hence, $\mathcal{M}, w \models \psi$ for every $w \in R(s_{i+1}) \cup R(d_i)$. Let $y = r_i < x$. It follows from lemma 7.24(2) that $g((y,x)) = R(d_i)$. By lemma 7.24(3) there is $z > x$ such that $g((x,z)) \subseteq R(s_{i+1})$. Of course, $\{g(x)\} \subseteq R(s_{i+1})$ too. So

$$g((y,z)) = g((y,x)) \cup \{g(x)\} \cup g((x,z)) \subseteq R(d_i) \cup R(s_{i+1}).$$

Choose $t \in (y,z)$ arbitrarily. By the above, $g(t) \in R(d_i) \cup R(s_{i+1})$. So $\mathcal{M}, g(t) \models \psi$, and inductively (see (7.2)), $\mathcal{R}, t \models \psi$ as well. Since t was arbitrary, we obtain $\mathcal{R}, x \models \Box\psi$, as required.

This completes the induction and the proof. □

Now choose $w \in W$ with $\mathcal{M}, w \models \varphi$. By lemma 7.21, g is surjective, so there is $x \in \mathbb{R}$ with $g(x) = w$. Since $\varphi \in \Psi$, by the lemma we obtain $\mathcal{R}, x \models \varphi$. Thus, φ is satisfiable over $\mathbb{R}$, proving theorem 7.11.

7.6 Example

We give a brief example of the construction of $\mathcal{R}$. Recall from proposition 5.6 that the formula

$$\theta = H\neg a \wedge H\neg b \wedge \neg a \wedge \neg b \wedge \Diamond a \wedge \Diamond b \wedge G\neg(\Diamond a \wedge \Diamond b) \wedge FG\neg a$$

of (4.2) is true at world u_0 in the Ψ-linked Kripke model $\mathcal{M}$ shown in figure 3, where Ψ is the set of subformulas of θ.

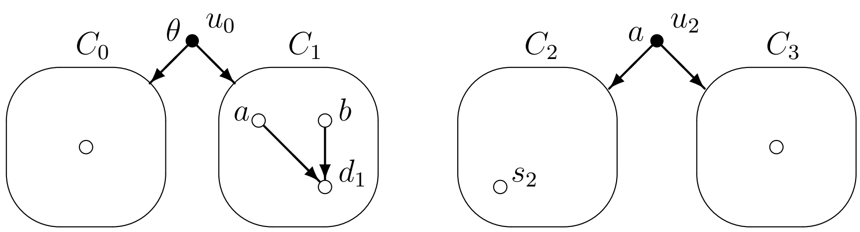

Figure 3: Kripke model $\mathcal{M}$ satisfying θ

Following the route taken in §7.5, we select points $-\infty = l_0 < r_0 = l_1 < r_1 < l_2 < r_2 = l_3 < r_3 = \infty$ in $\mathbb{R} \cup \{\pm\infty\}$. Theorem 7.10 yields maps $f_i^* : \mathbb{R} \to C_i$ for $i = 0, 1, 2, 3$, which we scale and chop to give maps $f_i : (l_i, r_i) \to C_i$. For $i \neq 1$ there is no choice for f_i, since $|C_i| = 1$. For $f_1^* : \mathbb{R} \to C_1$, the sequence a_i, b_i $(i \in \mathbb{Z})$ in the proof of theorem 7.10 looks like

$\cdots$	a_{-2}	b_{-2}	a_{-1}	b_{-1}	a_0	b_0	a_1	b_1	a_2	b_2	a_3	b_3	a_4	b_4	a_5	b_5	$\cdots$
$\cdots$	a	d_1	d_1	d_1	b	d_1	d_1	d_1	a	d_1	d_1	d_1	b	d_1	d_1	d_1	$\cdots$

so f_1^* is a $\mathbb{Z}$-sum of maps taking singletons to a, b, d_1 and open intervals of $\mathbb{R}$ to d_1. Up to isomorphism, it maps odd integers to the point satisfying a in C_1, even integers to the point satisfying b, and non-integers to d_1. We choose a non-integer, say -0.5, and scale $f_1^* \restriction (-\infty, -0.5)$ to (l_1, r_1), yielding our map $f_1 : (l_1, r_1) \to C_1$. The odd negative integers, when scaled, form a sequence of points in (l_1, r_1) converging to l_1 and mapping to a under f_1, and there is a similar sequence for b arising from the even negative integers. There is no choice over the maps f_i', since by the definitions we have $f_0' : [r_0, l_1] \to \{u_0\}$, $f_1' : [r_1, l_2) \to R(d_1) = \{d_1\}$, $f_1'(l_2) = s_2$, and $f_2' : [r_2, l_3] \to \{u_2\}$.

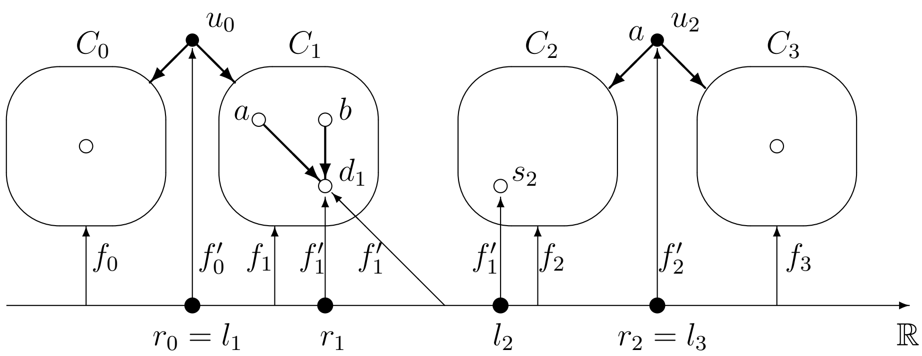

Figure 4: the maps f_i and f_i' $(i = 0, 1, 2)$ and f_3

Figure 4 sketches these maps. We let $g : \mathbb{R} \to \mathcal{M}$ be the union of all of them. The model over $\mathbb{R}$ induced by g is shown in figure 5. It should be clear that θ is true at l_1.

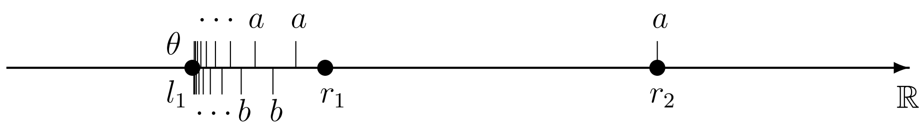

Figure 5: the map $g : \mathbb{R} \to \mathcal{M}$

8 Conclusion

We can now prove our main result:

THEOREM 8.1. *The system of §3.1 axiomatises the $\mathcal{L}$-logic of* $\mathbb{R}$.

Proof. Soundness was shown in theorem 3.1. Conversely, let φ_0 be a formula consistent with the system of §3.1. Let Ψ be the set of subformulas of $P\varphi_0$. By theorem 6.1, φ_0 is satisfied in a Ψ-linked Kripke model. By theorem 7.11, φ_0 is satisfiable over $\mathbb{R}$. Hence, any consistent formula is satisfiable over $\mathbb{R}$. $\square$

We have shown that the logic of $\mathbb{R}$ in the temporal language $\mathcal{L}$ with modalities G, H, and $\square$

- is finitely axiomatisable, answering an implicit problem of Shehtman [28],
- has PSPACE-complete complexity,
- has no strongly complete axiomatisation,
- is not Kripke complete.

We list some remaining open problems. First, some complexity problems.

PROBLEM 8.2. *For fixed $k \geq 0$, what is the complexity of the set of $\mathcal{L}$-formulas that are satisfiable over $\mathbb{R}$ and involve at most k $\Box$-operators?*

The methods of [23] may be helpful. If the answer is 'NP-complete', it might suggest that the language with F, P, and $\Box$ could be more tractable in practice than the more expressive language with Until and Since.

The operations of sum (+) and shuffle in §7, plus two more involving countable iterations, can be used to specify models over $\mathbb{R}$ in a finite way. By results in [4], any model over $\mathbb{R}$ can be specified up to any desired degree of first-order equivalence in such a way, so any satisfiable $\mathcal{L}$-formula has a model specified by these operations. This leads to the following problem.

PROBLEM 8.3. *Investigate the complexity of model checking for the language $\mathcal{L}$ for models over $\mathbb{R}$ specified by a finite sequence of operations of the above kinds.*

This problem was investigated in [6] for the language with Until and Since. It was shown to be in PSPACE in [19] and PSPACE-complete in [8]. One may also wish to develop alternative reasoning systems for $\mathcal{L}$ over $\mathbb{R}$, such as tableaux, and synthesis methods along the lines of [6, 7]. The end result of this research could justify the promotion of $\mathcal{L}$ as a viable language for specification and reasoning over the real line, possibly a more attractive one than the very expressive language with Until and Since.

It may be of interest to study the logic of $\mathbb{R}$ in the sublanguage of $\mathcal{L}$ without H: the only non-boolean connectives are G and $\Box$. This logic is PSPACE-complete, by the same argument as in theorem 4.1. Theorem 4.2 survives: there is no strongly complete axiomatisation. The proof of theorem 4.5 can be adapted to show that it is not Kripke complete, using the formula

$$F(p \wedge G\neg p \wedge \neg a \wedge \neg b \wedge \Diamond a \wedge \Diamond b \wedge G\neg(\Diamond a \wedge \Diamond b) \wedge FG\neg a) \wedge G(Fp \to \neg a \wedge \neg b).$$

The Prior axiom is no longer expressible, but a variant $Fp \wedge FG\neg p \to F(G\neg p \wedge \Diamond p)$ can be used instead.

PROBLEM 8.4 (N. Bezhanishvili)**.** Is the logic of $\mathbb{R}$ with connectives $G, \Box$ finitely axiomatisable?

An alternative and more expressive interpretation of $\Box$ is as 'derivative' $[d]$, so that $(\mathbb{R}, h), x \models [d]\varphi$ if there is an open neighbourhood O of x with $(\mathbb{R}, h), y \models \varphi$ for every $y \in O \setminus \{x\}$. Finite axiomatisations of the logic of $\mathbb{R}$ with $[d]$ alone (without G, H) and with $[d]$ and $\forall$ are given in [18] (see also Shehtman's habilitation thesis and [10]). Gatto [9] has recently shown that the logic of $\mathbb{R}$ in the language with G, H, and $[d]$ is finitely axiomatisable.

Other possible, non-first-order definable interpretations of $\Box$ over $\mathbb{R}$ are based on cardinality and Baire category. For example, one could define $(\mathbb{R}, h), x \models \Diamond\varphi$ to hold when every open neighbourhood of x contains uncountably many points y with $(\mathbb{R}, h), y \models \varphi$.

PROBLEM 8.5. Study the logic of $\mathbb{R}$ (and, dropping G, H, of other topological spaces) in the language with boxes based on cardinality or Baire category, plus $G, H, \Box, [d]$, and its sublanguages.

One final language consists of G, H and two modalities $\Box^+, \Box^-$, where $(\mathbb{R}, h), x \models \Box^+\varphi$ if there is $y > x$ such that $(\mathbb{R}, h), z \models \varphi$ for every $z \in (x, y)$, and $\Box^-$ is the mirror image. We could read $\Box^-$ and $\Box^+$ as 'recently' and 'imminently'. The corresponding diamonds have been written in the literature as K^-, K^+, respectively.

PROBLEM 8.6. Find axiomatisations of the logic of $\mathbb{R}$ in this language and in sublanguages such as $\{G, \Box^+\}$.

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