Sahlqvist theorem for modal fixed point logic

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Abstract

We define Sahlqvist fixed point formulas. By extending the technique of Sambin and Vaccaro we show that (1) for each Sahlqvist fixed point formula φ there exists an LFP-formula $\chi(\varphi)$, with no free first-order variable or predicate symbol, such that a descriptive μ-frame (an order-topological structure that admits topological interpretations of least fixed point operators as intersections of clopen pre-fixed points) validates φ iff $\chi(\varphi)$ is true in this structure, and (2) every modal fixed point logic axiomatized by a set Φ of Sahlqvist fixed point formulas is sound and complete with respect to the class of descriptive μ-frames satisfying $\{\chi(\varphi) : \varphi \in \Phi\}$. We also give some concrete examples of Sahlqvist fixed point logics and classes of descriptive μ-frames for which these logics are sound and complete.

Keywords: modal mu-calculus, completeness, correspondence, descriptive frame

1. Introduction

Modal μ-calculus, or synonymously, modal fixed point logic is obtained by adding to the basic modal logic the least and greatest fixed point operators. The attractive feature of modal μ-calculus is that it is very expressive, but still decidable, e.g., [9, Section 5]. Many expressive modal and temporal logics such as **PDL**, **CTL** and **CTL*** are all embeddable into the modal μ-calculus, e.g., [9, Section 4.1].

In [21] Kozen defined the syntax and semantics of modal μ-calculus, gave its axiomatization using the so-called fixed point rule (see Definition 2.13), and showed soundness of this axiomatization. Walukiewicz [28] proved completeness of Kozen's axiomatization using automata and tableaux. His proof, however, is complicated and has not been generalized to other axiomatic systems of μ-calculus. Ambler, Kwiatkowska, Measer [1] proved soundness and completeness of Kozen's axiomatization of modal μ-calculus with respect to non-standard, order-topological semantics. They also extended this result to all normal fixed point logics – logics obtained by adding extra axioms to Kozen's axiomatization of μ-calculus. Later Bonsangue and Kwiatkowska [8] showed that in this semantics the least fixed point can be computed as the intersection of clopen pre-fixed points. Hartonas [18] extended these completeness results to the systems of positive (negation-free) modal μ-calculus. Santocanale [25] proved that modal operators have adjoints on free modal μ-algebras and that the canonical embedding of the free modal μ-algebra into its Dedekind-MacNeille completion preserves all the operations in the class of the fixed point alternation hierarchy. Later Santocanale and Venema [26] used these results and coalgebraic methods to prove completeness for flat modal fixed point logic. Flat modal fixed point logic is obtained by replacing the fixed point operators by logical connectives; this has (among other things) the effect of severely restricting nesting of fixed point operators. Ten Cate and Fontaine [10] used non-standard semantics of modal fixed point logics for proving the finite model property result for the modal fixed point logic axiomatized by the formula $\mu x \Box x$. Van Benthem [2], [3] also investigated this logic and posed a question whether a version of the Sahlqvist theorem holds for the systems of μ-calculus.

Sahlqvist's completeness and correspondence theorem is one of the most fundamental results of classical modal logic (see e.g., [7, Sections 3.6 and 5.6]). It states that every modal logic obtained by adding Sahlqvist formulas (a large class of formulas of a particular syntactic shape) to the basic modal logic **K** is sound and complete with respect to

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a first-order definable class of Kripke frames. In [24] Sambin and Vaccaro gave an elegant proof of Sahlqvist's theorem using order-topological methods. An important ingredient of their proof is the Esakia lemma of [14]. Generalizations of the Sahlqvist completeness and correspondence (first-order definability) result to larger classes of modal formulas can be found in [16], [17] and [20]. Other generalizations of Sahlqvist correspondence for modal formulas (definability in first-order logic with fixed point operator) were obtained in [22], [2], [3] and [11]. See Remarks 5.14 and 5.15 below for more information on some of these generalizations.

In this paper we prove a version of Sahlqvist's theorem for modal fixed point logic. Our language is the modal language extended with the least fixed point operator μ (we do not have the greatest fixed point operator ν in our language). Following [1] we consider the order-topological semantics of modal μ-calculus. Descriptive frames are order-topological structures extensively used in modal logic, e.g., [7, Chapter 5]. In [1] the authors define, what we call in this paper, descriptive μ-frames – those descriptive frames that admit a topological interpretation of the least fixed point operator. Unlike the classical semantics of fixed point logics, in this semantics, the least fixed point operator is interpreted as the intersection of not all pre-fixed points, but of all *clopen* pre-fixed points. We prove that for this semantics of modal fixed point logic an analogue of the Esakia lemma still holds (Lemma 4.6). We also define Sahlqvist fixed point formulas (Definition 5.1) and extend the Sambin–Vaccaro method [24] of proving Sahlqvist's completeness and correspondence results (Theorems 5.3 and 5.11) from modal logic to modal fixed point logic.

More specifically, let LFP denote first-order logic with the least fixed point operator. (Again the least fixed point operator is interpreted topologically, that is, as the intersection of clopen pre-fixed points.) We prove that for every Sahlqvist fixed point formula φ there exists an LFP-formula $\chi(\varphi)$, with no free first-order variable or predicate symbol, such that a descriptive μ-frame validates φ iff $\chi(\varphi)$ is true in this structure. Our main result (Theorem 5.13) states that every modal fixed point logic axiomatized by a set Φ of Sahlqvist fixed point formulas is sound and complete with respect to the class of descriptive μ-frames satisfying $\{\chi(\varphi) : \varphi \in \Phi\}$. We also give some concrete examples of Sahlqvist fixed point logics and classes of descriptive μ-frames for which these logics are sound and complete.

Note that these results can also be formulated without mentioning any topology. A general frame is a Kripke frame with a distinguished set $\mathfrak{F}$ of 'admissible' subsets of this frame. A *general μ-frame* is a general frame in which all modal μ-formulas are assigned to admissible sets under any assignment of propositional variables to admissible sets, when the least fixed point operator is interpreted as the intersection of all the *admissible* pre-fixed points. A descriptive μ-frame can be seen as a general μ-frame where $\mathfrak{F}$ is the collection of all clopen sets. In this paper we show (Theorem 5.13(1)) that the Sahlqvist completeness and correspondence results also hold for this general-frame semantics of modal fixed point logic.

It needs to be stressed that our Sahlqvist completeness and correspondence results apply only to order-topological structures (descriptive μ-frames) and general μ-frames, and do not imply that every Sahlqvist modal fixed point logic is sound and complete with respect to Kripke frames (we discuss this in detail after Theorem 5.3). Sahlqvist correspondence for the classical semantics for a larger class of modal fixed point formulas with respect to LFP-definable classes of Kripke frames is investigated in [4]. For the preservation result of Sahlqvist fixed point formulas in (relativized) completions of modal μ-algebras we refer to [6].

Our work is a contribution to the study of modal fixed point logic and as such fits into a long tradition of computer science research on fixed point logics. Axiomatization and completeness results bring extra power and flexibility to applications of fixed point logics in computer science, and have already been extensively discussed in the computer science literature e.g., [21], [1],[18], [28], [10]. Our work can be seen as a continuation of this line of research. Our aim is to go beyond the basic modal fixed point logic, and provide a method of axiomatization (indeed completeness and correspondence results) for a wide range of (Sahlqvist) fixed point logics. More recent developments of this viewpoint can be found in [4] and [6] .

The paper is organized as follows: in Section 2 we recall a duality between modal μ-algebras and descriptive μ-frames and also the completeness of normal modal fixed point logics with respect to modal μ-algebras and descriptive μ-frames. In Section 3 we compare different kinds of order-topological semantics of modal μ-formulas. A modal fixed point analogue of the Esakia lemma is proved in Section 4. In Section 5 we define Sahlqvist fixed point formulas and prove the Sahlqvist completeness and correspondence results for modal fixed point logic. In Section 6 we discuss a few examples of Sahlqvist fixed point logics and their frame correspondents and conclude the paper with some remarks in Section 7.

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the presentation of the paper.

2. Preliminaries

In this section we set up the scene: we introduce the basic definitions of a modal μ-algebra and descriptive μ-frame, discuss a duality between them and the consequences of this duality for the completeness of axiomatic systems of modal fixed point logic.

2.1. *Classical fixed points*

Let $(L, \leq)$ be a complete lattice and $f : L \to L$ a monotone map, that is, for each $a, b \in L$ with $a \leq b$ we have $f(a) \leq f(b)$. Then the celebrated *Knaster–Tarski theorem* states that f has a least fixed point $LFP(f)$. Moreover, this fixed point can be computed as follows:

$$LFP(f) = \bigwedge \{a \in L : f(a) \leq a\}$$

There is another way of computing $LFP(f)$. In particular, for an ordinal α we let $f^0(0) = 0$, $f^\alpha(0) = f(f^\beta(0))$ if $\alpha = \beta + 1$, and $f^\alpha(0) = \bigvee_{\beta<\alpha} f^\beta(0)$, if α is a limit ordinal. Then $LFP(f) = f^\alpha(0)$, for some ordinal α such that $f^{\alpha+1}(0) = f^\alpha(0)$.

We briefly recall the syntax and Kripke semantics for the modal μ-calculus. The language of modal μ-calculus consists of

- a countably infinite set of propositional variables (x, y, p, q, x_0, x_1, etc),
- constants $\perp$ and $\top$,
- connectives $\wedge$, $\vee$, $\neg$,
- modal operators $\Diamond$ and $\Box$,
- $\mu x \varphi(x, x_1, \ldots, x_n)$ for all formulas $\varphi(x, x_1, \ldots, x_n)$, where x occurs under the scope of an even number of negations.

Formulas of modal μ-calculus will be called *modal μ-formulas*. A formula that does not contain any μ-operators will be called a *modal formula*. A *Kripke frame* is a pair (W, R), where W is a non-empty set and $R \subseteq W^2$ a binary relation. Given a Kripke frame (W, R), an *assignment* h is a map from the propositional variables to the powerset $\mathcal{P}(W)$ of W. The satisfiability and validity of a modal formula in a Kripke model and frame, respectively, are defined in a standard way (see, e.g., [7]). For each modal formula φ we denote by $[\![\varphi]\!]_h$ the set of points satisfying φ under the assignment h.

A propositional variable x is *bound* in a modal μ-formula φ if it occurs in the scope of some μx. A variable is *free* if it is not bound. We say that a modal μ-formula $\varphi(x, x_1, \ldots x_n)$ is *positive in* x if all the free occurrences of the variable x are under the scope of an even number of negations. A modal μ-formula $\varphi(x, x_1, \ldots x_n)$ is called *negative in* x if all the free occurrences of the variable x are under the scope of an odd number of negations.

Let (W, R) be a Kripke frame. For each modal μ-formula φ and an assignment h, we define the semantics $[\![\varphi]\!]_h$ of φ by induction on the complexity of φ. If φ is a propositional variable, a constant, or is of the form $\psi \wedge \chi$, $\psi \vee \chi$, $\neg\psi$, $\Box\psi$ or $\Diamond\psi$, then the semantics of φ is defined in a standard way. Now assume that $\varphi(x, x_1, \ldots, x_n)$ is a modal μ-formula positive in x. Then by the induction hypothesis, the semantics of φ is already defined for each assignment h. Let h be a fixed assignment. Then φ and h give rise to a map $f_{\varphi,h} : \mathcal{P}(W) \to \mathcal{P}(W)$ defined by $f_{\varphi,h}(U) = [\![\varphi]\!]_{h_x^U}$, where $h_x^U(x) = U$ and $h_x^U(y) = h(y)$ for each variable $y \neq x$. It is well known that if φ is positive in x, then $f_{\varphi,h}$ is monotone with respect to the inclusion order. It is also well known that $(\mathcal{P}(X), \subseteq)$ is a complete lattice where meets and joins are the set-theoretic intersections and unions, respectively. Thus, by the Knaster–Tarski theorem $f_{\varphi,h}$ has a least fixed point and $[\![\mu x \varphi]\!]_h$ is defined to be the least fixed point of $f_{\varphi,h}$.

2.2. Modal algebras and descriptive frames

We assume an elementary knowledge of general topology. We will not define standard concepts such as compact and Hausdorff spaces, closed and open sets etc. For all these definitions we refer to e.g., [13]. To keep notations simple, we will also follow the well-established convention to denote a topological space as, say, X instead of (X, τ). Whether a given letter X, Z or W, stands for a topological space or just a set will always be clear from the context.

Given a Kripke frame (W, R) we let $R^0 = \{(w, w) : w \in W\}$ and for each $d \geq 0$ we let R^d denote the dth iteration of R. That is, for $w, v \in W$ we have $wR^d v$ iff there exists $u \in W$ such that $wR^{d-1}u$ and uRv. For each $w \in W$ and $d \in \omega$ we let $R^d(w) = \{v \in W : wR^d v\}$. We will write $R(w)$ instead of $R^1(w)$. Also for each $U \subseteq W$ we let $[R]U = \{v \in W : R(v) \subseteq U\}$ and $\langle R \rangle U = \{v \in W : R(v) \cap U \neq \emptyset\}$.

Recall that a *Stone space* is a compact and Hausdorff topological space with a basis of clopen sets. A *descriptive frame* is a pair (W, R) such that W is a Stone space and R a binary relation on W such that $R(w)$ is a closed set for each $w \in W$ and the collection $\mathsf{Clop}(W)$ of all clopen subsets of W is closed under the operations $[R]$ and $\langle R \rangle$. The latter condition is equivalent to $\langle R \rangle C \in \mathsf{Clop}(W)$ for each $C \in \mathsf{Clop}(W)$. We also note that $\mathsf{Clop}(W)$ is a Boolean algebra with the operations $\cup$, $\cap$, $\setminus$, and constants W and $\emptyset$. We denote by $\mathsf{Cl}(W)$ and $\mathsf{Op}(W)$ the collections of all closed and all open subsets of W, respectively. We also note that $\mathsf{Cl}(W)$ and $\mathsf{Op}(W)$ are complete lattices (see, e.g., [27]). For $\mathsf{Cl}(W)$ the meet is the intersection and the join the closure of the union and for $\mathsf{Op}(W)$ the meet is the interior of the intersection and the join the union. The next lemma is well known e.g., [14] or [23]. It will be used in the subsequent sections.

Lemma 2.1. *Let (W, R) be a descriptive frame. Then*

1. $\langle R \rangle F \in \mathsf{Cl}(W)$ *for each* $F \in \mathsf{Cl}(W)$ *and* $\langle R \rangle U \in \mathsf{Op}(W)$ *for each* $U \in \mathsf{Op}(W)$,
2. $[R]F \in \mathsf{Cl}(W)$ *for each* $F \in \mathsf{Cl}(W)$ *and* $[R]U \in \mathsf{Op}(W)$ *for each* $U \in \mathsf{Op}(W)$,
3. $R^d(w) \in \mathsf{Cl}(W)$ *for each* $w \in W$ *and* $d \geq 0$.

Recall that a *modal algebra* is a pair $\mathfrak{B} = (B, \Diamond)$, where B is a Boolean algebra and $\Diamond$ a unary operation on B satisfying for each $a, b \in B$, (1) $\Diamond 0 = 0$ and (2) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$. Now we will briefly spell out the constructions establishing a duality between modal algebras and descriptive frames. For each descriptive frame $\mathcal{F} = (W, R)$ the algebra $\mathfrak{Clop}(\mathcal{F}) = (\mathsf{Clop}(W), \langle R \rangle)$ is a modal algebra. For each modal algebra $\mathfrak{B} = (B, \Diamond)$ we consider the set $W_{\mathfrak{B}}$ of all ultrafilters of B and define a topology on $W_{\mathfrak{B}}$ by declaring the set $\{\widehat{a} : a \in B\}$, where $\widehat{a} = \{w \in W_{\mathfrak{B}} : a \in w\}$, as a basis of the topology. We define a relation $R_{\mathfrak{B}}$ on $W_{\mathfrak{B}}$ by $wR_{\mathfrak{B}}v$ iff $\Diamond a \in w$ for each $a \in v$ (for $w, v \in W_{\mathfrak{B}}$). Then $(W_{\mathfrak{B}}, R_{\mathfrak{B}})$ is a descriptive frame and this correspondence is (up to isomorphism) one-to-one. That is, $\mathfrak{B} \cong (\mathsf{Clop}(W_{\mathfrak{B}}), \langle R_{\mathfrak{B}} \rangle)$ and $\mathcal{F} \cong (W_{\mathfrak{Clop}(\mathcal{F})}, R_{\mathfrak{Clop}(\mathcal{F})})$.

2.3. Modal μ-algebras and descriptive μ-frames

Definition 2.2.

1. *Let $\mathfrak{B} = (B, \Diamond)$ be a modal algebra. A map h from propositional variables to B is called an* algebra assignment. *We define a (possibly partial) semantics for modal μ-formulas by the following inductive definition.*

$$[\bot]_h = 0$$
$$[\top]_h = 1$$
$[x]_h = h(x)$, *where x is a propositional variable,*
$$[\varphi \wedge \psi]_h = [\varphi]_h \wedge [\psi]_h,$$
$$[\varphi \vee \psi]_h = [\varphi]_h \vee [\psi]_h,$$
$$[\neg\varphi]_h = \neg[\varphi]_h,$$
$$[\Diamond\varphi]_h = \Diamond[\varphi]_h,$$
$$[\Box\varphi]_h = \Box[\varphi]_h,$$

For $a \in B$ we denote by h^a_x a new algebra assignment such that $h^a_x(x) = a$ and $h^a_x(y) = h(y)$ for each propositional variable $y \neq x$.

If $\varphi(x, x_1, \dots, x_n)$ is positive in x, then

$$[\mu x\varphi(x, x_1, \dots, x_n)]_h = \bigwedge\{a \in B : [\varphi(x, x_1, \dots, x_n)]_{h_x^a} \leq a\},$$

if this meet exists; otherwise, the semantics for $\mu x\varphi(x, x_1, \dots, x_n)$ is undefined.

2. *A modal algebra $(B, \Diamond)$ is called a* modal μ-algebra *if $[\varphi]_h$ is defined for any modal μ-formula φ and any algebra assignment h.*

Notation: To simplify the notations instead of $[\varphi(x_1, \dots, x_n)]_h$ with $h(x_i) = a_i$, $1 \leq i \leq n$, we will simply write $\varphi(a_1, \dots, a_n)$.

Recall that a modal algebra $(B, \Diamond)$ is called *complete* if B is a complete Boolean algebra; that is, for each subset S of B the meet $\bigwedge S$ and the join $\bigvee S$ exist. It is straightforward to see that every complete modal algebra is a modal μ-algebra.

Lemma 2.3. *Let $\mathfrak{B} = (B, \Diamond)$ be a modal μ-algebra and h an algebra assignment. Then for each modal μ-formula φ positive in x, $[\mu x\varphi]_h$ is the least fixed point of the map $(a \mapsto [\varphi]_{h_x^a})$ for $a \in B$.*

Proof. The result follows from the definition of $[\mu x\varphi]_h$ and the standard argument of the proof of the Knaster–Tarski theorem. □

Definition 2.4. *Let (W, R) be a descriptive frame, $\mathfrak{F} \subseteq \mathcal{P}(W)$ and h an arbitrary assignment, that is, a map from the propositional variables to $\mathcal{P}(W)$. We define the semantics for modal μ-formulas by the following inductive definition.*

$[\![\bot]\!]_h^{\mathfrak{F}} = \emptyset$,

$[\![\top]\!]_h^{\mathfrak{F}} = W$,

$[\![x]\!]_h^{\mathfrak{F}} = h(x)$, *where x is a propositional variable,*

$[\![\varphi \wedge \psi]\!]_h^{\mathfrak{F}} = [\![\varphi]\!]_h^{\mathfrak{F}} \cap [\![\psi]\!]_h^{\mathfrak{F}}$,

$[\![\varphi \vee \psi]\!]_h^{\mathfrak{F}} = [\![\varphi]\!]_h^{\mathfrak{F}} \cup [\![\psi]\!]_h^{\mathfrak{F}}$,

$[\![\neg\varphi]\!]_h^{\mathfrak{F}} = W \setminus [\![\varphi]\!]_h^{\mathfrak{F}}$,

$[\![\Diamond\varphi]\!]_h^{\mathfrak{F}} = \langle R\rangle[\![\varphi]\!]_h^{\mathfrak{F}}$,

$[\![\Box\varphi]\!]_h^{\mathfrak{F}} = [R][\![\varphi]\!]_h^{\mathfrak{F}}$,

We denote by h_x^U a new assignment such that $h_x^U(x) = U$ and $h_x^U(y) = h(y)$ for each propositional variable $y \neq x$ and $U \in \mathcal{P}(W)$.

Let $\varphi(x, x_1, \dots, x_n)$ be positive in x, then

$$[\![\mu x\varphi(x, x_1, \dots, x_n))]\!]_h^{\mathfrak{F}} = \bigcap\{U \in \mathfrak{F} : [\![\varphi(x, x_1, \dots, x_n)]\!]_{h_x^U}^{\mathfrak{F}} \subseteq U\}.$$

We assume that $\bigcap \emptyset = W$.

Let (W, R) be a descriptive frame. We call a map h from the propositional variables to $\mathcal{P}(W)$ a *set-theoretic assignment*. If h maps each propositional variable to $\mathsf{Cl}(W)$, then h is called a *closed assignment*, and if h maps each propositional variable to $\mathsf{Clop}(W)$, then h is called a *clopen assignment*. Let h be any assignment. Then $[\![\cdot]\!]_h^{\mathfrak{F}}$ is called the *clopen semantics* if $\mathfrak{F} = \mathsf{Clop}(W)$, $[\![\cdot]\!]_h^{\mathfrak{F}}$ is called the *closed semantics* if $\mathfrak{F} = \mathsf{Cl}(W)$ and $[\![\cdot]\!]_h^{\mathfrak{F}}$ is called the *classical or set-theoretic semantics* if $\mathfrak{F} = \mathcal{P}(W)$.

Notation: To simplify the notations instead of $[\![\varphi(x_1, \dots, x_n)]\!]_h^{\mathfrak{F}}$ with $h(x_i) = U_i$, $1 \leq i \leq n$, we will simply write $\varphi(U_1, \dots, U_n)^{\mathfrak{F}}$. Moreover, we will skip the index $\mathfrak{F}$ if it is clear from the context or is irrelevant (e.g., when φ is a modal formula).

A set C such that $\varphi(C, h(x_1), \dots, h(x_n)) \subseteq C$ is called a *pre-fixed point*.

Lemma 2.5. *Let* (W, R) *be a descriptive frame,* $\mathfrak{F} \subseteq \mathcal{P}(W)$ *and* h *an arbitrary assignment. Then for each modal* μ*-formula* $\varphi(x, x_1 \dots, x_n)$ *positive in* x*,* $\varphi(\cdot, h(p_1), \dots, h(p_n))^{\mathfrak{F}}$ *is monotone. That is, for* $U, V \subseteq W$*,*

$$U \subseteq V \text{ implies } \varphi(U, h(p_1), \dots, h(p_n))^{\mathfrak{F}} \subseteq \varphi(V, h(p_1), \dots, h(p_n))^{\mathfrak{F}}.$$

Proof. We will prove the lemma by induction on the complexity of φ. As agreed above we will skip the index $\mathfrak{F}$. Our induction hypothesis is: 1) if $\varphi(x, x_1 \dots, x_n)$ is positive in x, then $\varphi(\cdot, h(p_1), \dots, h(p_n))$ is monotone and 2) if $\varphi(x, x_1 \dots, x_n)$ is negative in x, then $\varphi(\cdot, h(p_1), \dots, h(p_n))$ is anti-tone. The cases $\varphi = \bot$, $\varphi = \top$, φ is a propositional variable, $\varphi = \psi \wedge \chi$, $\varphi = \psi \vee \chi$, $\varphi = \neg\psi$, $\varphi = \Diamond\psi$ and $\varphi = \Box\psi$ are proved as in standard modal logic (see, e.g., [7]). Now let $\varphi = \mu y \psi(y, x, x_1, \dots, x_n)$ be positive in x and inductively assume the result for ψ. Then, by the induction hypothesis, for each $U, V \subseteq W$ with $U \subseteq V$ and $C \in \mathfrak{F}$ we have $\psi(C, U, h(p_1), \dots, h(p_n)) \subseteq \psi(C, V, h(p_1), \dots h(p_n))$. So if $\psi(C, V, h(p_1), \dots, h(p_n)) \subseteq C$, then $\psi(C, U, h(p_1), \dots, h(p_n)) \subseteq C$. Therefore, the set $\{C \in \mathfrak{F} : \psi(C, U, h(p_1), \dots, h(p_n)) \subseteq C\}$ contains the set $\{C \in \mathfrak{F} : \psi(C, V, h(p_1), \dots, h(p_n)) \subseteq C\}$. But this means that $\mu y \psi(y, U, h(p_1), \dots, h(p_n)) = \bigcap\{C \in \mathfrak{F} : \psi(C, U, h(p_1), \dots, h(p_n)) \subseteq C\} \subseteq \bigcap\{C \in \mathfrak{F} : \psi(C, V, h(p_1), \dots, h(p_n)) \subseteq C\} = \mu y \psi(y, V, h(p_1), \dots, h(p_n))$. Therefore, we obtained that $\varphi(U, h(p_1), \dots, h(p_n)) \subseteq \varphi(V, h(p_1), \dots, h(p_n))$. The case of φ negative in x is similar. $\square$

Definition 2.6. *A descriptive frame* (W, R) *is called a* descriptive μ-frame *if for each clopen assignment* h *and for each modal* μ*-formula* φ*, the set* $[\![\varphi]\!]_h^{\mathsf{Clop}(W)}$ *is clopen.*

Example 2.7. We will give an example of a descriptive frame which is not a descriptive μ-frame. Let $W = \mathbb{N} \cup \{\omega\}$ be the Alexandroff compactification of the set $\mathbb{N}$ of natural numbers with discrete topology. Then, the clopen sets of W are finite subsets of $\mathbb{N}$ and cofinite subsets of $\mathbb{N}$ together with the point ω. Let R be such that $\omega R \omega$ and nRm if $n, m \in \mathbb{N}$ and $m + 1 = n$. It is easy to see that (W, R) is a descriptive frame. Consider the formula $\mu x(\Box\bot \vee \Diamond\Diamond x)$. It is easy to see that every clopen pre-fixed point of this formula is a cofinite subset of $\mathbb{N}$ containing the set E of all even numbers (as $\Box\bot$ is true at point 0) and containing the point ω. So the intersection of all these pre-fixed points is the set $E \cup \{\omega\}$, which is not clopen. Therefore, (W, R) is not a descriptive μ-frame.

Obviously, each finite descriptive frame is a descriptive μ-frame. We will see more examples of descriptive μ-frames later in this section and in the following section.

Now we will discuss a duality between modal μ-algebras and descriptive μ-frames. This duality was first obtained in [1] and later improved in [8]. A generalization of this duality to positive modal μ-algebras can be found in [18].

Lemma 2.8. *Let* (W, R) *be a descriptive* μ*-frame. Then the modal algebra* $(\mathsf{Clop}(W), \langle R \rangle)$ *is a modal* μ*-algebra.*

Proof. In order to show that $(\mathsf{Clop}(W), \langle R \rangle)$ is a modal μ-algebra, we need to prove that $[\varphi]_h$ exists for each modal μ-formula φ and each algebra assignment h. Note that in this case an algebra assignment for $(\mathsf{Clop}(W), \langle R \rangle)$ is the same as a clopen assignment for (W, R). So we will not distinguish them. We prove the lemma by induction on the complexity of φ. Our induction hypothesis is: for any clopen assignment h, $[\varphi]_h$ is defined and

$$[\varphi]_h = [\![\varphi]\!]_h^{\mathsf{Clop}(W)},$$

where $[\![\varphi]\!]_h^{\mathsf{Clop}(W)}$ is the clopen semantics of φ in the descriptive μ-frame (W, R) with the clopen assignment h.

The cases $\varphi = \bot$, $\varphi = \top$, φ is a propositional variable, $\varphi = \psi \wedge \chi$, $\varphi = \psi \vee \chi$, $\varphi = \neg\psi$, $\varphi = \Diamond\psi$ and $\varphi = \Box\psi$ are proved as in the duality theorem for modal algebras and descriptive frames. Now assume $\varphi(x, x_1, \dots, x_n)$ is a modal μ-formula positive in x for which the induction hypothesis holds. We consider any clopen assignment h. By the induction hypothesis, for each $C \in \mathsf{Clop}(W)$, we have $[\varphi]_{h_x^C} = [\![\varphi]\!]_{h_x^C}^{\mathsf{Clop}(W)}$. We will denote this set by $\varphi(C, h(x_1), \dots, h(x_n))$. Let

$$\mathbb{C} = \{C \in \mathsf{Clop}(W) : \varphi(C, h(x_1), \dots, h(x_n)) \subseteq C\}.$$

Since (W, R) is a descriptive μ-frame, $\bigcap \mathbb{C}$ is clopen. We will show that $\bigwedge \mathbb{C} = \bigcap \mathbb{C}$. That $\bigwedge \mathbb{C}$ exists will be an obvious consequence of this. Let $G = \bigcap \mathbb{C}$. So $G \in \mathsf{Clop}(W)$. Then G is a lower bound of $\mathbb{C}$. On the other hand, for each $C \in \mathbb{C}$ we have $G \subseteq C$. By monotonicity, $\varphi(G, h(x_1), \dots, h(x_n)) \subseteq \varphi(C, h(x_1), \dots, h(x_n)) \subseteq C$. Thus, $\varphi(G, h(x_1), \dots, h(x_n)) \subseteq \bigcap \mathbb{C} = [\![\varphi]\!]_h^{\mathsf{Clop}(W)} = G$. Therefore, G belongs to $\mathbb{C}$. So G is a lower bound that belongs to the set, which means that $G = \bigwedge \mathbb{C}$. Thus, $[\mu x \varphi]_h$ is defined and is equal to $[\![\mu x \varphi]\!]_{h_x^C}^{\mathsf{Clop}(W)}$. This completes the induction, and so $(\mathsf{Clop}(W), \langle R \rangle)$ is a modal μ-algebra. $\square$

Let (W, R) be a descriptive μ-frame, h a clopen assignment and φ be a modal μ-formula positive in x. Let $(C \mapsto [\![\varphi]\!]^{\mathsf{Clop}(W)}_{h^C_x})$ be the map from $\mathsf{Clop}(W)$ to $\mathsf{Clop}(W)$ sending each clopen set C to $[\![\varphi]\!]^{\mathsf{Clop}(W)}_{h^C_x}$. It is easy to see that this map is well defined and, by Lemma 2.5, monotone.

Corollary 2.9. *Let (W, R) be a descriptive μ-frame and h a clopen assignment. Then for each modal μ-formula φ positive in x, $[\![\mu x\varphi]\!]^{\mathsf{Clop}(W)}_h$ is the least fixed point of the map $(C \mapsto [\![\varphi]\!]^{\mathsf{Clop}(W)}_{h^C_x})$ for $C \in \mathsf{Clop}(W)$.*

Proof. The result follows immediately from the proof of Lemma 2.8. It follows from the proof that $G = [\![\mu x\varphi]\!]^{\mathsf{Clop}(W)}_h$ is a fixed point of $(C \mapsto [\![\varphi]\!]^{\mathsf{Clop}(W)}_{h^C_x})$ and by the definition of G, it is contained in every (pre-)fixed point. □

Lemma 2.10. *Let $\mathfrak{B} = (B, \Diamond)$ be a modal μ-algebra. Then the corresponding descriptive frame $(W_{\mathfrak{B}}, R_{\mathfrak{B}})$ is a descriptive μ-frame.*

Proof. We need to show that for each modal μ-formula φ and each clopen assignment h, the set $[\![\varphi]\!]^{\mathsf{Clop}(W_{\mathfrak{B}})}_h$ is clopen. We prove this by induction on the complexity of φ. By the definition of a modal μ-algebra and duality we have $\mathfrak{B} \cong (\mathsf{Clop}(W_{\mathfrak{B}}), \langle R_{\mathfrak{B}}\rangle)$. Therefore, as in the proof of Lemma 2.8, we will identify algebra assignments for $(\mathsf{Clop}(W_{\mathfrak{B}}), \langle R_{\mathfrak{B}}\rangle)$ with clopen assignments for $(W_{\mathfrak{B}}, R_{\mathfrak{B}})$. Our induction hypothesis is: for any clopen assignment h we have

$$[\![\varphi]\!]^{\mathsf{Clop}(W_{\mathfrak{B}})}_h = [\varphi]_h,$$

where $[\varphi]_h$ is the semantics of φ in the algebra $(\mathsf{Clop}(W_{\mathfrak{B}}), \langle R_{\mathfrak{B}}\rangle)$.

As in the proof of Lemma 2.8, the cases $\varphi = \bot$, $\varphi = \top$, φ is a propositional variable, $\varphi = \psi \wedge \chi$, $\varphi = \psi \vee \chi$, $\varphi = \neg\psi$, $\varphi = \Diamond\psi$ and $\varphi = \Box\psi$ are proved as in the duality theorem for modal algebras and descriptive frames. Now let $\varphi(x, x_1, \dots, x_n)$ be a modal μ-formula positive in x and let h be any clopen assignment. By the assumed induction hypothesis for φ, for each $C \in \mathsf{Clop}(W_{\mathfrak{B}})$, we have $[\![\varphi]\!]^{\mathsf{Clop}(W_{\mathfrak{B}})}_{h^C_x} = [\varphi]_{h^C_x}$. As in the proof of Lemma 2.8, we denote this set by $\varphi(C, h(x_1), \dots, h(x_n))$. We also denote the set $\{C \in \mathsf{Clop}(W_{\mathfrak{B}}) : \varphi(C, h(x_1), \dots, h(x_n)) \subseteq C\}$ by $\mathbb{C}$. Since $\mathfrak{B}$ is a modal μ-algebra, $(\mathsf{Clop}(W_{\mathfrak{B}}), \langle R_{\mathfrak{B}}\rangle)$ is also a modal μ-algebra. Therefore, $D = \bigwedge \mathbb{C}$ exists and is a clopen set. Thus, D is contained in $\bigcap \mathbb{C}$. Moreover, the same argument as in the proof of Lemma 2.8 shows that $\varphi(D, h(x_1), \dots h(x_n)) \subseteq D$. So $D \in \mathbb{C}$ and hence $\bigcap \mathbb{C} \subseteq D$. Therefore, $[\![\varphi]\!]^{\mathsf{Clop}(W_{\mathfrak{B}})}_h = \bigcap \mathbb{C} = D$ is clopen. This completes the induction, and thus $(W_{\mathfrak{B}}, R_{\mathfrak{B}})$ is a descriptive μ-frame. □

As every complete modal algebra is a modal μ-algebra, it follows from Lemma 2.10 that a descriptive frame dual to a complete modal algebra is a descriptive μ-frame. Descriptive μ-frames of this kind will be heavily used in the next section.

Remark 2.11. From now on, we will identify clopen assignments of a descriptive frame (W, R) with algebra assignments of $(\mathsf{Clop}(W), \langle R\rangle)$.

It is easy to see that the correspondence between descriptive μ-frames and modal μ-algebras is (up to the isomorphism) one-to-one. Putting everything together we obtain the following theorem.

Theorem 2.12. *([1]) The correspondence between modal algebras and descriptive frames restricts to a one-to-one correspondence between modal μ-algebras and descriptive μ-frames.*

We note that the duality result of [1] is a bit different than ours since in [1] descriptive μ-frames are defined as those descriptive frames where meets of clopen pre-fixed points are clopen. It was later observed in [8] that these meets are in fact the intersections of clopen pre-fixed points. In [18] the duality is obtained for distributive modal μ-lattices (algebraic models of negation-free μ-calculus). Our duality result can be seen as a restricted case of [18] when the distributive μ-lattice is a (Boolean) modal μ-algebra. In [1] and [18] the above correspondence between modal μ-algebras and descriptive μ-frames is also extended to a dual equivalence of the corresponding categories.

2.4. Axiomatic systems of modal fixed point logic

Next we briefly discuss the connection of modal μ-algebras and descriptive μ-frames with the axiomatic systems of μ-calculus. If φ and ψ are formulas and x a variable, we will denote by $\varphi[\psi/x]$ the formula obtained by freely replacing in φ each free occurrence of x by ψ.

Definition 2.13.

1. *[21] The axiomatization of Kozen's system* $\mathbf{K}^\mu$ *can be taken to consist of the following axioms and rules:*

   propositional tautologies,
   If $\vdash \varphi$ *and* $\vdash \varphi \to \psi$*, then* $\vdash \psi$ (*Modus Ponens*),
   If $\vdash \varphi$*, then* $\vdash \varphi[\psi/x]$ (*Substitution*),
   If $\vdash \varphi$*, then* $\vdash \Box\varphi$ (*Necessitation*),
   $\vdash \Box(x \to y) \to (\Box x \to \Box y)$ ($\mathbf{K}$*-axiom*),
   $\vdash \varphi[\mu x\varphi/x] \to \mu x\varphi$ (*Fixed Point axiom*),
   If $\vdash \varphi[\psi/x] \to \psi$*, then* $\vdash \mu x\varphi \to \psi$ (*Fixed Point rule*),

   where x *is not a bound variable of* φ *and no free variable of* ψ *is bound in* φ.

2. *[1, 10] Let* Φ *be a set of modal* μ*-formulas. We write* $\mathbf{K}^\mu + \Phi$ *for the smallest set of formulas which contains both* $\mathbf{K}^\mu$ *and* Φ *and is closed under the Modus Ponens, Substitution, Necessitation and Fixed Point rules. We say that* $\mathbf{K}^\mu + \Phi$ *is the* extension of $\mathbf{K}^\mu$ by Φ. *We also call* $\mathbf{K}^\mu + \Phi$ *a* normal modal fixed point logic.

Let $L = \mathbf{K}^\mu + \Phi$ be a normal modal fixed point logic. A modal μ-algebra $(B, \Diamond)$ is called an *L-algebra* if it validates all the formulas in Φ. A descriptive μ-frame (W, R) is called an *L-frame* if (W, R) validates all the formulas in Φ with respect to clopen assignments.

Theorem 2.14. *[1, 10] Let L be a normal modal fixed point logic. Then*

1. *L is sound and complete with respect to the class of modal* μ*-L-algebras.*
2. *L is sound and complete with respect to the class of descriptive* μ*-L-frames.*

We note that [1] prove this result using the Lindenbaum-Tarski algebra and canonical model constructions, while [10] give an alternative proof using the so-called replacement map and translations.

3. A comparison of different semantics of fixed point operators

In this section we investigate the connections between different kinds of semantics of modal μ-formulas. The results and examples discussed here are not directly relevant for the Sahlqvist completeness and correspondence theorem proved in Section 5. Thus, the reader interested only in the Sahlqvist theorem for modal fixed point logic can skip this section.

In the previous section we introduced various (e.g. clopen, closed, set-theoretic) semantics for modal μ-formulas. An obvious question is: how different are all these semantics? In this section we will address this question. We will first consider classes of descriptive μ-frames for which the semantics coincide. After that we will give examples of descriptive μ-frames for which the semantics differ.

Recall that a modal algebra is called *locally finite* if its every finitely generated subalgebra is finite. Let (W, R) be a descriptive frame and h a clopen assignment. Let also $\mathfrak{B} = (\mathsf{Clop}(W), \langle R\rangle)$. Then for each modal μ-formula φ whose only free variables are $x_1, \ldots, x_n$ we associate a modal subalgebra of $\mathfrak{B}$ generated by the elements $h(x_1), \ldots, h(x_n)$ and denote it by $\mathfrak{B}^\varphi_h$.

Theorem 3.1. *Let* $(B, \Diamond)$ *be a locally finite modal algebra and* (W, R) *its dual descriptive frame. Then for each formula* φ*, clopen assignment* h*, and* $\mathfrak{F}$ *such that* $\mathsf{Clop}(W) \subseteq \mathfrak{F} \subseteq \mathcal{P}(W)$*, we have*

$$[\![\varphi]\!]^{\mathfrak{F}}_h = [\![\varphi]\!]^{\mathcal{P}(W)}_h \in \mathfrak{B}^\varphi_h.$$

Consequently, (W, R) *is a descriptive* μ*-frame and* $(B, \Diamond)$ *is a modal* μ*-algebra.*

Proof. Since $(B, \Diamond)$ is locally finite and $(B, \Diamond)$ is isomorphic to $\mathfrak{B} = (\mathsf{Clop}(W), \langle R\rangle)$, for formula ψ we have that $\mathfrak{B}^\psi_h$ is finite. We now prove by induction on the complexity of any subformula ψ of φ that for any clopen assignment h and any $\mathfrak{F}$ with $\mathsf{Clop}(W) \subseteq \mathfrak{F} \subseteq \mathcal{P}(W)$ we have:

$$[\![\psi]\!]^{\mathfrak{F}}_h = [\![\psi]\!]^{\mathcal{P}(W)}_h \in \mathfrak{B}^\psi_h. \tag{1}$$

If $\psi = \bot$, $\psi = \top$, ψ is a propositional variable, $\psi = \chi_1 \wedge \chi_2$, $\psi = \chi_1 \vee \chi_2$, $\psi = \neg\chi$, $\psi = \Diamond\chi$ or $\psi = \Box\chi$, then (1) easily follows from the induction hypothesis. Now let $\psi = \mu x \chi$, where $\chi(x, x_1, \ldots, x_n)$ is a modal μ-formula positive in x. Let g be any clopen assignment and $\mathfrak{F}$ such that $\mathsf{Clop}(W) \subseteq \mathfrak{F} \subseteq \mathcal{P}(W)$. For each $l \in \omega$ we let:

$$S_0 = \emptyset \text{ and } S_{l+1} = [\![\chi]\!]^{\mathcal{P}(W)}_{g_x^{S_l}}. \tag{2}$$

Claim 3.2. *$S_l \in \mathfrak{B}^\psi_g$, for each $l \in \omega$.*

Proof. Since $\mathfrak{B}^\psi_g$ is a modal subalgebra of $\mathfrak{B}$, obviously $S_0 = \emptyset \in \mathfrak{B}^\psi_g$. Now we assume that $S_l \in \mathfrak{B}^\psi_g$ and prove that $S_{l+1} \in \mathfrak{B}^\psi_g$. Since $S_l \in \mathfrak{B}^\psi_g$ and g is a clopen assignment, the assignment $g_x^{S_l}$ is also clopen. Therefore, by our assumption, (1) holds for $g_x^{S_l}$ and χ. So $[\![\chi]\!]^{\mathcal{P}(W)}_{g_x^{S_l}} \in \mathfrak{B}^\chi_{g_x^{S_l}}$. Now $S_l \in \mathfrak{B}^\psi_g$ yields that the subalgebra of $\mathfrak{B}$ generated by the elements $S_l, g(x_1), \ldots, g(x_n)$ is equal to the subalgebra of $\mathfrak{B}$ generated by the elements $g(x_1), \ldots, g(x_n)$. Thus, $\mathfrak{B}^\chi_{g_x^{S_l}} = \mathfrak{B}^\psi_g$. So $S_{l+1} = [\![\chi]\!]^{\mathcal{P}(W)}_{g_x^{S_l}} \in \mathfrak{B}^\psi_g$, which completes the induction and the proof of the claim. $\Box$

It follows from Lemma 2.5, that $S_l \subseteq S_{l+1}$ for all l. Therefore, as $\mathfrak{B}^\psi_g$ is finite, there is $m \in \omega$ such that $S_l = S_m$ for all $l > m$. Let $U \in \mathfrak{F}$ be such that $[\![\chi]\!]^{\mathfrak{F}}_{g_x^U} \subseteq U$. By induction on l, we show that $S_l \subseteq U$, for all $l \in \omega$. Obviously, $S_0 \subseteq U$. Now assume $S_l \subseteq U$. Then by (2), (1) and Lemma 2.5, we have $S_{l+1} = [\![\chi]\!]^{\mathcal{P}(W)}_{g_x^{S_l}} = [\![\chi]\!]^{\mathfrak{F}}_{g_x^{S_l}} \subseteq [\![\chi]\!]^{\mathfrak{F}}_{g_x^U} \subseteq U$. So $S_l \subseteq U$, for all $l \in \omega$. By (1) and (2), $[\![\chi]\!]^{\mathfrak{F}}_{g_x^{S_m}} = [\![\chi]\!]^{\mathcal{P}(W)}_{g_x^{S_m}} = S_{m+1} = S_m$. As $\mathfrak{B}^\psi_g \subseteq \mathsf{Clop}(W) \subseteq \mathfrak{F}$ we obtain that $S_m \in \mathfrak{F}$. Therefore, S_m is a pre-fixed point that is contained in every pre-fixed point. So $S_m = [\![\mu x \chi]\!]^{\mathfrak{F}}_g$. As $\mathfrak{F}$ was arbitrary, we also have $S_m = [\![\mu x \chi]\!]^{\mathcal{P}(W)}_g$, which together with the fact that $S_m \in \mathfrak{B}^\psi_g$ completes the induction. Finally, as $\mathsf{Clop}(W) \subseteq \mathfrak{F}$, we deduce that $[\![\varphi]\!]^{\mathsf{Clop}(W)}_h \in \mathsf{Clop}(W)$. So (W, R) is a descriptive μ-frame and by Lemma 2.8, $(B, \Diamond)$ is a modal μ-algebra. This finishes the proof of the theorem. $\Box$

Next we will show that for descriptive μ-frames corresponding to complete modal algebras closed and clopen semantics coincide. For this we will first recall a topological characterization of the Stone spaces dual to complete Boolean algebras.

Theorem 3.3. (*see e.g. [27]*) *Let B be a Boolean algebra and W its dual Stone space. Then B is complete iff for each closed subset $F \subseteq W$, the interior of F is clopen iff for each open subset $U \subseteq W$, the closure of U is clopen.*

Stone spaces satisfying the condition of Theorem 3.3 are called *extremally disconnected* [27].

Lemma 3.4. *Let W be a non-empty set with the discrete topology and let R be a binary relation on W. Then*

1. *The Stone-Čech compactification $\beta(W)$ of W is extremally disconnected.*
2. *The Boolean algebra $\mathsf{Clop}(\beta(W))$ is isomorphic to the Boolean algebra $\mathcal{P}(W)$.*
3. *Let $(W_{\mathfrak{B}}, R_{\mathfrak{B}})$ be the dual space of $\mathfrak{B} = (\mathcal{P}(W), \langle R \rangle)$. Then $W_{\mathfrak{B}}$ is (up to isomorphism) the Stone-Čech compactification of W, W is the subset of $W_{\mathfrak{B}}$ consisting of all the isolated points and $R_{\mathfrak{B}} \cap W^2 = R$.*

Proof. The proofs of (1) and (2) can be found in [27]. The proof of (3) can be easily derived from (2) using the duality of descriptive frames and modal algebras. $\Box$

Theorem 3.5. *Let (W, R) be a descriptive μ-frame dual to a complete modal algebra. Then for each modal μ-formula φ and each clopen assignment h, we have*

$$[\![\varphi]\!]^{\mathsf{Clop}(W)}_h = [\![\varphi]\!]^{\mathsf{Cl}(W)}_h.$$

Proof. We will prove the theorem by induction on the complexity of φ. Our inductive hypothesis is: For any clopen assignment h and any subformula ψ of φ, we have

$$[\![\psi]\!]^{\mathsf{Clop}(W)}_h = [\![\psi]\!]^{\mathsf{Cl}(W)}_h. \tag{3}$$

If ψ is a constant, propositional variable or of the form $\psi = \chi_1 \wedge \chi_2$, $\psi = \chi_1 \vee \chi_2$, $\psi = \neg\chi$, $\psi = \Diamond\chi$, $\psi = \Box\chi$, then (3) easily follows from the induction hypothesis. Now let $\psi = \mu x \chi$, where $\chi(x, x_1, \ldots, x_n)$ is a modal μ-formula positive in x. Then

$$
\begin{array}{rcl}
[\![\mu x\chi]\!]_h^{\mathsf{Cl}(W)} & = & \bigcap\{F \in \mathsf{Cl}(W) : [\![\chi]\!]_{h_x^F}^{\mathsf{Cl}(W)} \subseteq F\} \text{ (by definition)} \\
& \subseteq & \bigcap\{U \in \mathsf{Clop}(W) : [\![\chi]\!]_{h_x^U}^{\mathsf{Cl}(W)} \subseteq U\} \text{ (as } \mathsf{Clop}(W) \subseteq \mathsf{Cl}(W)) \\
& = & \bigcap\{U \in \mathsf{Clop}(W) : [\![\chi]\!]_{h_x^U}^{\mathsf{Clop}(W)} \subseteq U\} \text{ (by (3))} \\
& = & [\![\mu x\chi]\!]_h^{\mathsf{Clop}(W)}.
\end{array}
$$

So it remains to prove that

$$[\![\mu x\chi]\!]_h^{\mathsf{Clop}(W)} \subseteq [\![\mu x\chi]\!]_h^{\mathsf{Cl}(W)}.$$

Let $\mathbb{F} = \{F \in \mathsf{Cl}(W) : [\![\chi]\!]_{h_x^F}^{\mathsf{Cl}(W)} \subseteq F\}$. We also let $G = [\![\mu x\chi]\!]_h^{\mathsf{Cl}(W)}$ and $D = \mathrm{Int}(G)$, where $\mathrm{Int}(G)$ is the interior of G. Since W corresponds to a complete algebra, by Theorem 3.3, D is clopen. So, by (3), $[\![\chi]\!]_{h_x^D}^{\mathsf{Cl}(W)} = [\![\chi]\!]_{h_x^D}^{\mathsf{Clop}(W)}$. Obviously for each $F \in \mathbb{F}$ we have $D \subseteq F$. So, by Lemma 2.5, $[\![\chi]\!]_{h_x^D}^{\mathsf{Cl}(W)} \subseteq [\![\chi]\!]_{h_x^F}^{\mathsf{Cl}(W)} \subseteq F$. Therefore, $[\![\chi]\!]_{h_x^D}^{\mathsf{Cl}(W)} \subseteq \bigcap \mathbb{F} = G$ and, by (3), we obtain $[\![\chi]\!]_{h_x^D}^{\mathsf{Clop}(W)} \subseteq G$. But since (W, R) is a descriptive μ-frame, h is a clopen assignment and D is a clopen, $[\![\chi]\!]_{h_x^D}^{\mathsf{Clop}(W)}$ is clopen. Hence, $[\![\chi]\!]_{h_x^D}^{\mathsf{Clop}(W)} \subseteq \mathrm{Int}(G) = D$. Thus, D is a clopen pre-fixed point contained in G, which implies that $[\![\mu x\chi]\!]_h^{\mathsf{Clop}(W)} \subseteq G = [\![\mu x\chi]\!]_h^{\mathsf{Cl}(W)}$. This finishes the induction and the proof of the theorem. □

It is still an open problem whether Theorem 3.5 holds for descriptive μ-frames not corresponding to complete or locally finite algebras. Next we will give an example of a descriptive μ-frame, a closed assignment and a modal μ-formula φ for which closed and clopen semantics differ.

Example 3.6. Let $\mathbb{Z}$ be the set of integers with the discrete topology. We define a relation R on $\mathbb{Z}$ by zRy iff $y = z + 1$ or $y = z - 1$ for $z, y \in \mathbb{Z}$. Then $\mathfrak{A} = (\mathcal{P}(\mathbb{Z}), \langle R \rangle)$ is a complete modal algebra and therefore it is a modal μ-algebra. Let (W, R^*) be its dual descriptive frame. By Lemma 3.4, the subframe consisting of all principal ultrafilters in W will be isomorphic to $(\mathbb{Z}, R)$ and every singleton consisting of a principal ultrafilter will be clopen in W. We will denote this subspace with the restricted order by $(\mathbb{Z}^*, R^*)$. (In fact, topologically, as mentioned in Lemma 3.4, W is the Stone–Čech compactification of $\mathbb{Z}^*$ with the discrete topology.) Let $M = W \setminus \mathbb{Z}^*$ denote the closed set of all non-principal ultrafilters of $\mathfrak{A}$. For each $z \in \mathbb{Z}$ we let $F_z = \{U \subseteq \mathbb{Z} : z \in U\}$. Obviously, F_z is a principal ultrafilter of $\mathfrak{A}$ and each principal ultrafilter of $\mathfrak{A}$ is of the form F_z for some $z \in \mathbb{Z}$.

Claim 3.7.

1. *For each principal ultrafilter $F_z \in \mathbb{Z}^*$ and non-principal ultrafilter $F \in M$ we have $\neg(F_z R^* F)$ and $\neg(F R^* F_z)$.*
2. *For each non-principal ultrafilter F, there exists a non-principal ultrafilter F' such that $F' R^* F$.*
3. $\langle R^* \rangle M = M$.

Proof. (1) Since F is a non-principal ultrafilter, it contains all cofinite subsets of $\mathbb{Z}$. Let $V = \mathbb{Z} \setminus \langle R \rangle(\{z\})$. Then V is cofinite and therefore $V \in F$. Moreover, $(z+1) \notin V$ and $(z-1) \notin V$. Thus, $z \notin \langle R \rangle(V)$ and so $\langle R \rangle(V) \notin F_z$. This implies that $\neg(F_z R^* F)$. On the other hand, $\{z\} \in F_z$. But $\langle R \rangle(\{z\}) = \{z+1, z-1\} \notin F$. Therefore, $\neg(F R^* F_z)$.

(2) Let $F \in M$. We consider the set $S = \{\langle R \rangle U : U \in F\}$. We generate a filter by S and then extend it to a maximal filter. The filter generated by S is proper. To see this, assume $\langle R \rangle U_1 \cap \cdots \cap \langle R \rangle U_n \in S$. Then $U_1, \ldots, U_n \in F$ and since $\bigcap_{i=1}^n U_i \neq \emptyset$, there exists $z \in \mathbb{Z}$ such that $z \in \bigcap_{i=1}^n U_i$. But then $(z+1) \in \bigcap_{i=1}^n \langle R \rangle U_i$. Now we extend this filter to a maximal filter F'. By the definition, $\langle R \rangle U$ belongs to F' for each $U \in F$. So we have $F' R^* F$. By (1) F' must be non-principal. (Alternatively, we could take the filter $F' = \{u + 1 : u \in F\}$ and show that it satisfies condition (2) of the claim.)

(3) Follows directly from (2) and (1). □

Next we define a closed (not clopen) assignment h on W by $h(p) = \{F_0\} \cup M$. Consider the formula $\varphi(x, p) = p \vee \Diamond\Diamond x$. Then, using the claim, it is easy to see that the only closed pre-fixed points of $\varphi(x, h(p))$ are the whole space W and the set $E_M = \{F_z : z \text{ is even or negative even}\} \cup M$. However, the only clopen pre-fixed point of $\varphi(x, h(p))$ is the whole space W. Therefore, $[\![\mu x\varphi]\!]_h^{\mathsf{Clop}(W)} = W \neq [\![\mu x\varphi]\!]_h^{\mathsf{Cl}(W)} = E_M$. It is also easy to see that $E_M = \varphi(E_M, h(p))$. Thus, E_M is a closed fixed point of the map $(F \mapsto [\![\varphi]\!]_{h_x^F}^{\mathsf{Clop}(W)})$, for $F \in \mathsf{Cl}(W)$. This implies that $[\![\mu x\varphi]\!]_h^{\mathsf{Clop}(W)}$ is not the least closed fixed point of $(F \mapsto [\![\varphi]\!]_{h_x^F}^{\mathsf{Clop}(W)})$, for $F \in \mathsf{Cl}(W)$.

Example 3.8. We note that if, in the previous example, we consider the clopen assignment $h(p) = \{F_0\}$, then $[\![\mu x\varphi]\!]_h^{\mathcal{P}(W)} = E = \{F_z : z \text{ is even or negative even}\}$. Thus every pre-fixed point of $\varphi(x, h(p))$ contains E. Moreover, it is easy to see that E is an open set and thus, by Theorem 3.3, the closure of E, which we denote by $\overline{E}$, is a clopen set. It is not hard to see that $\overline{E}$ is a pre-fixed point of $\varphi(x, h(p))$. Therefore, $\overline{E}$ is the least clopen pre-fixed point of $\varphi(x, h(p))$. So $\overline{E} = [\![\mu x\varphi]\!]_h^{\mathsf{Clop}(W)} \neq [\![\mu x\varphi]\!]_h^{\mathcal{P}(W)}$.

In Example 3.8 we have that $[\![\mu x\varphi]\!]_h^{\mathsf{Clop}(W)}$ is the closure of $[\![\mu x\varphi]\!]_h^{\mathcal{P}(W)}$. The next example shows not only that this is not the case in general, but also that the closure of $[\![\mu x\varphi]\!]_h^{\mathcal{P}(W)}$ may not be even a fixed point of $\varphi(x, h(p))$.

Example 3.9. We will give an example of a descriptive μ-frame (W, R), a clopen assignment h and a modal formula $\varphi(x, p)$ such that the closure of $[\![\mu x\varphi]\!]_h^{\mathcal{P}(W)}$ is not a fixed point of $\varphi(x, h(p))$. Let $\mathbb{Z}$ be the set of integers with the discrete topology. Let $W = \beta(\mathbb{Z})$ be the Stone–Čech compactification of $\mathbb{Z}$. We define a relation R on W by zRy iff ($z, y \in \mathbb{Z}$ and $y = z + 1$ or $y = z - 1$ or $z \in W$ and $y \in \beta(\mathbb{Z}) \setminus \mathbb{Z}$). It is easy to check that (W, R) is a descriptive frame. Moreover, by Lemma 3.4, (W, R) is a descriptive μ-frame. Now we define a clopen assignment $h(p) = \{0\}$. Consider the formula $\varphi(x, p) = p \vee \Diamond\Diamond x$. Then $[\![\mu x(p \vee \Diamond\Diamond x)]\!]_h^{\mathcal{P}(W)}$ is equal to the set of all even and negative even numbers. The closure of this set contains a proper subset of $\beta(\mathbb{Z}) \setminus \mathbb{Z}$ and, as is easy to check, is not a fixed point of $\varphi(x, h(p))$. Note that in this case we have $[\![\mu x(p \vee \Diamond\Diamond x)]\!]_h^{\mathsf{Clop}(W)} = [\![\mu x(p \vee \Diamond\Diamond x)]\!]_h^{\mathsf{Cl}(W)} = W$.

In addition, if we demand that $\neg(0R1)$, $\neg(0R(-1))$ and $\neg(0Ry)$ for each $y \in \beta(\mathbb{Z}) \setminus \mathbb{Z}$, then the same argument shows that the clopen semantics of the formula $\varphi' = \mu x(\Box\bot \vee \Diamond\Diamond x)$, under any assignment (assignments play no role as φ' has no free variables), is W, whereas the set-theoretic semantics of φ', under any assignment, is equal to the set of even and negative even numbers. We deduce that φ' is valid on (W, R) as a descriptive μ-frame, but is not valid on (W, R) seen as a Kripke frame.

Remark 3.10. We can combine Examples 3.6 and 3.9 by taking the disjoint union of the frames defined in these examples. This will give us an example of a (single) descriptive μ-frame (W, R), a closed assignment h and a modal μ-formula φ such that all the three semantics of φ differ and, moreover, neither closed nor clopen semantics of φ is the closure of the set-theoretic semantics of φ. We skip the details.

4. The intersection lemma

In this section we address two issues. We prove the analogue of the Esakia–Sambin–Vaccaro lemma, which will play an essential role in Section 5 in proving Sahlqvist's completeness and correspondence results for modal fixed point logic. We also discuss whether clopen semantics gives rise to fixed points for closed and set-theoretic assignments. We use the analogue of the Esakia–Sambin–Vaccaro lemma in proving that the clopen semantics gives a fixed point for closed assignments. We also show that in general the clopen semantics does not provide a fixed point for set-theoretic assignments. Note that the only fact in this section that will be used in the proof of the Sahlqvist theorem for modal fixed point logic (Section 5) is Lemma 4.6.

In more detail, let (W, R) be a descriptive μ-frame, h a clopen assignment, and $\varphi(x, x_1, \dots, x_n)$ a modal μ-formula positive in x. Then, by Corollary 2.9, $[\![\mu x\varphi]\!]_h^{\mathsf{Clop}(W)}$ is the least fixed point of the map $(C \mapsto [\![\varphi]\!]_{h_x^C}^{\mathsf{Clop}(W)})$, for $C \in \mathsf{Clop}(W)$. On the other hand, Example 3.6 shows that there exist a descriptive μ-frame (W, R), a closed assignment h and a modal formula φ positive in x such that $[\![\mu x\varphi]\!]_h^{\mathsf{Clop}(W)}$ is not the least fixed point of the map $(F \mapsto [\![\varphi]\!]_{h_x^F}^{\mathsf{Clop}(W)})$, for $F \in \mathsf{Cl}(W)$. The next question we are going to address is whether $[\![\mu x\varphi]\!]_h^{\mathsf{Clop}(W)}$ is a (not necessarily least) fixed point for the maps $(F \mapsto [\![\varphi]\!]_{h_x^F}^{\mathsf{Clop}(W)})$ and $(U \mapsto [\![\varphi]\!]_{g_x^U}^{\mathsf{Clop}(W)})$ for $F \in \mathsf{Cl}(W)$, $U \in \mathcal{P}(W)$, a closed assignment h, and set-theoretic assignment g, respectively. In fact, we will prove that for a closed assignment h, $[\![\mu x\varphi]\!]_h^{\mathsf{Clop}(W)}$ is a fixed point of the map $(F \mapsto [\![\varphi]\!]_{h_x^F}^{\mathsf{Clop}(W)})$ for $F \in \mathsf{Cl}(W)$. We will also show that there exist a descriptive μ-frame (W, R) and a set-theoretic assignment g such that $[\![\mu x\varphi]\!]_g^{\mathsf{Clop}(W)}$ is not a fixed point of the map $(U \mapsto [\![\varphi]\!]_{g_x^U}^{\mathsf{Clop}(W)})$ for $U \in \mathcal{P}(W)$.

Definition 4.1. *We call a modal μ-formula φ* positive *if it does not contain any negation. φ is called* negative *if $\neg\varphi$ is positive.*

Remark 4.2. We note that φ is positive implies that φ is positive in each variable, but not vice versa.

Lemma 4.3. *Let (W, R) be a descriptive frame, h a closed assignment and $\varphi(x, x_1, \ldots, x_n)$ a positive modal μ-formula. Then the set $[\![\varphi]\!]_h^{\mathsf{Clop}(W)}$ is closed. Consequently, the map $(F \mapsto [\![\varphi]\!]_{h_x^F}^{\mathsf{Clop}(W)})$ mapping each closed set F to $[\![\varphi]\!]_{h_x^F}^{\mathsf{Clop}(W)}$ is well defined and monotone.*

Proof. We will prove the result by induction on the complexity of φ. If φ is a constant or propositional variable, then as h is closed, $[\![\varphi]\!]_h^{\mathsf{Clop}(W)}$ is obviously closed. The cases $\varphi = \psi \wedge \chi$ and $\varphi = \psi \vee \chi$ are trivial since finite unions and intersections of closed sets are closed. The cases $\varphi = \Diamond\psi$ and $\varphi = \Box\psi$ follow directly from Lemma 2.1(1),(2). Finally, the case $\varphi = \mu x\psi$ is also easy since any intersection of clopen sets is closed. Therefore, $(F \mapsto [\![\varphi]\!]_{h_x^F}^{\mathsf{Clop}(W)})$ is a well-defined map from $\mathsf{Cl}(W)$ to $\mathsf{Cl}(W)$. Monotonicity of this map follows from Lemma 2.5. $\Box$

Remark 4.4. Since $\mathsf{Cl}(W)$ is a complete lattice, by the Knaster–Tarski theorem, the map $(F \mapsto [\![\varphi]\!]_{h_x^F}^{\mathsf{Clop}(W)})$ will have a least fixed point. As the meet in $\mathsf{Cl}(W)$ coincides with the intersection, the least point will be the intersection of all closed pre-fixed points. However, as was shown in Example 3.6, this least fixed point may be different from $[\![\mu x\varphi]\!]_h^{\mathsf{Clop}(W)}$.

Next we prove an auxiliary lemma which is an extension of the so-called intersection lemma of Esakia–Sambin–Vaccaro [14], [24] to the modal μ-case. This lemma will be an essential ingredient in the proof of the Sahlqvist completeness result in Section 5. We will be concerned only with the clopen semantics. So we will skip the sup index $\mathsf{Clop}(W)$ everywhere. We first recall Esakia's lemma. Let W be any set. A set $\mathbb{F} \subseteq \mathcal{P}(W)$ is called *downward directed* if for each $F, F' \in \mathbb{F}$, there exists $F'' \in \mathbb{F}$ such that $F'' \subseteq F \cap F'$.

Lemma 4.5. *[14](Esakia) Let (W, R) be a descriptive frame and $\mathbb{F} \subseteq \mathsf{Cl}(W)$ a downward directed set. Then*

$$\langle R\rangle \bigcap\{F : F \in \mathbb{F}\} = \bigcap\{\langle R\rangle F : F \in \mathbb{F}\}$$

Next we prove a modal μ-analogue of the Intersection Lemma of [24].

Lemma 4.6. *Let (W, R) be a descriptive frame.[1] Let also $F, F_1, \ldots, F_n \subseteq W$ be closed sets and let $\mathcal{A} \subseteq \mathsf{Clop}(W)$ be a downward directed set such that $\bigcap \mathcal{A} = F$. Then for each positive modal μ-formula $\varphi(x, x_1, \ldots, x_n)$ we have*

$$\varphi(F, F_1, \ldots, F_n) = \bigcap\{\varphi(U, F_1, \ldots, F_n) : U \in \mathcal{A}\}.$$

Proof. We will prove the lemma by induction on the complexity of φ. The modal cases are already proved in [24]. We briefly recall these proofs to make the paper self contained.

If $\varphi = \bot$ or $\varphi = \top$, then the lemma is obvious. If φ is a propositional variable, then the lemma is again obvious since every closed set is the intersection of the clopen sets containing it.

First let $\varphi = \psi \wedge \chi$. Then

$$\begin{aligned}
\varphi(F, F_1, \ldots, F_n) &= \psi(F, F_1, \ldots, F_n) \cap \chi(F, F_1, \ldots, F_n) \\
&= \bigcap\{\psi(U, F_1, \ldots, F_n) : U \in \mathcal{A}\} \cap \bigcap\{\chi(U, F_1, \ldots, F_n) : U \in \mathcal{A}\} \text{ (ind)} \\
&= \bigcap\{\psi(U, F_1, \ldots, F_n) \cap \chi(U, F_1, \ldots, F_n) : U \in \mathcal{A}\} \\
&= \bigcap\{(\psi \wedge \chi)(U, F_1, \ldots, F_n) : U \in \mathcal{A}\} \\
&= \bigcap\{\varphi(U, F_1, \ldots, F_n) : U \in \mathcal{A}\}.
\end{aligned}$$

Now let $\varphi = \psi \vee \chi$. Since φ is positive we have that $\varphi(F, F_1, \ldots, F_n) \subseteq \varphi(U, F_1, \ldots, F_n)$ for each $U \in \mathcal{A}$. Thus, $\varphi(F, F_1, \ldots, F_n) \subseteq \bigcap\{\varphi(U, F_1, \ldots, F_n) : U \in \mathcal{A}\}$. Now suppose $w \notin \varphi(F, F_1, \ldots, F_n)$. Then, by the induction hypothesis, $w \notin \bigcap\{\psi(C, F_1, \ldots, F_n) : C \in \mathcal{A}\} \cup \bigcap\{\chi(D, F_1, \ldots, F_n) : D \in \mathcal{A}\}$. So $w \notin \bigcap\{\psi(C, F_1, \ldots, F_n) : C \in \mathcal{A}\}$ and $w \notin \bigcap\{\chi(D, F_1, \ldots, F_n) : D \in \mathcal{A}\}$. Therefore, there exists $C, D \in \mathcal{A}$ such that $w \notin \psi(C, F_1, \ldots, F_n)$ and $w \notin \chi(D, F_1, \ldots, F_n)$. Since $\mathcal{A}$ is downward directed, there exists $E \in \mathcal{A}$ such that $E \subseteq C \cap D$. As both ψ and χ are positive, by Lemma 2.5, we have $w \notin \psi(E, F_1, \ldots, F_n)$ and $w \notin \chi(E, F_1, \ldots, F_n)$. Thus, $w \notin \bigcap\{\psi(E, F_1, \ldots, F_n) \cup \chi(E, F_1, \ldots, F_n) : E \in \mathcal{A}\}$ and therefore, $w \notin \bigcap\{\varphi(U, F_1, \ldots, F_n) : U \in \mathcal{A}\}$. This means that $\bigcap\{\varphi(U, F_1, \ldots, F_n) : U \in \mathcal{A}\} \subseteq \varphi(F, F_1, \ldots, F_n)$. So $\varphi(F, F_1, \ldots, F_n) = \bigcap\{\varphi(U, F_1, \ldots, F_n) : U \in \mathcal{A}\}$.

[1] Note that we do not require that (W, R) is a descriptive μ-frame.

Now suppose $\varphi = \Diamond\psi$. We will need to use the following fact, which easily follows from Lemma 2.5: if $\mathcal{A}$ is downward directed, then $\{\psi(U, F_1, \dots, F_n) : U \in \mathcal{A}\}$ is also downward directed. So

$$\begin{array}{rcl}\varphi(F, F_1, \dots, F_n) & = & \langle R\rangle\psi(F, F_1, \dots, F_n) \\ & = & \langle R\rangle \bigcap\{\psi(U, F_1, \dots, F_n) : U \in \mathcal{A}\} \text{ (ind hyp)} \\ & = & \bigcap\{\langle R\rangle\psi(U, F_1, \dots, F_n) : U \in \mathcal{A}\} \text{ (Esakia's lemma)} \\ & = & \bigcap\{\varphi(U, F_1, \dots, F_n) : U \in \mathcal{A}\}.\end{array}$$

Now assume $\varphi = \Box\psi$. We recall that $\langle R\rangle$ commutes with all unions. Then

$$\begin{array}{rcl}\varphi(F, F_1, \dots, F_n) & = & [R]\psi(F, F_1, \dots, F_n) \\ & = & W \setminus \langle R\rangle \bigcup\{W \setminus \psi(U, F_1, \dots, F_n) : U \in \mathcal{A}\} \text{ (ind hyp)} \\ & = & W \setminus \bigcup\{\langle R\rangle(W \setminus \psi(U, F_1, \dots, F_n)) : U \in \mathcal{A}\} \\ & = & \bigcap\{[R]\psi(U, F_1, \dots, F_n) : U \in \mathcal{A}\} \\ & = & \bigcap\{\varphi(U, F_1, \dots, F_n) : U \in \mathcal{A}\}.\end{array}$$

Finally, let $\varphi = \mu x\psi(x, y, x_1, \dots, x_n)$. Then we need to show

$$\mu x\psi(x, F, F_1, \dots, F_n) = \bigcap\{\mu x\psi(x, U, F_1, \dots, F_n) : U \in \mathcal{A}\}.$$

By Lemma 2.5, for each $U \in \mathcal{A}$ we have $\mu x\psi(x, F, F_1, \dots, F_n) \subseteq \mu x\psi(x, U, F_1, \dots, F_n)$. Therefore, $\mu x\psi(x, F, F_1, \dots, F_n) \subseteq \bigcap\{\mu x\psi(x, U, F_1, \dots, F_n) : U \in \mathcal{A}\}$.

Now suppose $w \in \bigcap\{\mu x\psi(x, C, F_1, \dots, F_n) : C \in \mathcal{A}\}$. Then we have that $w \in \mu x\psi(x, C, F_1, \dots, F_n)$ for each $C \in \mathcal{A}$. So for each $C \in \mathcal{A}$ and each $V \in \mathsf{Clop}(W)$ with $\psi(V, C, F_1, \dots, F_n) \subseteq V$ we have $w \in V$. Assume $U \in \mathsf{Clop}(W)$ is such that $\psi(U, F, F_1, \dots, F_n) \subseteq U$. By the induction hypothesis we have $\psi(U, F, F_1, \dots, F_n) = \bigcap\{\psi(U, C, F_1, \dots F_n) : C \in \mathcal{A}\}$. Thus $\bigcap\{\psi(U, C, F_1, \dots, F_n) : C \in \mathcal{A}\} \subseteq U$. By Lemma 4.3, each $\psi(U, C, F_1, \dots, F_n)$ is a closed set. Therefore, as U is open, by compactness, there exist finitely many $C_1, \dots, C_k \in \mathcal{A}$ such that $\bigcap_{i=1}^k \psi(U, C_i, F_1, \dots, F_n) \subseteq U$. As $\mathcal{A}$ is downward directed, there exists $C' \in \mathcal{A}$ such that $C' \subseteq \bigcap_{i=1}^k C_i$. Then, by Lemma 2.5, $\psi(U, C', F_1, \dots, F_n) \subseteq U$. But then $w \in U$. Thus, $w \in \bigcap\{U \in \mathsf{Clop}(W) : \psi(U, F, F_1, \dots, F_n) \subseteq U\} = \mu x\psi(x, F, F_1, \dots, F_n)$, which finishes the proof of the lemma. □

Corollary 4.7. *Let (W, R) be a descriptive frame, $F_1, \dots, F_n, G_1, \dots, G_k \subseteq W$ closed sets and $\varphi(x_1, \dots, x_n, y_1, \dots, y_k)$ a positive modal μ-formula. Then*

1. $\varphi(F_1, \dots, F_n, G_1, \dots, G_k) = \bigcap\{\varphi(C_1, \dots, C_n, G_1, \dots, G_k) : F_i \subseteq C_i \in \mathsf{Clop}(W), 1 \leq i \leq n\}$.
2. $\varphi(F_1, \dots, F_n, G_1, \dots, G_k) = \bigcap\{\varphi(C_1, \dots, C_n, G_1, \dots, G_k) : F_i \subseteq C_i \in \mathcal{A}_i, 1 \leq i \leq n\}$*, where $\mathcal{A}_i \subseteq \mathsf{Clop}(W)$ is downward directed and $\bigcap \mathcal{A}_i = F_i$, for each $1 \leq i \leq n$.*

Proof. The result follows from Lemma 4.6 by a trivial induction. □

Next we will apply Lemma 4.6 to show that for each descriptive frame (W, R), positive modal formula φ, and a closed assignment h, the set $[\![\mu x\varphi]\!]_h^{\mathsf{Clop}(W)}$ is a fixed point of the map $(F \mapsto [\![\varphi]\!]_{h_x^F}^{\mathsf{Clop}(W)})$ for $F \in \mathsf{Cl}(W)$.

Lemma 4.8. *Let (W, R) be a descriptive μ-frame and $\varphi(x, x_1, \dots, x_n)$ a positive modal μ-formula. Let $G = \mu x\varphi(x, F_1, \dots, F_n)$, where $F_1, \dots F_n \subseteq W$ be closed sets. Then $\varphi(G, F_1, \dots, F_n) = G$, that is, G is a fixed point of the map $(F \mapsto [\![\varphi]\!]_{h_x^F}^{\mathsf{Clop}(W)})$ for $F \in \mathsf{Cl}(W)$.*

Proof. We first show that G is a pre-fixed point, that is, $\varphi(G, F_1, \dots, F_n) \subseteq G$. Let V be an arbitrary clopen pre-fixed point: that is, $\varphi(V, F_1, \dots, F_n) \subseteq V$. Then, by the definition of G, we have $G \subseteq V$. By Lemma 2.5 we obtain $\varphi(G, F_1, \dots, F_n) \subseteq \varphi(V, F_1, \dots, F_n) \subseteq V$. Therefore, $\varphi(G, F_1, \dots, F_n) \subseteq \bigcap\{V \in \mathsf{Clop}(W) : \varphi(V, F_1, \dots, F_n) \subseteq V\} = G$.

Conversely, as G is the intersection of closed sets, G is closed. Therefore, by Corollary 4.7, we have

$$\varphi(G, F_1, \dots, F_n) = \bigcap\{\varphi(U, U_1, \dots, U_n) : G \subseteq U \in \mathsf{Clop}(W), F_i \subseteq U_i \in \mathsf{Clop}(W), 1 \leq i \leq n\}. \quad (4)$$

Let U and $U_1, \dots, U_n$ be arbitrary clopen sets with $G \subseteq U$ and $F_i \subseteq U_i$ for $1 \leq i \leq n$. We show that $G \subseteq \varphi(U, U_1, \dots, U_n)$. The fact that $G \subseteq U$ means that $\bigcap\{V \in \mathsf{Clop}(W) : \varphi(V, F_1, \dots, F_n) \subseteq V\} \subseteq U$. Therefore, the same argument as in the proof of Lemma 4.6 shows that there exists a clopen set $V' \subseteq U$ such that $\varphi(V', F_1, \dots, F_n) \subseteq V'$. By

Corollary 4.7, $\varphi(V', F_1, \ldots, F_n) = \bigcap\{\varphi(V', C_1, \ldots, C_n) : C_i \in \mathsf{Clop}(W), F_i \subseteq C_i \subseteq U_i, 1 \le i \le n\}$. But then a similar argument as in the proof of Lemma 4.6 shows that there exist clopen sets $C'_1, \ldots, C'_n$ such that $F_i \subseteq C'_i \subseteq U_i$ for $1 \le i \le n$ and $\varphi(V', C'_1, \ldots, C'_n) \subseteq V'$. By monotonicity we have $\varphi(\varphi(V', C'_1, \ldots, C'_n), F_1, \ldots, F_n) \subseteq \varphi(V', F_1, \ldots, F_n) \subseteq \varphi(V', C'_1, \ldots, C'_n)$. Since (W, R) is a descriptive μ-frame, $\varphi(V', C'_1, \ldots, C'_n)$ is a clopen set. Thus $\varphi(V', C'_1, \ldots, C'_n)$ is a clopen pre-fixed point of $\varphi(\cdot, F_1, \ldots, F_n)$. This means that $G \subseteq \varphi(V', C'_1, \ldots, C'_n)$. But since φ is monotone and $C'_i \subseteq U_i$ for $1 \le i \le n$ we have $\varphi(V', C'_1, \ldots, C'_n) \subseteq \varphi(U, U_1, \ldots, U_n)$. Thus, as $U, U_1, \ldots, U_n$ were arbitrary, we obtain by (4) that $G \subseteq \bigcap\{\varphi(U, U_1, \ldots, U_n) : G \subseteq U, F_i \subseteq U_i, 1 \le i \le n\} = \varphi(G, F_1, \ldots, F_n)$. □

Next we will see that an analogue of Lemma 4.8 does not hold for set-theoretic assignments.

Example 4.9. We will give an example of a descriptive μ-frame (W, R), a set-theoretic (neither clopen nor closed) assignment h, and a formula $\varphi(x, p)$ such that $\bigcap\{C \in \mathsf{Clop}(W) : \varphi(C, h(p)) \subseteq C\}$ is no longer a fixed point of $\varphi(x, h(p))$. Let $\mathbb{N}$ be the set of natural numbers with the discrete topology. Let $W = \beta(\mathbb{N})$ be the Stone–Čech compactification of $\mathbb{N}$. Let $M = \beta(\mathbb{N}) \setminus \mathbb{N}$. We define a relation R on W by zRy iff $z \in W$ and $y \in M$. It is easy to check that (W, R) is a descriptive frame. Moreover, by Lemma 3.4, (W, R) is a descriptive μ-frame. Now define an assignment $h(p) = E$, where E is the set of all even numbers. Obviously, h is neither clopen nor closed. Consider a formula $\varphi(x, p) = p \vee \Box x$. Then the clopen semantics $[\![\mu x\varphi]\!]_h^{\mathsf{Clop}(W)}$ of $\varphi(x, h(p))$ is equal to $\overline{E}$, the closure of E. To see this, note that every clopen containing E must contain $\overline{E}$. Thus, $\overline{E} \subseteq \bigcap\{C \in \mathsf{Clop}(W) : \varphi(C, h(p)) \subseteq C\}$. On the other hand, W is extremally disconnected. So $\overline{E}$ is clopen. Also note that $(W \setminus \overline{E}) \cap M \neq \emptyset$. Thus, $\langle R\rangle(W \setminus \overline{E}) = W$. Then $[R]\overline{E} = W \setminus \langle R\rangle(W \setminus \overline{E}) = W \setminus W = \emptyset$. Therefore, $\varphi(\overline{E}, h(p)) = E \cup [R]\overline{E} = E \subseteq \overline{E}$. So $\overline{E}$ is a clopen pre-fixed point of $\varphi(x, h(p))$ and we have that $\bigcap\{C \in \mathsf{Clop}(W) : \varphi(C, h(p)) \subseteq C\} \subseteq \overline{E}$. Finally, note that, as computed above, $\varphi(\overline{E}, h(p)) = E \neq \overline{E}$. Thus, $\overline{E}$ is not a fixed point of $\varphi(x, h(p))$.

5. Sahlqvist fixed point formulas

In this section we extend the proof of the Sahlqvist completeness and correspondence results of [24] from modal logic to modal μ-calculus.

5.1. Completeness

For each $m \in \omega$ we let $\Box^0 x = x$ and $\Box^{m+1} x = \Box(\Box^m x)$.

Definition 5.1. *A formula $\varphi(x_1, \ldots, x_n)$ is called a* Sahlqvist fixed point formula *if it is obtained from formulas of the form $\neg\Box^m x_i$ ($m \in \omega$, $i \le n$) and positive formulas (in the language with the μ-operator) by applying the operations $\vee$ and $\Box$.*

Remark 5.2. We note that when considering the language without fixed point operators, the above definition of the Sahlqvist formula is different from the 'standard' definition (see e.g., [7]), but is equivalent to it. Any Sahlqvist formula of [7] is equivalent to a conjunction of Sahlqvist formulas in the aforementioned sense.

Theorem 5.3. *Let (W, R) be a descriptive frame,*[2] *$w \in W$ and $\varphi(x_1, \ldots, x_l)$ a Sahlqvist fixed point formula. If $w \in [\![\varphi]\!]_f^{\mathsf{Clop}(W)}$, for each clopen assignment f, then $w \in [\![\varphi]\!]_h^{\mathsf{Clop}(W)}$, for each set-theoretic assignment h.*

Proof. Since $\varphi(x_1, \ldots, x_l)$ is a Sahlqvist fixed point formula, there exists a formula $\alpha(p_1, \ldots, p_n, q_1, \ldots, q_m)$ using only $\vee$ and $\Box$ such that all listed propositional variables occur and no propositional variable occurs twice in α, and there exist positive formulas $\pi_1, \ldots, \pi_m$ and formulas $\psi_1, \ldots, \psi_n$, where each ψ_i is of the form $\neg\Box^{d_i} s_i$, for some $d_i \in \omega$ and $s_i \in \{x_1, \ldots, x_l\}$ such that

$$\varphi(x_1, \ldots x_l) = \alpha(\psi_1/p_1, \ldots, \psi_n/p_n, \pi_1/q_1, \ldots, \pi_m/q_m). \tag{5}$$

Let h be an assignment such that $w \notin [\![\varphi]\!]_h^{\mathsf{Clop}(W)}$. For each subformula β of α we define a world $w_\beta \in W$ by induction such that

$$w_\beta \notin [\![\beta(\psi_1/p_1, \ldots, \psi_n/p_n, \pi_1/q_1, \ldots, \pi_m/q_m)]\!]_h^{\mathsf{Clop}(W)}. \tag{6}$$

As the basic step of the induction we put $w_\alpha = w$. Now assume β is a subformula of α and w_β is already defined and satisfies (6). There are three possible cases:

[2] Note again that we do not require that (W, R) is a descriptive μ-frame.

1. β is atomic. Then there is nothing to define.
2. $\beta = \gamma \vee \delta$. Then we put $w_\gamma = w_\delta = w_\beta$. We obviously have $w_\gamma \notin [\![\gamma(\psi_1, \dots, \psi_n, \pi_1, \dots, \pi_m)]\!]_h^{\mathsf{Clop}(W)}$ and $w_\delta \notin [\![\delta(\psi_1, \dots, \psi_n, \pi_1, \dots, \pi_m)]\!]_h^{\mathsf{Clop}(W)}$.
3. $\beta = \Box\gamma$. Then we have $w_\beta \notin [\![\Box\gamma(\psi_1, \dots, \psi_n, \pi_1, \dots, \pi_m)]\!]_h^{\mathsf{Clop}(W)}$. So there exists $v \in W$ such that $w_\beta R v$ and $v \notin [\![\gamma(\psi_1, \dots, \psi_n, \pi_1, \dots, \pi_m)]\!]_h^{\mathsf{Clop}(W)}$. Then we put $w_\gamma = v$.

By the construction, for each atomic subformula p_i ($i \leq n$) and q_j ($j \leq m$) of α we have

$$w_{p_i} \in [\![\Box^{d_i} s_i]\!]_h^{\mathsf{Clop}(W)} \text{ and } w_{q_j} \notin [\![\pi_j]\!]_h^{\mathsf{Clop}(W)} \tag{7}$$

Recall that $\psi_i = \neg\Box^{d_i} s_i$ and that for each $z \in W$ and $d \in \omega$ we let $R^d(z) = \{u \in W : zR^d u\}$. We are now ready to define a 'minimal' closed assignment g. For each propositional variable x we let

$$g(x) = \bigcup\{R^{d_i}(w_{p_i}) : 1 \leq i \leq n, s_i = x\}. \tag{8}$$

Claim 5.4.

1. *g is a closed assignment.*
2. *For $1 \leq i \leq n$ we have $w_{p_i} \notin [\![\psi_i]\!]_g^{\mathsf{Clop}(W)}$.*
3. *For each propositional variable x we have $g(x) \subseteq h(x)$.*

Proof. (1) follows from Lemma 2.1 and the fact that a finite union of closed sets is closed.

For (2) note that $w_{p_i} \notin [\![\psi_i]\!]_g^{\mathsf{Clop}(W)}$ iff $w_{p_i} \in [\![\Box^{d_i} s_i]\!]_g^{\mathsf{Clop}(W)}$ iff $R^{d_i}(w_{p_i}) \subseteq g(s_i)$. By definition, $g(s_i) = \bigcup\{R^{d_j}(w_{p_j}) : 1 \leq j \leq n, s_j = s_i\}$. If we take $j = i$, then we obtain $R^{d_i}(w_{p_i}) \subseteq g(s_i)$ and, thus, $w_{p_i} \notin [\![\psi_i]\!]_g^{\mathsf{Clop}(W)}$.

For (3) note that by (7) we have $w_{p_i} \in [\![\Box^{d_i} s_i]\!]_h^{\mathsf{Clop}(W)}$ for all $1 \leq i \leq n$ such that $s_i = x$. So $R^{d_i}(w_{p_i}) \subseteq h(x)$, for all i with $1 \leq i \leq n$ and $s_i = x$. Hence, $g(x) = \bigcup\{R^{d_i}(w_{p_i}) : 1 \leq i \leq n, s_i = x\} \subseteq h(x)$. □

Claim 5.5. *$w_{q_i} \notin [\![\pi_i]\!]_g^{\mathsf{Clop}(W)}$, for each i with $1 \leq i \leq m$.*

Proof. By (7), we have $w_{q_i} \notin [\![\pi_i]\!]_h^{\mathsf{Clop}(W)}$. By Claim 5.4(3) and Lemma 2.5 we obtain that $w_{q_i} \notin [\![\pi_i]\!]_g^{\mathsf{Clop}(W)}$. □

Let $F_s = g(x_s)$ for each $1 \leq s \leq l$. Then each F_s is a closed set. We fix j with $1 \leq j \leq m$. By Claim 5.5, $w_{q_j} \notin \pi_j(F_1, \dots, F_l)^{\mathsf{Clop}(W)}$. Therefore, by Corollary 4.7, we have $w_{q_j} \notin \bigcap\{\pi_j(C_1, \dots, C_l)^{\mathsf{Clop}(W)} : C_s \in \mathsf{Clop}(W), F_s \subseteq C_s, 1 \leq s \leq l\}$. Thus, there exist $C_1^j, \dots C_l^j \in \mathsf{Clop}(W)$ such that $F_s \subseteq C_s^j$ for each s with $1 \leq s \leq l$ and

$$w_{q_j} \notin \pi_j(C_1^j, \dots, C_l^j)^{\mathsf{Clop}(W)}. \tag{9}$$

We put $f(x_s) = \bigcap\{C_s^j : 1 \leq j \leq m\}$, for $1 \leq s \leq l$. Then $F_s = g(x_s) \subseteq f(x_s)$ and by Lemma 2.5 and (9), $w_{q_j} \notin [\![\pi_j]\!]_f^{\mathsf{Clop}(W)}$, for $1 \leq j \leq m$. As $g(x_s) \subseteq f(x_s)$ for each $1 \leq s \leq l$, the same argument as in the proof of Claim 5.4(2) shows that $w_{p_i} \notin [\![\psi_i]\!]_f^{\mathsf{Clop}(W)}$, for $1 \leq i \leq n$. Putting everything together we obtain that $w_{p_i} \notin [\![\psi_i]\!]_f^{\mathsf{Clop}(W)}$, for $1 \leq i \leq n$, and $w_{q_j} \notin [\![\pi_j]\!]_f^{\mathsf{Clop}(W)}$, for $1 \leq j \leq m$. Finally, a straightforward induction shows that for each subformula β of α, we have $w_\beta \notin [\![\beta(\psi_1, \dots, \psi_n, \pi_1, \dots, \pi_m)]\!]_f^{\mathsf{Clop}(W)}$. So $w \notin [\![\alpha(\psi_1, \dots, \psi_n, \pi_1, \dots, \pi_m)]\!]_f^{\mathsf{Clop}(W)}$, which means that $w \notin [\![\varphi(x_1, \dots, x_l)]\!]_f^{\mathsf{Clop}(W)}$. □

The modal logic analogue of Theorem 5.3 immediately implies that every modal logic axiomatized by Sahlqvist modal formulas is Kripke complete. For modal fixed point logic, however, this is not the case for at least two reasons. First, in standard modal logic, if a modal formula φ is not valid on a descriptive frame (W, R), it is also not valid on (W, R) seen as a Kripke frame. This is not the case for modal μ-formulas and descriptive μ-frames. Indeed, in descriptive μ-frames we have the clopen semantics $[\![\cdot]\!]_h^{\mathsf{Clop}(W)}$ (see Definition 2.6), whereas in Kripke frames we have the set-theoretic semantics $[\![\cdot]\!]_h^{\mathcal{P}(W)}$ (see Section 2.1). So, if φ is an arbitrary modal μ-formula, then $[\![\varphi]\!]_h^{\mathsf{Clop}(W)} \neq W$ may not imply $[\![\varphi]\!]_h^{\mathcal{P}(W)} \neq W$. Thus, from the fact that a modal fixed point logic L is complete with respect to a class K of descriptive μ-frames, it does not follow that L is complete with respect to Kripke-frame reducts of the frames in K (as mentioned above this property holds for ordinary modal logic).

Second, there is also a problem with soundness. The modal logic analogue of Theorem 5.3 implies that every Sahlqvist modal formula is *d-persistent* (see e.g., [7, Section 5.6]). This means that if L' is a modal logic axiomatized by Sahlqvist modal formulas, then the underlying Kripke frame of a descriptive L'-frame is also an L'-frame. We cannot claim this for modal fixed point logic as, by Theorem 5.3, we only know that validity under clopen semantics of every Sahlqvist fixed point formula is preserved under set-theoretic assignments. As Example 3.9 shows, the validity of Sahlqvist fixed point formulas may not be preserved under moving from the clopen semantics $[\![\cdot]\!]_h^{\mathsf{Clop}(W)}$ to the set-theoretic semantics $[\![\cdot]\!]_h^{\mathcal{P}(W)}$. So Sahlqvist fixed point formulas are not d-persistent in the standard sense.

Finally, one may try to consider a new 'hybrid' semantics, where we keep the clopen semantics $[\![\cdot]\!]_h^{\mathsf{Clop}(W)}$, but consider any set-theoretic assignment h. The problems raised in the previous two paragraphs with respect to soundness and completeness will then be eliminated in this semantics. But a new problem with soundness may arise. In particular, the fixed point rule of Definition 2.13 may not be sound for this new semantics. Indeed, let h be a set-theoretic assignment on a descriptive μ-frame (W, R). This means that formulas may take non-clopen values under h. Let ψ be such a formula with an additional property that $[\![\varphi[\psi/x]]\!]_h^{\mathsf{Clop}(W)} \subseteq [\![\psi]\!]_h^{\mathsf{Clop}(W)}$ for some modal μ-formula φ. Then there is no guarantee that $[\![\mu x\varphi]\!]_h^{\mathsf{Clop}(W)} \subseteq [\![\psi]\!]_h^{\mathsf{Clop}(W)}$ as in the clopen semantics we only consider the intersection of all clopen pre-fixed points and the value of ψ is a non-clopen set. Consequently, it is not clear how to establish soundness of the fixed point rule.

Nevertheless, Theorem 5.3 will lead us to a variant of Sahlqvist completeness (and in the next section, Sahlqvist correspondence) for modal fixed point logic. For this we will make use of this new 'hybrid' semantics. We will use it in a way avoiding the soundness problem with the fixed point rule. We introduce general μ-frames and define strict validity in these structures. The concept of general frames in modal logic is quite well known (see e.g., [7, Section 5.5]).

Definition 5.6. *A triple* $(W, R, \mathfrak{F})$ *is called a* general frame *if* (W, R) *is a Kripke frame and* $\mathfrak{F} \subseteq \mathcal{P}(W)$*. Elements of* $\mathfrak{F}$ *are called* admissible sets*. An assignment h from the propositional variables to* $\mathfrak{F}$ *is called an* admissible *assignment. A general frame* $(W, R, \mathfrak{F})$ *is called a* general μ-frame *if for each modal* μ*-formula* φ *and an admissible assignment h we have that* $[\![\varphi]\!]_h^{\mathfrak{F}} \in \mathfrak{F}$.

Obviously, every descriptive μ-frame (W, R) can be viewed as a general μ-frame $(W, R, \mathsf{Clop}(W))$. Conversely, it is well known that if a general (μ-)frame is *refined* and *compact* (see e.g., [7, Section 5.5]), then it corresponds to a descriptive (μ-)frame.

Let $(W, R, \mathfrak{F})$ be a general μ-frame. Then a formula φ is called *valid* (resp. *strictly valid*) in $(W, R, \mathfrak{F})$ if $[\![\varphi]\!]_h^{\mathfrak{F}} = W$ for each admissible (resp. each set-theoretic) assignment h. A normal modal fixed point logic L is called *sound* (resp. *strictly sound*) with respect to a class K of general μ-frames if every formula in L is *valid* (resp. *strictly valid*) in each frame in K. L is called *complete* (resp. *strictly complete*) with respect to a class K of general μ-frames if every formula valid (resp. strictly valid) in K is in L.

Corollary 5.7. *Let* Φ *be a set of Sahlqvist fixed point formulas. Then the normal modal fixed point logic* $L = \mathbf{K}^\mu + \Phi$ *is sound and complete with respect to the class of general* μ*-frames where all the formulas in* Φ *are strictly valid.*

Proof. Let K be the class of general μ-frames where all the formulas in Φ are strictly valid. For the soundness of L we need to show that all the formulas in Φ and all the axioms of $\mathbf{K}^\mu$ are valid in K and that the fixed point rule preserves validity in K. Since strict validity implies validity, all the formulas in Φ are obviously valid in K. That the axioms of $\mathbf{K}^\mu$ are valid in K and the fixed point rule preserves validity is easy to check using the fact that in a general frame $(W, R, \mathfrak{F})$, formulas take values in $\mathfrak{F}$ and the fixed point operators involve only pre-fixed points from $\mathfrak{F}$.

For the completeness of L, let φ be a modal μ-formula such that $\varphi \notin L$. Then by Theorem 2.14, there exists a descriptive μ-L-frame (W, R) and a clopen assignment f such that $[\![\varphi]\!]_f^{\mathsf{Clop}(W)} \neq W$. We view the descriptive μ-frame (W, R) as a general μ-frame $(W, R, \mathsf{Clop}(W))$. Thus, f is an admissible assignment. It is left to be shown that all the formulas in Φ are strictly valid in $(W, R, \mathsf{Clop}(W))$. But this follows directly from Theorem 5.3. $\square$

It follows from Corollary 5.7 that each normal modal fixed point logic L axiomatized by Sahlqvist modal fixed point formulas is strictly complete with respect to general μ-frames. It is not clear, however, that L is strictly sound with respect to general μ-frames. We leave it as an open problem to find an axiomatization of modal fixed point logics that gives strict soundness and completeness for general μ-frames.

Corollary 5.8. *Let (W, R) be a descriptive μ-frame, $w \in W$ and φ a formula built from positive modal μ-formulas and negative modal μ-formulas using the operations $\vee$ and $\Box$. Then $w \in [\![\varphi]\!]_f^{\mathsf{Clop}(W)}$, for each clopen assignment f, implies $w \in [\![\varphi]\!]_g^{\mathsf{Clop}(W)}$, for each closed assignment g.*

Proof. The proof is analogous to the proof of Theorem 5.3. Assume $w \notin [\![\varphi]\!]_g^{\mathsf{Clop}(W)}$, for some closed assignment g. In the same way as in the proof of Theorem 5.3 we proceed by defining a clopen assignment f. The same reasoning as in the last two paragraphs of the proof of Theorem 5.3 guarantees that $w \notin [\![\varphi]\!]_f^{\mathsf{Clop}(W)}$, which finishes the proof of the corollary. $\Box$

Note that we can define a new validity of modal μ-formulas in descriptive μ-frames via closed assignments. For the formulas discussed in Corollary 5.8 we will then have a completeness result with respect to this semantics. However, as in the case of the strict soundness we may not have the soundness result for this semantics.

5.2. Correspondence

Let LFP be the first-order language with the least fixed point operator μ; see, e.g., [12, Section 8]. We assume that μ is applied to unary predicates only. For each propositional variable p we reserve a unary predicate symbol P. An LFP-formula ξ is said to be *an LFP-frame condition* if it does not contain free variables or predicate symbols. (A frame condition can contain a bound first-order variable or a unary predicate symbol bound by μ: for example, $\mu(Z, u)\,\xi$.)

Let $\mathfrak{M} = (W, R, \mathfrak{F})$ be a general μ-frame and h an admissible assignment. We view $\mathfrak{M}$ as an LFP-structure via $P^{\mathfrak{M}} = h(p) \subseteq W$, for each propositional variable p. Let g be a first-order assignment of variables. We adopt non-classical semantics of LFP. The notation $(\mathfrak{M}, h, g) \models \xi$ is defined by induction on ξ. The cases of atomic formulas, Booleans and quantifiers are standard (see, e.g., [12, Section 8]). The semantics of expressions of the type $(\mu(Z, u)\,\xi(u, Z))(v)$, where Z is a unary predicate symbol and u and v first-order variables, is defined as follows. (We assume that ξ may have some other free variables and predicate symbols). We let

$$F(U) = \{w \in W : (\mathfrak{M}, h_z^U, g_u^w) \models \xi(u, Z)\}, \tag{10}$$

where g_u^w is a first-order assignment mapping variable u to a point $w \in W$. Now we define

$$(\mathfrak{M}, h, g) \models (\mu(Z, u)\,\xi(u, Z))(v) \text{ iff } g(v) \in \bigcap\{U \in \mathfrak{F} : F(U) \subseteq U\}.$$

Note that for a sentence ξ, we can drop g, and for an LFP frame condition ξ we can drop h and g. That is, a frame condition is true in $(\mathfrak{M}, h, g)$ iff it is true in $\mathfrak{M}$.

Definition 5.9. *Let v be a first-order variable. We define the* standard translation *of modal μ-formulas into* LFP *as follows:*

$ST_v(\bot) = \bot$,

$ST_v(\top) = \top$,

$ST_v(p) = P(v)$*, where p is a propositional variable,*

$ST_v(\varphi \wedge \psi) = ST_v(\varphi) \wedge ST_v(\psi)$,

$ST_v(\varphi \vee \psi) = ST_v(\varphi) \vee ST_v(\psi)$,

$ST_v(\neg\varphi) = \neg ST_v(\varphi)$,

$ST_v(\Diamond\varphi) = \exists u(R(v, u) \wedge ST_u(\varphi))$,

$ST_v(\Box\varphi) = \forall u(R(v, u) \to ST_u(\varphi))$,

$ST_v(\mu z\varphi) = (\mu(Z, u)\,ST_u(\varphi))(v)$*, where φ is a modal μ-formula positive in z.*

Proposition 5.10. *Let $\mathfrak{M} = (W, R, \mathfrak{F})$ be a general μ-frame, h an admissible assignment and φ a modal μ-formula. Then*

1. *For each $w \in W$ and a first-order assignment g_v^w mapping variable v to w, we have*

$$w \in [\![\varphi]\!]_h^{\mathfrak{F}} \text{ iff } (\mathfrak{M}, h, g_v^w) \models ST_v(\varphi),$$

2. $W = [\![\varphi]\!]_h^{\mathfrak{F}}$ *iff* $(\mathfrak{M}, h) \models \forall v ST_v(\varphi)$.

Proof. The result is proved by an easy induction on the complexity of φ. ☐

We note that if we wanted to express strict validity of a modal μ-formula φ in a general μ-frame, then we would have to translate φ into a monadic second order formula obtained from the standard translation of $ST_v(\varphi)$ of φ by universally quantifying all the free unary predicate symbols. But for Sahlqvist fixed point formulas we can translate into LFP.

Theorem 5.11. *Let $(W, R, \mathfrak{F})$ be a general μ-frame and $\varphi(x_1, \ldots, x_l)$ a Sahlqvist fixed point formula. Then there is an LFP-frame condition $\chi(\varphi)$ such that*

$$(W, R, \mathfrak{F}) \models \chi(\varphi) \iff \varphi \text{ is strictly valid in } (W, R, \mathfrak{F}). \tag{11}$$

Proof. Since $\varphi(x_1, \ldots, x_l)$ is a Sahlqvist fixed point formula, there exists a formula $\alpha(p_1, \ldots, p_n, q_1, \ldots, q_m)$ using only $\vee$ and $\Box$ such that all propositional variables occur and no propositional variable occurs twice in α, and there exist positive formulas $\pi_1, \ldots, \pi_m$ and formulas $\psi_1, \ldots, \psi_n$, where each ψ_i is of the form $\neg\Box^{d_i} s_i$, for some $d_i \in \omega$ and $s_i \in \{x_1, \ldots, x_l\}$, such that

$$\varphi(x_1, \ldots x_l) = \alpha(\psi_1/p_1, \ldots, \psi_n/p_n, \pi_1/q_1, \ldots, \pi_m/q_m).$$

For each subformula β of α we introduce a new first-order variable v_β and define an LFP-formula $\widehat{\beta}$ by induction on β.

1. if $\beta = p_i$, for each $i = 1, \ldots, n$, then $\widehat{\beta} = P_i(v_\beta)$,
2. if $\beta = q_j$, for each $j = 1, \ldots, m$, then $\widehat{\beta} = Q_j(v_\beta)$,
3. if $\beta = \gamma \vee \delta$, then $\widehat{\beta} = ((v_\beta = v_\gamma) \wedge (v_\beta = v_\delta)) \to (\widehat{\gamma} \vee \widehat{\delta})$.
4. if $\beta = \Box\gamma$, then $\widehat{\beta} = R(v_\beta, v_\gamma) \to \widehat{\gamma}$.

Let ρ be a formula defined as follows: for each $d \in \omega$ we have $\rho^0(x, y) = (x = y)$ and $\rho^{d+1} = \exists z(R(x, z) \wedge \rho^d(z, y))$. Similarly to the proof of the Sahlqvist completeness theorem, for each propositional variable x, we define

$$\theta_x(v) = \bigvee\{\rho^{d_i}(v_{p_i}, v) : 1 \leq i \leq n, s_i = x\}. \tag{12}$$

We let

$$\chi'(\varphi) = \forall v_{\beta_1} \ldots \forall v_{\beta_k} \widehat{\alpha}(\bot/P_i(v_{p_i}), \overline{ST_{v_{q_j}}(\pi_j)}/Q_j(v_{q_j})), \text{ for } 1 \leq i \leq n, 1 \leq j \leq m, \tag{13}$$

where $\beta_1, \ldots, \beta_k$ enumerate all proper subformulas of α, and for any LFP-formula ξ, the formula $\overline{\xi}$ denotes the result of replacing each atomic subformula of ξ of the form $P(t)$ (where t is any first-order variable) by $\theta_p(t/v)$.

Finally we let

$$\chi(\varphi) = \forall v_\alpha \chi'(\varphi). \tag{14}$$

(For examples of frame conditions $\chi(\varphi)$ for specific modal μ-formulas φ see Section 6.)

Now it is only left to be shown that

Claim 5.12. $(W, R, \mathfrak{F}) \models \chi(\varphi)$ *iff* $[\![\varphi]\!]_h^{\mathfrak{F}} = W$, *for each (set-theoretic) assignment h.*

Proof. For $w \in W$ we write $(W, R, \mathfrak{F}) \models \neg\chi'(\varphi)[w]$ to mean that $\chi'(\varphi)$ is false in $(W, R, \mathfrak{F})$ when v_α is assigned to w. It follows from the proof of Theorem 5.3 and the definition of $\chi'(\varphi)$ that for each $w \in W$ we have $(W, R, \mathfrak{F}) \models \neg\chi'(\varphi)[w]$ iff there exists an assignment g such that $w \notin [\![\varphi]\!]_g^{\mathfrak{F}}$. Now suppose $[\![\varphi]\!]_h^{\mathfrak{F}} \neq W$ for some assignment h. Then, by the above equivalence, we obviously obtain that $(W, R, \mathfrak{F}) \models \neg\chi'(\varphi)[w]$ and thus $(W, R, \mathfrak{F}) \models \neg\chi(\varphi)$. Conversely, if $(W, R, \mathfrak{F}) \models \neg\chi(\varphi)$, then there is $w \in W$ such that $(W, R, \mathfrak{F}) \models \neg\chi'(\varphi)[w]$. So there exists an assignment g with $w \notin [\![\varphi]\!]_g^{\mathfrak{F}}$. Therefore, $[\![\varphi]\!]_g^{\mathfrak{F}} \neq W$, which finishes the proof of the claim. ☐

The theorem now follows immediately from Claim 5.12 and the definition of strict validity.

□

Theorem 5.13. (*Main Theorem*) *Let Φ be a set of Sahlqvist fixed point formulas. Then*

1. *The normal modal fixed point logic $L = \mathbf{K}^\mu + \Phi$ is sound and complete with respect to the class of general μ-frames satisfying the* LFP*-frame conditions $\{\chi(\varphi) : \varphi \in \Phi\}$.*
2. *The normal modal fixed point logic $L = \mathbf{K}^\mu + \Phi$ is sound and complete with respect to the class of descriptive μ-frames satisfying the* LFP*-frame conditions $\{\chi(\varphi) : \varphi \in \Phi\}$.*

Proof. The result follows directly from Corollary 5.7 and Theorems 5.3 and 5.11. □

Remark 5.14. In [16, 17] Goranko and Vakarelov define *inductive modal formulas*. Roughly speaking, inductive formulas are obtained by replacing negated boxed propositional variables in the definition of Sahlqvist formulas by so-called 'negated boxed formulas'. By modifying the technique of Sambin and Vaccaro they show that every inductive formula has a first-order correspondent and, moreover, the modal logic obtained by adding inductive formulas to the basic modal logic **K** is sound and complete with respect to the frames in which the correspondents are valid. All these results are proved by using the Esakia–Sambin–Vaccaro Lemma and 'minimal' assignments. The definition of negated boxed formulas is designed so that these minimal assignments always exist. For example, in the formula $(p \wedge \Box(\Diamond p \to \Box q)) \to \Diamond\Box q$, a minimal assignment making the antecedent true at a world w can be constructed first for p and then for q, using the assignment of p. [17, Theorem 57] shows that the minimal assignment is always closed and definable in first-order logic.

We note that we can define *inductive modal μ-formulas* by replacing positive formulas in the definition of [16],[17] by positive fixed point formulas. Then the fixed point analogue of the Esakia–Sambin–Vaccaro Lemma (Lemma 4.6) will yield a fixed point analogue of the Goranko–Vakarelov result. We skip the details.

Remark 5.15. In [2] and [3] van Benthem defined a syntactic class of the so-called *PIA*-formulas (standing for 'positive implies atomic') for first-order logic. The main property of PIA-formulas is the following: a first-order formula $\varphi(P)$ is preserved under arbitrary intersections of values of the predicate P (that is, if $\varphi(S_i)$ holds for each $i \in I$ then $\varphi(\bigcap_{i\in I} S_i)$ holds too) iff φ is equivalent to a PIA-formula. Van Benthem [2, 3] then defined a special class of modal formulas, which we call *generalized Sahlqvist formulas*, as the modal formulas of the form $\varphi \to \psi$, where φ is a modal analogue of a PIA-formula and ψ is positive. He showed that such formulas admit 'minimal' assignments that are expressible in LFP on Kripke frames. This implies that generalized Sahlqvist formulas have LFP-correspondents on Kripke frames. An algorithm computing LFP-correspondents for some modal formulas was discussed in [11].

We note that the definition of generalized Sahlqvist formulas can be extended to the language of modal μ-calculus if we replace 'positive' by 'positive in the modal μ-language'. In terms of Sahlqvist fixed point formulas this would amount to dropping in Definition 5.1 the clause about being closed under $\Box$'s and disjunctions and replacing negated boxed propositional variables by negated modal PIA-formulas. If we do this, we obtain LFP-correspondents of not only modal formulas (as in [2] and [3]), but of a wide class of modal μ-formulas containing all Sahlqvist fixed point formulas (see [4] for details). The issue of completeness, however, is unclear, as minimal assignments of [2], [3] and [4] are not necessarily topologically closed. Recall that the minimal assignments we considered in this paper are topologically closed, which, together with the modal fixed point version of the Esakia lemma, gave the completeness result.

6. Examples

In this section we discuss a few examples of Sahlqvist fixed point formulas and their frame correspondents.

Example 6.1. We first consider the formula $\mu x \Box x$. By adding this formula to $\mathbf{K}^\mu$ we obtain a Sahlqvist fixed point logic which in the standard semantics defines dually well-founded Kripke frames. We refer to [3] and [10] for the soundness and completeness results for this logic with respect to the Kripke semantics. We recall that the *Gödel–Löb* modal logic **GL** is obtained by adding the *Löb* axiom

$$\Box(\Box p \to p) \to \Box p \tag{15}$$

to the basic modal logic **K**. It is well known that **GL** is sound and complete with respect to the class of transitive dually well-founded Kripke frames; see, e.g., [7, Section 4.4]. Descriptive frames of **GL** were first characterized in

[15]. In particular, it was proved in [15] that a descriptive frame (W, R) is a **GL**-frame iff it is transitive and each non-empty clopen $U \subseteq W$ contains an irreflexive maximal point. We call a descriptive μ-frame validating the Löb axiom a *descriptive* **GL**-μ*-frame*. Recall also that a modal algebra is called a **K4**-algebra if it validates the formula $\Box p \to \Box\Box p$.

Theorem 6.2. $\mathbf{GL}^\mu = \mathbf{K}^\mu + (\Box p \to \Box\Box p) + (\mu x \Box x)$ *is sound and complete with respect to the class of descriptive* **GL**-μ*-frames.*

Proof. In [15] Esakia showed that a **K4**-algebra $(B, \Diamond)$ is a **GL**-algebra iff for each $a \in B$, $a \neq 0$ implies $m(a) \neq 0$, where $m(a) = a \wedge \neg\Diamond a$. Note that

$$\text{for each } a \in B,\ a \neq 0 \text{ implies } m(a) \neq 0 \tag{16}$$

is equivalent to

$$\text{for each } a \in B,\ a \neq 1 \text{ implies } \neg m(\neg a) \neq 1. \tag{17}$$

Now $\neg m(\neg a) = \neg(\neg a \wedge \Box a) = \Box a \to a$. Finally, $\Box a \to a \neq 1$ is equivalent to $\Box a \nleq a$. Therefore, a **K4**-algebra $(B, \Box)$ is a **GL**-algebra iff $a \neq 1$ implies $\Box a \nleq a$. This means that the only pre-fixed point of the map mapping each $a \in B$ to $\Box a$ is 1. (The equation $\Box 1 = 1$ obviously holds in each modal algebra.) Therefore, $\mathbf{GL}^\mu$-algebras are exactly those modal μ-algebras that validate $(\Box p \to \Box\Box p)$ and satisfy (16).

Translating this into descriptive μ-frames, we obtain that $\mathbf{GL}^\mu$ is sound and complete with respect to transitive descriptive μ-frames that are also **GL**-frames. $\Box$

Remark 6.3. We note that the Sahlqvist correspondent of the formula $\mu x \Box x$ is equivalent to the aforementioned condition on the maximal points. We also remark that Theorem 6.2 implies that $\mathbf{GL}^\mu = \mathbf{K}^\mu + (\Box(\Box p \to p) \to \Box p)$.

Example 6.4. Let $\varphi(x) = \Diamond x \to \Box\Diamond^* x$, where

$$\Diamond^* x = \mu z(x \vee \Diamond z), \tag{18}$$

and let $L_\varphi = \mathbf{K}^\mu + \varphi$. Note that $\varphi(x)$ is equivalent to $\neg\Diamond x \vee \Box\Diamond^* x$ which is equivalent to

$$\Box\neg x \vee \Box\Diamond^* x. \tag{19}$$

By Definition 5.1, the latter is a Sahlqvist fixed point formula.

It is well known that in the standard Kripke semantics L_φ defines a class of frames satisfying the following property: for each w, v, u in the frame, if wRv, then for each wRu with $u \neq v$, there exists a finite path from u to v. Next using the results of the previous section we will compute the frame correspondent for descriptive μ-frames.

Let (W, R) be a descriptive frame. We say that a set $U \subseteq W$ is a *downset* if for each $w, v \in W$, $w \in U$ and vRw imply $v \in U$. Obviously, U is a downset iff $\langle R \rangle U \subseteq U$. The least clopen downset containing a set U (if it exists) will be denoted by $\mathbb{D}(U)$. If U is a singleton $\{u\}$ we will write $\mathbb{D}(u)$ instead of $\mathbb{D}(\{u\})$.

Lemma 6.5. *Let (W, R) be a descriptive μ-frame, $U \in \mathsf{Clop}(W)$ and h a clopen assignment such that $h(x) = U$. Then*

$$[\![\Diamond^* x]\!]_h^{\mathsf{Clop}(W)} = \mathbb{D}(U). \tag{20}$$

Proof. The lemma follows directly from the definition of the clopen semantics. A set S is a pre-fixed point of $(X \mapsto [\![x \vee \Diamond z]\!]_{h_z^X}^{\mathsf{Clop}(W)})$ iff S contains U and S is a downset. This implies (20). As (W, R) is a descriptive μ-frame and $\Diamond^* x$ is (a shorthand of) a modal μ-formula, we deduce that $\mathbb{D}(U)$ exists. $\Box$

Next we observe that

$$ST_v(\Diamond^* x) = ST_v(\mu z(x \vee \Diamond z)) = \mu(Z, u)\,(X(u) \vee \exists y(R(u, y) \wedge Z(y)))(v). \tag{21}$$

Now as in the proof of Theorem 5.11, we will compute the frame condition $\chi(\varphi)$ corresponding to φ. Recall that φ is equivalent to $\Box\neg x \vee \Box\Diamond^* x$. So $\alpha(p, q) = \Box p \vee \Box q$, $\psi = \neg x$ and $\pi = \Diamond^* z$. Then

$$\begin{aligned}
\widehat{\alpha} &= ((v_\alpha = v_{\Box p}) \wedge (v_\alpha = v_{\Box q})) \to (\widehat{\Box p} \vee \widehat{\Box q}) \\
&\equiv (v_\alpha = v_{\Box p} = v_{\Box q}) \to ((R(v_{\Box p}, v_p) \to P(v_p)) \vee (R(v_{\Box q}, v_q) \to Q(v_q))).
\end{aligned}$$

Now

$$\theta_x(z) = (z = v_p). \tag{22}$$

Thus,

$$\chi'(\varphi) = \forall v_{\Box p} \forall v_{\Box q} \forall v_p \forall v_q ((v_\alpha = v_{\Box p} = v_{\Box q}) \to ((R(v_{\Box p}, v_p) \to \bot) \vee (R(v_{\Box q}, v_q) \to \overline{ST_{v_q}(\Diamond^* x)})),$$

and so using (21) and (22),

$$\begin{array}{ll} & \chi'(\varphi) \\ \equiv & \forall v_p \forall v_q ((R(v_\alpha, v_p) \to \bot) \vee (R(v_\alpha, v_q) \to \overline{ST_{v_q}(\Diamond^* x)})) \\ \equiv & \forall v_p \forall v_q (\neg R(v_\alpha, v_p) \vee (R(v_\alpha, v_q) \to \overline{ST_{v_q}(\Diamond^* x)})) \\ \equiv & \forall v_p \forall v_q (\neg R(v_\alpha, v_p) \vee (R(v_\alpha, v_q) \to \overline{\mu(Z, u)\, (X(u) \vee \exists y (R(u, y) \wedge Z(y)))(v_q)})) \\ \equiv & \forall v_p \forall v_q ((R(v_\alpha, v_p) \wedge R(v_\alpha, v_q)) \to \mu(Z, u)\, ((u = v_p) \vee \exists y (R(u, y) \wedge Z(y)))(v_q)). \end{array}$$

Finally, we obtain

$$\chi(\varphi) \equiv \forall v_\alpha \forall v_p \forall v_q ((R(v_\alpha, v_p) \wedge R(v_\alpha, v_q)) \to \mu(Z, u)\, (u = v_p \vee \exists y (R(u, y) \wedge Z(y)))(v_q)). \tag{23}$$

But in order to get a shorter and more intuitive condition, using Lemma 6.5 and Proposition 5.10(1) we can rewrite (23) as

$$\forall t \forall u \forall v\, ((R(t, u) \wedge R(t, v)) \to v \in \mathbb{D}(u)), \tag{24}$$

where by '$v \in \mathbb{D}(u)$' we mean that the interpretation of the variable v belongs to the least clopen downset containing the interpretation of the variable u.

Thus, we arrived at the following corollary of Theorem 5.13.

Corollary 6.6. *The modal fixed point logic L_φ is sound and complete with respect to the class of descriptive μ-frames satisfying* (24)*.*

Example 6.7. Let $\psi(x) = x \to \Diamond^+ x$, where

$$\Diamond^+ x = \mu z \Diamond (x \vee z) \tag{25}$$

and let $L_\psi = \mathbf{K}^\mu + \psi$. Clearly ψ is equivalent to $\neg x \vee \Diamond^+ x$ and thus is a Sahlqvist fixed point formula.

It is well known that in the standard Kripke semantics L_ψ defines a class of frames satisfying the following property: each point in the frame is a part of a finite R-loop. Now we will compute the frame correspondent for descriptive μ-frames.

Lemma 6.8. *Let (W, R) be a descriptive μ-frame, $U \in \mathsf{Clop}(W)$ and h a clopen assignment such that $h(x) = U$. Then*

$$[\![\Diamond^+ x]\!]_h^{\mathsf{Clop}(W)} = \mathbb{D}(\langle R \rangle U). \tag{26}$$

Proof. The proof of the lemma is similar to the proof of Lemma 6.5. □

Unlike the previous example, here we will use more intuitive reasoning. We observe that if a descriptive μ-frame (W, R) with an assignment h refutes ψ, then there is a point w such that $w \in [\![x]\!]_h^{\mathsf{Clop}(W)}$ and $w \notin [\![\Diamond^+ x]\!]_h^{\mathsf{Clop}(W)}$. Then the 'minimal' valuation g is given by $g(x) = \{w\}$. Therefore, we have

$$\chi(\psi) = \forall v \overline{ST_v(\Diamond^+ x)}, \tag{27}$$

and so

$$\chi(\psi) \equiv \forall v \mu(Z, s)\, (\exists y (R(s, y) \wedge (y = v \vee Z(y))))(v). \tag{28}$$

As before in order to get shorter and more intuitive condition, using (27), Lemma 6.8 and Proposition 5.10(1), we can replace (28) with

$$\forall v\, (v \in \mathbb{D}(\langle R \rangle v)), \tag{29}$$

where by '$v \in \mathbb{D}(\langle R \rangle u)$' we mean that the interpretation of the variable v belongs to the least clopen downset containing $\langle R \rangle$ applied to the interpretation of the variable u.

Thus, we arrived at the following corollary of Theorem 5.13.

Corollary 6.9. *The modal fixed point logic L_ψ is sound and complete with respect to the class of descriptive μ-frames satisfying* (29)*.*

In both Examples 6.4 and 6.7, we see that the classical notion of reflexive transitive closure $R^*(x, y)$ has been replaced by a topological analogue, namely that x is in the 'topological downward closure' of $\{y\}$, i.e., $x \in \mathbb{D}(y)$, as defined above. We feel this to be a rather natural and intuitive condition that we expect to be useful in applications involving spatial or topological reasoning.

7. Conclusions

In this paper we proved a version of Sahlqvist's theorem for modal fixed point logic by extending the Sambin–Vaccaro technique [24] from modal logic to modal fixed point logic. Following [1] we considered an order-topological semantics of modal fixed point logic. In this semantics the least fixed point operator is interpreted as the intersection of clopen pre-fixed points. Descriptive μ-frames are those order-topological structures that admit this topological interpretation of fixed point operators. We defined Sahlqvist fixed point formulas and proved that for every Sahlqvist fixed point formula φ there exists an LFP-formula $\chi(\varphi)$, with no free first-order variable or predicate symbol, such that a descriptive μ-frame validates φ iff $\chi(\varphi)$ is true in this structure. Our main result states that every modal fixed point logic axiomatized by a set Φ of Sahlqvist fixed point formulas is sound and complete with respect to the class of descriptive μ-frames satisfying $\{\chi(\varphi) : \varphi \in \Phi\}$. This result also applies to general μ-frames (general frames in which all modal μ-formulas have admissible semantics: see Definition 5.6). We also gave some concrete examples of Sahlqvist fixed point logics and classes of descriptive μ-frames for which these logics are sound and complete. It needs to be stressed again that our Sahlqvist completeness and correspondence result applies only to descriptive μ-frames and general μ-frames, and does not imply that every Sahlqvist modal fixed point logic is sound and complete with respect to Kripke frames.

From the viewpoint of the standard theory of fixed point logics, the interpretation of the least fixed point operator as the intersection of clopen pre-fixed points might look a bit complex and unnatural. The results of this paper, however, (together with the other results obtained on this semantics of fixed point logics) show that, when it comes to the issue of completeness of axiomatic systems of modal fixed point logic, the order-topological semantics is much better behaved than the classical semantics. Indeed, order-topological semantics guarantees that adding any axioms to the basic modal fixed point logic $\mathbf{K}^{\mu}$ results in a sound and complete system [1]. Moreover, as we have shown here, if the extra axioms are Sahlqvist, then the frame class for which this logic is sound and complete is LFP-definable, with order-topological interpretation of LFP. The examples discussed in Section 6 illustrate that the class of descriptive μ-frames for which particular Sahlqvist fixed point logics are sound and complete, can have neat and 'sensible' descriptions. Also from the topological perspective it seems to us quite natural to consider the interpretation of the least fixed point operator as the intersection of not arbitrary, but particular, 'topological' (clopen) pre-fixed points. In order-topological semantics of modal logic all formulas are interpreted as clopen sets. Thus, it is only natural to demand that in the modal language enriched with fixed point operators, the operation of taking least fixed points involves only clopen sets. All these features underline that order-topological semantics of modal fixed point logic is quite a rich and promising area.

Finally, we finish by mentioning a number of open problems and topics for possible future work. We start with some technical questions already raised in this paper. The results of this paper are restricted to the language of modal fixed point logic with only the least fixed point operator. An obvious question is whether there is a way to extend these results to the language of modal fixed point logic with the greatest fixed point operator. Another interesting question is whether the definition of Sahlqvist fixed point formulas can be expanded to allow least fixed point operators to occur in more places in a formula.

As identified in Section 3, it is still open whether there exists a descriptive μ-frame, a clopen assignment on it and a modal μ-formula φ such that the clopen semantics for φ differs from the closed semantics for φ. The other open problem mentioned in Section 5.1 is whether the results of this paper can be extended to encompass strict validity. Solving this problem may require an introduction of a new axiomatic system for modal fixed point logic.

Another direction for future research is to investigate the possibility of proving an analogue for order-topological semantics of the Janin–Walukiewicz [19] characterization of modal μ-calculus as the bisimulation invariant fragment of monadic second-order logic. Bisimulations of descriptive frames have already been introduced and studied in [5]. Also a more general question would be whether the methods of automata and game semantics, which have proven to be very successful in the classical theory of fixed point logics, can be adjusted to the order-topological setting. Since, unlike classical fixed points, order-topological fixed points do not admit iterative approximation, answering this question is by no means straightforward.

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