CONSTRUCTING INVERSE SYSTEMS OF GRAPHS FOR BARELY CANONICAL VARIETIES

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ABSTRACT. Monk proved that the variety RRA of representable relation algebras is canonical — closed under taking canonical extensions. Hodkinson and Venema (2005) proved that RRA is only 'barely canonical': every first-order axiomatisation of it has infinitely many non-canonical axioms. Central to their proof was a probabilistic construction, building on a famous one of Erdős (1959), of an inverse system of finite graphs of chromatic number k for any finite $k \geq 2$, and of unbounded odd girth, so that its inverse limit is bipartite. The connecting maps are surjective bounded morphisms.

Here, we recast this construction using a rather simpler method of Hell and Nešetřil (1992).

This paper is dedicated to Robert Goldblatt, and involves several of his results.

1. INTRODUCTION

1.1. BACKGROUND. Our story starts long, long ago, in algebraic logic, and in particular, in that part of algebraic logic that concerns *algebras of binary relations.* Specifically, a *proper relation algebra* is an algebra

$$(A, \cup, -, \emptyset, W, Id_U, ^{-1}, |)$$

of binary relations on some set U, where W is some equivalence relation on U (hence $W \subseteq U \times U$); $A \subseteq \wp(W)$, the power set of W; $Id_U = \{(u,u) : u \in U\}$ denotes the identity relation on U; and for binary relations $a, b \in A$: $-a = W \setminus a$ is the complement of a relative to W; $a^{-1} = \{(u,v) : (v,u) \in a\}$ is the converse of a; and $a \mid b = \{(u,v) : \exists x((u,x) \in a \wedge (x,v) \in b)\}$ is the relative product of a and b. Of course, A should be closed under these operations. The class PRA of proper relation algebras was introduced by Jónsson and Tarski in 1948 [21].

A *relation algebra* is an algebra of the signature of proper relation algebras that satisfies certain equations given in [21] and not relevant here, except to say that every proper relation algebra is a relation algebra, and every relation algebra is a boolean algebra with operators as per [22]. A relation algebra is said to be *representable* if it is isomorphic to a proper relation algebra, and RRA denotes the class of *representable*

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relation algebras — the closure of PRA under isomorphism. RRA is an important class in algebraic logic. It is an elementary class, and Tarski [31] proved that it is a variety, defined by equations. Readers can find information about it in many references and surveys, such as [16; 17; 27; 28; 29].

1.2. BARE CANONICITY. Of interest here is that RRA is *canonical* — closed under taking canonical extensions. The *canonical extension* of a relation algebra, and more generally of a boolean algebra with operators, was introduced by Jónsson and Tarski in [22; 23] under the name of 'perfect extension'. It can be formed from the ultrafilters of the algebra, analogously to the *canonical models* ubiquitous in modal logic. Canonicity of RRA was proved by J. D. Monk: he did not publish his proof, but it appeared in outline as theorem 2.12 in the doctoral thesis of his student McKenzie [30]. The first full proof was published by Maddux [26], using a different method. The theorem can also be derived by model-theoretic saturation [17, theorem 3.36], and from Goldblatt's generalisation of Fine's modal canonicity theorem [8] to boolean algebras with operators [9; 11].

In the light of this, Venema asked whether RRA has a *canonical* axiomatisation. A first-order sentence of the signature of relation algebras (or more generally, some signature of boolean algebras with operators) is said to be *canonical* if it is preserved by taking canonical extensions of boolean algebras with operators. A canonical axiomatisation of RRA would consist of first-order sentences that individually are canonical. Of course, the existence of such an axiomatisation would imply Monk's result.

In [33], Venema proved that RRA cannot be axiomatised by Sahlqvist equations. Since Sahlqvist equations are a prime source of canonical sentences, after this point he rather gave up on the hope that RRA has any canonical axiomatisation. This view was soon vindicated:

Theorem 1. [20, theorem 6.8] *Every first-order axiomatisation of* RRA *contains infinitely many non-canonical sentences.*

Thus, the canonicity of RRA is not to be found in the individual axioms of any axiomatisation of it, but emerges only when all axioms are taken together. This phenomenon is called *canonicity in the limit* in [24]. We can say that RRA is only 'barely' canonical, where an elementary class of boolean algebras with operators is said to be *barely canonical* if it is canonical but every first-order axiomatisation of it involves infinitely many non-canonical sentences.

Theorem 1 has since been extended. In modal logic, [12] showed that the McKinsey –Lemmon logic KM^∞ is barely canonical in a similar sense. Some other well-known canonical varieties in algebraic logic have been shown barely canonical: the variety RCA_n of representable cylindric algebras [4; 5], and the varieties RDf_n, RPA_n, and RPEA_n of representable diagonal-free algebras, polyadic algebras, and polyadic equality algebras, respectively [5], all for finite $n \geq 3$; the proofs are extensions of the one for relation algebras.

As well as contributing to our knowledge about particular varieties, these results illuminate the boundaries of the concept of canonicity itself. They played a role in the remarkable dichotomy for modal logics developed by Kikot and Zolin [24; 25]

— [24, p.1066] comments that 'what was thought pathological [bare canonicity] for elementarily generated modal logics can now be seen to be the norm'.

1.3. INVERSE SYSTEMS OF GRAPHS. Over time, it has become apparent that parts of the proof of theorem 1 above in [20] were somewhat overcomplicated. The finite combinatorics in [20, propositions 6.4 and 6.6] can be replaced by first-order compactness arguments, of which there are already a number in the paper. This was done by Bulian in [4], and later in [5]. A gentle outline of this simplified argument for RRA can be found in [19], which serves as a kind of companion piece to the current article.

The purpose of the current article is to describe a second simplification — of the 'combinatorial heart' of [20] (and of [4; 5]). Let me quote it (definitions and background facts will be given in §2):

> **Theorem 2. [20, theorem 2.3]** *Let $k \geq 2$. There are finite* [undirected loop-free] *graphs $G_0, G_1, \ldots$, and surjective homomorphisms $\rho_i \colon G_{i+1} \to G_i$ for $i < \omega$, such that for each i,*
>
> 1. *for each edge xy of G_i and each $x' \in \rho_i^{-1}(x)$, there is $y' \in \rho_i^{-1}(y)$ such that $x'y'$ is an edge of G_{i+1},*
> 2. *G_i has no odd cycles of length $\leq i$,*
> 3. *$\chi(G_i) = k$* [i.e., G_i has chromatic number k].

Modal logicians will recognise that the maps $\rho_0, \rho_1, \ldots$ are *surjective bounded morphisms,* also known as *surjective p-morphisms*. The theorem gives us an inverse system[1]

$$G_0 \xleftarrow{\rho_0} G_1 \xleftarrow{\rho_1} \cdots \tag{1}$$

of finite graphs of chromatic number k, whose shortest odd cycles get longer and longer as we move rightwards in (1). The inverse limit G_ω of the system projects by a bounded morphism onto each G_i, and hence has no odd cycles at all, so is bipartite with chromatic number ≤ 2. By an interpretation of relation algebras in graphs from [18], and duality results of Goldblatt [10, §1.10–1.11], this can be parlayed into a direct system of finite relation algebras and embeddings whose direct limit $\mathcal{A}$ has chromatic number k (in a certain sense), while the canonical extension of $\mathcal{A}$ has chromatic number 2. Now the larger its chromatic number, the nearer the algebra is to RRA. The arbitrarily large gap between k and 2, combined with compactness for first-order logic, will establish that RRA has no canonical first-order axiomatisation. See [20; 19] for details.

A minor strengthening of this will show that RRA is barely canonical, as per theorem 1. We simply need to construct, for each k, l with $2 \leq l \leq k < \omega$, an inverse system

$$G'_0 \xleftarrow{\rho'_0} G'_1 \xleftarrow{\rho'_1} \cdots$$

of finite graphs of chromatic number k and surjective bounded morphisms ρ'_i, whose inverse limit G'_ω has chromatic number l (instead of 2). This can be done by defining

[1] To complete the definition, for $j \leq i < \omega$ we define the surjective bounded morphism $\rho^i_j \colon G_i \to G_j$ to be the identity map on G_i if $i = j$, and $\rho_j \circ \cdots \circ \rho_{i-1}$ if $i > j$.

(for each $i < \omega$) the graph G_i' to be the disjoint union of G_i from (1) and a fixed complete graph K with l vertices. The ρ_i are extended to ρ_i' by adjoining the identity map : $K \to K$. See [20; 19] for how bare canonicity follows from this.

1.4. A SIMPLER CONSTRUCTION. The proof in [20] of theorem 2 constructed the G_i and ρ_i in the inverse system (1) by induction on $i \geq 2$. The graphs G_0, G_1, G_2 are a fixed complete graph on k vertices, with ρ_0 and ρ_1 being the identity map. In the inductive step, given G_i for $i \geq 2$, the graph G_{i+1} and bounded morphism ρ_i were built by a probabilistic construction [20, proposition 2.2], elaborating on a famous construction by Erdős [7] of finite graphs with arbitrarily large chromatic number and girth.

This article focuses on that inductive step. It seems that the elaboration may have been too elaborate. Trotta [32, footnote 3] observed that there was already a similar but 'slightly simpler' construction in the literature, by Hell and Nešetřil [13, theorem 4]. Again, I quote it:

> **Theorem 3. (Hell–Nešetřil, [13, theorem 4])** *Let G be [a] connected [graph], $\chi(G) > 2$, and let $g > 0$. For every graph G' with $G' < G$ there is a connected graph H of girth at least g which admits a homomorphism to G but not to G'.*

Here, graphs are finite and can be taken to be either directed or undirected. The notation $G' < G$ means that there is a homomorphism from G' to G but not vice versa.

Hell and Nešetřil's proof of their theorem is also probabilistic, with the exact same definition of the probability p as in [20] — compare [20, equation (1)] with [13, line 7 of proof of theorem 4] and (7) below. Perhaps that is to be expected, as this probability often crops up in similar proofs, including of Erdős's original result — see, e.g., [6, line 1 of proof of theorem 11.2.2] — and in a similar form in Erdős's original proof in [7].

However, Hell–Nešetřil's argument does indeed seem simpler than the one in [20]. It takes just one page (in rather condensed form). The homomorphisms are handled differently and more simply than was chromatic number in [20], even though homomorphisms are more general in that chromatic number can be characterised by existence and non-existence of homomorphisms to complete graphs.

Let us see how theorem 3 might help with the inductive step in theorem 2. Assume inductively that G_i in theorem 2 has been constructed and is connected, and in theorem 3 take $g = i+1$, $G = G_i$, and G' to be a complete graph with $k-1$ vertices. Since $\chi(G_i) = k$, there is no homomorphism from G to G'. Then the H given by theorem 3 has no odd (or even) cycles of length $\leq i$; and H admits a homomorphism to G_i but not to G', from which it follows that $\chi(H) = k$ as well. Also, H is connected. So we could try taking $G_{i+1} = H$.

As it stands, this argument is not correct. We do not know that all the assumptions of theorem 3 are met: in particular, that $k = \chi(G_i) > 2$ and there is a homomorphism from G' to G_i. Nor do we know that $G' < H$, which would be needed to continue the induction.

These problems are minor. A more serious one is that the homomorphism from H to G given by theorem 3 is not stated to be a bounded morphism. This is not

a concern in [13] and is not mentioned there, but in fact the proof does not yield bounded morphisms. In more detail, the proofs of theorem 2 in [20] and theorem 3 in [13] each have two stages.

1. The first stage constructs a ballpark approximation, say A^*, to G_{i+1} or H, respectively. It is a graph with few short cycles, and with large chromatic number or no homomorphism to G', respectively. In each case, there is a 'projection' map from A^* to G_i or G, respectively, which is in fact a surjective bounded morphism.
2. The second stage removes all remaining short cycles from A^* — odd cycles in [20], and all cycles in [13]. The way this is done in [13] seems to destroy the bounded morphism property, and perhaps unavoidably. We will comment on this in §4. The second stage in [20] proceeds differently, precisely in order to keep the bounded morphism property, but at the cost of removing only short *odd* cycles (which is sufficient for the canonicity argument).

In spite of, or because of these problems, it seems a worthwhile exercise — and it is no more than that — to reprove theorem 2 with Hell and Nešetřil's methods, using the approach in [20] for the second stage. We do this in §3 of this article. §4 gives some evidence that the loss of the bounded morphism property in [13] may be inevitable. §2 gives definitions and notation, and §5 lists some possibilities for further work.

2. Definitions, notation, and background facts

Here is a quick rundown of the definitions and notation used later, plus some handy basic facts.

2.1. Graphs. All graphs from here on in are *undirected, loop-free, and finite* (though some statements also hold for infinite graphs, and the inverse-limit graphs G_ω, G'_ω that we met in §1.3 are uncountable). Formally, we take a *graph* to be a finite model-theoretic structure G for the signature comprising a single binary relation symbol E, such that $G \vDash \forall xy(Exy \to Eyx \wedge x \neq y)$. It is standard in model theory that structures are non-empty. We often identify G notationally with its domain $\mathrm{dom}\, G$, so that for example, $|G|$ denotes the cardinality of $\mathrm{dom}\, G$. Elements of $\mathrm{dom}\, G$ are called *nodes* or *vertices* of G.

The interpretation $E^G \subseteq \mathrm{dom}\, G \times \mathrm{dom}\, G$ of E in G is called the *edge relation* on G. An *edge* of G is a set $\{a,b\}$ of nodes of G such that $G \vDash Eab$. We generally write an edge $\{a,b\}$ as simply ab; note that ab is the *same* edge as ba, and that a subset $X \subseteq G$ *contains* an edge ab of G iff $ab = \{a,b\} \subseteq X$, iff $a,b \in X$. We say that $a,b \in G$ are *adjacent* if ab is an edge of G.

A *(spanning) subgraph* of G is a graph H with $\mathrm{dom}\, H \subseteq \mathrm{dom}\, G$ (respectively, $\mathrm{dom}\, H = \mathrm{dom}\, G$) and $E^H \subseteq E^G$. An *induced subgraph* of G is just a substructure H of G in the usual model-theoretic sense: $\mathrm{dom}\, H \subseteq \mathrm{dom}\, G$ and $E^H = E^G \cap (\mathrm{dom}\, H \times \mathrm{dom}\, H)$.

Homomorphisms and bounded morphisms. Homomorphisms of graphs are as usual in model theory: thus, a *homomorphism* from a graph G to a graph H is a map $f\colon \mathrm{dom}\, G \to \mathrm{dom}\, H$ such that for each edge ab of G, $f(a)f(b)$ is an edge of H. Following [13], we write $G \to H$ if there exists a homomorphism from G to H (it need

not be injective or surjective), and $G < H$ if $G \to H \not\to G$. The relation $\to$ is reflexive and transitive, and $<$ is irreflexive and transitive.

We borrow a standard notion from modal logic. For graphs G, H, a *bounded morphism* $f\colon G \to H$ is a homomorphism with the additional 'back' property that for every $a \in G$ and $b \in H$ such that $f(a)b$ is an edge of H, there is $a' \in G$ such that aa' is a edge of G and $f(a') = b$. Plainly, the composition $f \circ g$ of two bounded morphisms f, g is also a bounded morphism.

Cycles and girth. For finite $n \geq 3$, an *n-cycle,* or a *cycle of length n,* is a graph C with precisely n vertices, say $c_1, \ldots, c_n$, and whose edges are precisely c_1c_2, c_2c_3, $\ldots$, $c_{n-1}c_n$, and c_nc_1. An *even (odd) cycle* is one of even (respectively, odd) length. A *cycle in a graph G* is a subgraph of G that is a cycle.

The *(odd) girth* of a graph is the length of its shortest (respectively, odd) cycle, and ∞ if it has no (respectively, odd) cycles. We will be using mainly odd girth, and we write $og(G)$ for the odd girth of the graph G. As an example, if C is an odd cycle then $og(C) = |C|$. It is well known that

$$\text{if } G \to H \text{ then } og(G) \geq og(H). \tag{2}$$

For a proof, see, e.g., [20, lemma 2.9]. This can fail for ordinary girth. For example, a 4-cycle G can wrap homomorphically onto a single edge in any graph H that has at least one edge. There is no constraint at all on the girth of such graphs H.

Chromatic number. For a graph G and a set K (of 'colours'), a *K-colouring of G* is a map $k\colon G \to K$ such that $k(a) \neq k(b)$ for each edge ab of G. We take the *chromatic number* $\chi(G)$ of G to be the least cardinal κ such that G has a κ-colouring. Since our graphs are loopfree (irreflexive), the identity map $\iota : G \to G$ is a $(\mathrm{dom}\, G)$-colouring of G, and it follows that $\chi(G)$ is well defined and $\chi(G) \leq |G|$.

Chromatic number has a connection to cycles. Graphs of chromatic number at most 2 are said to be *bipartite,* and it is a standard fact that a graph is bipartite iff it has no odd cycles: that is, it has odd girth ∞. (This also holds for infinite graphs, such as the inverse limit G_ω seen earlier.) For a proof, see, e.g., [6, proposition 1.6.1]. We will not use this fact here, but its application to the inverse limit G_ω of (1) is a critical part of the proof of theorem 1.

Let G, H be graphs and $f\colon G \to H$ a homomorphism. If $k : H \to K$ is a K-colouring of H, then $k \circ f\colon G \to K$ is one of G. So in contrast to (2),

$$\text{if } G \to H \text{ then } \chi(G) \leq \chi(H). \tag{3}$$

A graph H is *complete* if $H \vDash \forall xy(x \neq y \to Exy)$. In that case, $\chi(H) = |H|$, and an H-colouring (i.e., a $(\mathrm{dom}\, H)$-colouring) of a graph G is the same thing as a homomorphism $: G \to H$. If $\chi(G) \leq \chi(H)$ then G has an H-colouring, so if H is complete then $G \to H$. Combining this with (3), we obtain:

$$G \to H \quad \text{iff} \quad \chi(G) \leq \chi(H) \qquad (\text{if } H \text{ is complete}). \tag{4}$$

2.2. PROBABILITY. We assume some familiarity with this topic, but it takes little space to rehearse the relevant points. Given a finite graph A and a real number p with $0 < p < 1$, we will be choosing a *random subgraph* A^* of A, with the same nodes as A (i.e., a spanning subgraph of A), and with each edge chosen independently with probability p. In more detail, let E be the set of edges of A, and $\mathcal{P}$ be the finite set of

all spanning subgraphs of A. Let $A^* \in \mathcal{P}$ and let the set of edges of A^* be E^*, say, so that $\mathsf{E}^* \subseteq \mathsf{E}$. Then the *probability of* A^* is defined to be the real number

$$\mathbb{P}(A^*) \;=\; \prod_{e\in\mathsf{E}^*} p \cdot \prod_{e\in\mathsf{E}\setminus\mathsf{E}^*} (1-p) \;=\; p^{|\mathsf{E}^*|}\cdot(1-p)^{|\mathsf{E}\setminus\mathsf{E}^*|}.$$

Then $0 < \mathbb{P}(A^*) \leq 1$ (we can have $\mathbb{P}(A^*) = 1$ if $\mathsf{E} = \emptyset$). For a property φ of graphs in $\mathcal{P}$, also called an *event,* we define its *probability*

$$\mathbb{P}(\varphi) = \sum\{\mathbb{P}(A^*)\colon A^* \in \mathcal{P},\, A^* \vDash \varphi\}.$$

Let I be a finite set and φ, φ_i $(i \in I)$ be properties. Clearly, $0 \leq \mathbb{P}(\varphi) \leq 1$, $\mathbb{P}(\bot) = 0$, $\mathbb{P}(\top) = 1$, $\mathbb{P}(\neg\varphi) = 1 - \mathbb{P}(\varphi)$, and $\mathbb{P}(\bigvee_{i\in I}\varphi_i) \leq \sum_{i\in I}\mathbb{P}(\varphi_i)$. For $e \in \mathsf{E}$, let ε_e be the property 'e is an edge (of A^*)'. If $I, J \subseteq \mathsf{E}$ are disjoint, then one can check, for example by induction on $|\,\mathsf{E}\setminus(I\cup J)|$, that

$$\mathbb{P}\Big(\bigwedge_{e\in I}\varepsilon_e \wedge \bigwedge_{e\in J}\neg\varepsilon_e\Big) = p^{|I|}\cdot(1-p)^{|J|}. \tag{5}$$

Hence, we say that the edges of A^* are *chosen independently.*

For a map $f\colon \mathcal{P} \to \mathbb{R}$ taking only non-negative values, the *expected* (or *mean*) *value* of f is defined to be

$$E(f) = \sum\{\mathbb{P}(A^*)\cdot f(A^*)\colon A^* \in \mathcal{P}\}.$$

Let $v \in \mathbb{R}$ with $v > 0$. Clearly, $\mathbb{P}(f(A^*) \geq v)$ decreases as v grows, since fewer A^* have the cited property. In fact, if we put $\mathcal{Q} = \{A^* \in \mathcal{P}\colon f(A^*) \geq v\}$, then *Markov's inequality* says:

$$\begin{array}{lcll}
\mathbb{P}(f(A^*)\geq v) & = & \sum\{\mathbb{P}(A^*)\colon A^*\in\mathcal{Q}\} & \text{by definition of } \mathbb{P}(-) \text{ and } \mathcal{Q}\\
& \leq & \sum\{\mathbb{P}(A^*)\cdot f(A^*)/v\colon A^*\in\mathcal{Q}\} & \text{since } f(A^*)/v \geq 1 \text{ for } A^*\in\mathcal{Q}\\
& \leq & \sum\{\mathbb{P}(A^*)\cdot f(A^*)/v\colon A^*\in\mathcal{P}\} & \text{since } f(A^*)\geq 0 \text{ for } A^*\in\mathcal{P}\setminus\mathcal{Q}\\
& = & E(f)/v & \text{by definition of } E(f).
\end{array}$$

3. Theorem

The following adapts Hell and Nešetřil's [13, theorem 4] to our setting. We reiterate that graphs are undirected, loop-free, and finite.

Theorem 4. *Let $0 < c < \omega$, let G be a graph with no odd cycles of length $< c$ (that is, $og(G) \geq c$), and let K be a graph with $G \not\to K$. Then there exists a graph H such that*

- H1. *H has no odd cycles of length $\leq c$ (that is, $og(H) > c$),*
- H2. *there is a surjective bounded morphism $\rho : H \to G$,*
- H3. *$H \not\to K$.*

One can see the similarity of this to [13, theorem 4], quoted as theorem 3 above, but there are also differences. Theorem 4 is solely for undirected graphs. Some of theorem 3's hypotheses have been dropped, but we must assume that $G \not\to K$ because (H2) and (H3) imply it. The hypothesis $og(G) \geq c$ is met in our application and has been added as a convenience to simplify the proof, but it could be eliminated by iteration. The conclusion is also different: some parts have been dropped and (H1) is weaker than in [13], but (H2) is stronger.

To prove theorem 2 using theorem 4, we construct the G_i and ρ_i by induction on i. Take G_0 to be a complete graph with k vertices. Let $i < \omega$ and assume inductively that G_i is given, with $og(G_i) > i$ and $\chi(G_i) = k \geq 2$. In theorem 4, take $c = i+1$, $G = G_i$, and K to be a complete graph with $k-1$ vertices. By (3), $G \not\to K$. Put $G_{i+1} = H$ and $\rho_i = \rho$ from the theorem. Then $og(G_{i+1}) > i+1$ by (H1); and (H3) with (4), and (H2) with (3), yield

$$k-1 = \chi(K) < \chi(H) \leq \chi(G) = k,$$

so $\chi(G_{i+1}) = k$, as required.

We will be quoting some parts of Hell and Nešetřil's proof, but we have not stuck religiously to their presentation, and we give much more detail to allow easier comparison with [20], parts of which we also quote.

Proof. If c is even, we can take $H = G$, since (H2) is then trivial and (H1) and (H3) follow from the hypotheses. So from now on, we suppose that c is odd. This is used later only in establishing (H1).

Let $|G| = r$ and let the vertices of G be $a_0, \ldots, a_{r-1}$, say. Also fix a real number δ with

$$0 < \delta < 1/c. \tag{6}$$

This is well defined because $c > 0$. Now let n be a variable with $2 \leq n < \omega$. Many definitions below depend implicitly on n.

As in [13], which we partially quote here, let A be the graph obtained from G by replacing each a_i by a set A_i of size n. (These sets are pairwise disjoint, and whenever a_i and a_j are adjacent in G, every vertex in A_i is adjacent in A to every vertex in A_j. There are no $A_i - A_j$ edges in A when a_i and a_j are not adjacent in G.)

As in §2.2 and [13], consider a random subgraph A^* of A, on the same nodes (a spanning subgraph), with each edge chosen independently with the probability

$$p = n^{\delta - 1}. \tag{7}$$

Since $n \geq 2$ and $\delta < 1$, we have $0 < p < 1$.

Let $\pi : A^* \to G$ be the projection (a term we use loosely) given by $\pi(x) = a_i$ for each $i < r$ and $x \in A_i$. Because A^* is a subgraph of A, π is a homomorphism, and it is plainly surjective.

Claim 1. *The probability (say, $\alpha(n)$) that π is a surjective bounded morphism goes to 1 as $n \to \infty$.*

Proof of Claim 1. This is as in [20, lemma 2.7]. Since π is a surjective homomorphism, we need only check the 'back' property, and to do this, we consider its negation: *there is an edge $a_i a_j$ of G and a node $x \in A_i$ with no edge (in A^*) to any node in A_j.* Consider some $n \geq 2$. The number of edges $a_i a_j$ of G is at most $|G \times G| = r^2$. Given such an edge $a_i a_j$, the number of $x \in A_i$ is n. And given $x \in A_i$, for each $y \in A_j$ the probability of xy not being an edge of A^* is $1-p$. There are n such y, and edges are chosen independently (see (5)), so the probability that no such xy is an edge of A^* is $(1-p)^n$. So, writing E^G for the set of edges of G, the probability $1 - \alpha(n)$ of the

whole negation satisfies

$$\begin{aligned} 1-\alpha(n) &= \mathbb{P}\Big(\bigvee_{a_ia_j\in \mathsf{E}^G}\bigvee_{x\in A_i}(\text{there is no edge in } A^* \text{ from } x \text{ to } A_j)\Big) \\ &\leq \sum_{a_ia_j\in \mathsf{E}^G}\sum_{x\in A_i}\mathbb{P}(\text{there is no edge in } A^* \text{ from } x \text{ to } A_j) \qquad (8)\\ &\leq r^2\cdot n\cdot(1-p)^n. \end{aligned}$$

Here and below, $\mathbb{P}$ denotes probability, as in §2.2.

To prove the claim, we now use natural logarithms. It is an exercise that $e^x \geq 1+x$ for all $x\in\mathbb{R}$. So $1-p\leq e^{-p}$, and hence by (7),

$$\ln(1-p)\leq -p = -(n^{\delta-1}). \qquad (9)$$

The left-hand side exists since $1-p>0$. Hence,

$$\begin{aligned} 0\leq 1-\alpha(n) &\leq r^2\cdot n\cdot(1-p)^n && \text{by (8)}\\ &= \exp[2\ln r+\ln n+n\ln(1-p)] && \text{using } x=\exp[\ln x] \text{ for } x>0\\ &\leq \exp[2\ln r+\ln n-n(n^{\delta-1})] && \text{by (9)}\\ &= \exp[2\ln r+\ln n-n^{\delta}]. \end{aligned}$$

Now r is fixed, and $\delta>0$ by (6). Hence, as $n\to\infty$, the term $-n^{\delta}$ dominates the expression inside the 'exp', which therefore tends to $-\infty$, so that the right-hand side tends to 0; so $1-\alpha(n)$ also tends to 0 and $\alpha(n)$ to 1, proving the claim. ⊣

Definition 5. Let $|K|=k$, say. We say that a set X of vertices of A is *large* if X intersects each of A_i and A_j in at least n/k vertices, for some edge a_ia_j of G.

We say that X is *minimal large* if it is large but no proper subset of it is large. In that case, there is a unique edge a_ia_j of G with $X\subseteq A_i\cup A_j$, we have $|X\cap A_i|=|X\cap A_j|=\lceil n/k\rceil$, and X contains exactly $\lceil n/k\rceil^2$ edges of A.

We write $\mathcal{S}$ for the set of all pairs (X,S), where $X\subseteq A$ is minimal large and S is a set of $<n$ edges of the induced subgraph of A with domain X.

Claim 2. *The probability (say, $\beta(n)$) that every large set contains $\geq n$ edges of A^* goes to 1 as $n\to\infty$.*

Proof of Claim 2. This is as in [13], and is the most complicated claim. We write $\beta(n)$ as just β here. Again, we consider some $n\geq 2$, and analyse the negation of the property in the claim: *some large set contains $<n$ edges of A^**. This holds iff some minimal large set contains $<n$ edges of A^*, iff there is some $(X,S)\in\mathcal{S}$ such that all edges of A^* that are contained in X are in S. So

$$\begin{aligned} 1-\beta &= \mathbb{P}(\textstyle\bigvee_{(X,S)\in\mathcal{S}}(\text{every edge of } A^* \text{ contained in } X \text{ lies in } S))\\ &\leq \textstyle\sum_{(X,S)\in\mathcal{S}}\mathbb{P}(\text{every edge of } A^* \text{ contained in } X \text{ lies in } S). \end{aligned} \qquad (10)$$

We compute an upper bound for this expression.

- The number of minimal large X is trivially $\leq|\wp(A)|=2^{nr}$; and if $X\subseteq A_i\cup A_j$, say, the number of S with $(X,S)\in\mathcal{S}$ is clearly at most the number of sequences of elements of $A_i\times A_j$ of length n, which is $(n^2)^n$. So $|\mathcal{S}|\leq 2^{nr}\cdot n^{2n}$.

- For a given $(X,S) \in \mathcal{S}$, the probability γ, say, that *every edge of A^* contained in X lies in S* is the probability that every edge of A contained in X but not in S comes out as a non-edge in A^*. There are $\lceil n/k \rceil^2$ edges of A contained in X, so $\lceil n/k \rceil^2 - |S|$ edges of A contained in X are not in S. For each one, its probability of not being an edge of A^* is $1-p$. Again, edges are chosen independently (5), so $\gamma = (1-p)^{\lceil n/k \rceil^2 - |S|}$. Since $1-p<1$, $n^2/k^2 \leq \lceil n/k \rceil^2$, and $|S|<n$, it follows that $\gamma \leq (1-p)^{n^2/k^2-n}$.

So by (10),

$$1-\beta \;\leq\; |\mathcal{S}| \cdot \gamma \;\leq\; 2^{nr} \cdot n^{2n} \cdot (1-p)^{n^2/k^2-n}. \tag{11}$$

To finish, we use logs again:

$$\begin{aligned} 0 \leq 1-\beta &\leq \exp[nr\ln 2 + 2n\ln n + (n^2/k^2 - n)\ln(1-p)] && \text{by (11)} \\ &\leq \exp[nr\ln 2 + 2n\ln n + (n^2/k^2 - n)(-(n^{\delta-1}))] && \text{by (9)} \\ &= \exp[nr\ln 2 + 2n\ln n + n^{\delta} - n^{\delta+1}/k^2]. \end{aligned}$$

Since $\delta > 0$ by (6), the term $n^{\delta+1}/k^2$ is a polynomial of degree >1 and dominates the term inside the 'exp', which therefore tends to $-\infty$ as $n \to \infty$. So as $n \to \infty$, the right-hand side above tends to 0, so also $1-\beta \to 0$ and $\beta \to 1$, proving the claim. $\dashv$

We now turn to the odd cycles in A^* of length $\leq c$. Recall that $\pi\colon A^* \to G$ is a homomorphism witnessing $A^* \to G$. So by (2) and the theorem's assumptions,

$$og(A^*) \geq og(G) \geq c. \tag{12}$$

Thus already, A^* has no odd cycles of length $<c$. We would like to choose it with no c-cycles either, so that $og(A^*) > c$. We have no way to do this directly, but the next claim lets us choose it with few c-cycles, which we can then safely eliminate.

The claims so far used *probabilities* to obtain *lower* bounds for some quantities (numbers of edges), with limiting probability 1. We now want to obtain an *upper* bound, with limiting probability 1, for the number of edges in c-cycles, and we will use *expected values* instead.

Claim 3. *The probability that A^* has $<n$ edges lying in cycles of length c tends to 1 as $n \to \infty$.*

Proof of Claim 3. Again this is much as in [13] (and also [20, lemma 2.6]). Fix $n \geq 2$ and let $\mathcal{C}$ be the set of all c-cycles in A (see §2.1 above for the definition of c-cycle). Recalling that $|A| = nr$, we certainly have $|\mathcal{C}| \leq (nr)^c$, the number of sequences of points in A of length c. For each $C \in \mathcal{C}$ and each A^*, define

$$\xi(C,A^*) = \begin{cases} 1, & \text{if } C \text{ is a subgraph of } A^* \text{ (i.e., a cycle in it)}, \\ 0, & \text{otherwise.} \end{cases}$$

So the number of c-cycles in A^* is $\sum_{C \in \mathcal{C}} \xi(C,A^*)$. Now a given $C \in \mathcal{C}$ is a cycle in A^* iff each of its c edges is an edge of A^*, and again since edges are chosen independently (5), we obtain $\mathbb{P}(C \text{ is a cycle}) = p^c$. The expected (or mean) number E of

c-cycles in A^* (see §2.2) is then

$$\begin{array}{rcll} E & = & \sum_{A^*}(\mathbb{P}(A^*)\cdot\sum_{C\in\mathcal{C}}\xi(C,A^*)) & \text{by definition of } E \\ & = & \sum_{C\in\mathcal{C}}(\sum_{A^*}\mathbb{P}(A^*)\cdot\xi(C,A^*)) & \\ & = & \sum_{C\in\mathcal{C}}\mathbb{P}(C \text{ is a cycle}) & \text{by definition of } \mathbb{P} \\ & = & \sum_{C\in\mathcal{C}}p^c & \text{by the above} \\ & = & |\mathcal{C}|\cdot p^c & \\ & \leq & (nr)^c\cdot(n^{\delta-1})^c & \text{by the above and (7)} \\ & = & r^c\cdot n^{c\delta}. & \end{array}$$

If the property in the claim fails and A^* has $\geq n$ edges lying in c-cycles, then since every c-cycle has c edges, A^* has $\geq n/c$ c-cycles. By Markov's inequality (see §2.2), this event has probability at most Ec/n, which from the above is at most $r^c\cdot c\cdot n^{c\delta-1}$. Since $\delta < 1/c$ by (6), we have $c\delta-1<0$, so this probability tends to 0 as $n\to\infty$, which proves the claim. ⊣

By claims 1–3, we can choose n so large that each of the three properties cited in them has probability $> 2/3$. Then the probability that one or more of the properties fails is $< 3(1-2/3) = 1$, so we can choose and fix A^* so that the following all hold:

P1. the projection $\pi\colon A^*\to G$ is a surjective bounded morphism,
P2. each large set contains $\geq n$ edges of A^*,
P3. A^* has $< n$ edges lying in c-cycles (and $og(A^*)\geq c$ by (12)).

This concludes the first stage of the proof. In the second stage, we eliminate the c-cycles from A^*. As we said, Hell and Nešetřil do this by deleting apparently all edges in cycles of length $\leq c$, yielding something like the graph B below. This approach seems not to preserve (P1) above — see §4 for discussion — so we proceed differently, as in [20].

Write $\mathbb{Z}_2$ for the cyclic group $(\{0,1\},+)$ of order 2, where $+$ is taken modulo 2. For each edge xy of A^*, define $\varepsilon(xy)\in\mathbb{Z}_2$ by

$$\varepsilon(xy) = \begin{cases} 1, & \text{if } xy \text{ lies in a } c\text{-cycle in } A^*, \\ 0, & \text{otherwise.} \end{cases}$$

Define a graph H by:

- the set of nodes of H is $\mathrm{dom}(A^*)\times\mathbb{Z}_2$,
- $(x,i)(y,j)$ is an edge of H iff xy is an edge of A^* and $i+j=\varepsilon(xy)$.

Essentially, H comprises two disjoint copies of A^*, except that edges in c-cycles are diverted to run between the copies. We show that H is as required by the theorem.

Let $\zeta\colon H\to A^*$ be the projection given by $(x,i)\mapsto x$, for each $(x,i)\in H$. We have not lost any edges, so ζ is a surjective bounded morphism, and by (P1), the composition $\rho=\pi\circ\zeta\colon H\to G$ is also a surjective bounded morphism. For a full proof of this, see [20, lemma 2.8].

We show next that $og(H)>c$. This is as in [20, lemma 2.10], and the essence is that c-cycles in A^* are unwrapped to become $2c$-cycles in H.

As ζ witnesses $H \to A^*$, we have $og(H) \geq og(A^*) \geq c$ by (2) and (P3). So it suffices to show that H has no c-cycles.

Suppose for contradiction that C is a c-cycle in H. Because c is odd, $og(C) = c$. Let C have nodes $\{(x_t, i_t) : t < c\}$ and edges $\{(x_t, i_t)(x_{t+1}, i_{t+1}) : t < c\}$, say, where here and below, $t+1$ is taken modulo c. By definition of H, the graph D with domain $\{x_t : t < c\}$ and edges $\{x_t x_{t+1} : t < c\}$ is a subgraph of A^*. Also, $\zeta \restriction C$ and the inclusion map $\iota : D \hookrightarrow A^*$ are homomorphisms witnessing $C \to D \to A^*$, so by (2) and (P3),

$$c = og(C) \geq og(D) \geq og(A^*) \geq c.$$

Hence, $og(D) = c$, so D has a c-cycle, say C'. As C' has c nodes and edges and D has at most c nodes and edges, it follows that $C' = D$. Hence, each edge $x_t x_{t+1}$ of D lies in a c-cycle of A^* (namely, C'), so by definition of C, H, and ε, we obtain $i_t + i_{t+1} = \varepsilon(x_t x_{t+1}) = 1$ for each $t < c$. Then since $\mathbb{Z}_2 \vDash \forall x(x + x = 0)$ and c is odd,

$$\mathbb{Z}_2 \vDash 0 = \big(\sum_{t<c} i_t\big) + \big(\sum_{t<c} i_t\big) = \sum_{t<c} (i_t + i_{t+1}) = \sum_{t<c} 1 = 1,$$

a contradiction.

It remains only to prove that $H \not\to K$. To this end, we first let B be the spanning subgraph of A^* obtained by deleting all edges xy of A^* lying in c-cycles — those with $\varepsilon(xy) = 1$. By (P3), $< n$ edges are deleted.

By definition, xy is an edge of B iff it is an edge of A^* and $\varepsilon(xy) = 0$, iff $(x,0)(y,0)$ is an edge of H. Thus, the map $x \mapsto (x,0)$ witnesses $B \to H$, so if $H \to K$ then $B \to K$ by transitivity of $\to$. So to show that $H \not\to K$, it suffices to prove:

Claim 4. *$B \not\to K$.*

Proof of Claim 4. This is as in [13]. Suppose for contradiction that $f : B \to K$ is a homomorphism. Let $i < r$. Then $A_i \subseteq B = f^{-1}(K) = \bigcup_{z \in K} f^{-1}(z)$, so $A_i = \bigcup_{z \in K} (A_i \cap f^{-1}(z))$. Since $|A_i| = n$ and $|K| = k$, there must be some $z_i \in K$ such that

$$|A_i \cap f^{-1}(z_i)| \geq n/k. \tag{13}$$

Choose such a z_i for each $i < r$. Define a map $g : G \to K$ by $g(a_i) = z_i$, for each $i < r$.

We show that g is a homomorphism. Let $a_i a_j$ be an edge of G. By (13), the set $X = f^{-1}(\{z_i, z_j\})$ is large, so by (P2) contains $\geq n$ edges of A^*; and by (P3), $< n$ of them satisfy $\varepsilon(xy) = 1$ and were deleted in forming B. So X contains some edge of B. As f is a homomorphism, and since by (13), $f(X) = \{z_i, z_j\}$, we see that $\{z_i, z_j\}$ contains some edge of K, and it follows that $z_i z_j$ is an edge of K, as required. The existence of g contradicts our assumption that $G \not\to K$, and proves the claim and the theorem. ⊣

4. Bounded morphisms

I am unable to show that the projection $\pi : A^* \to G$ is with limiting probability 1 a bounded morphism after deleting edges of A^* in c-cycles, let alone edges in c'-cycles for all $c' \leq c$ as in [13, theorem 4]. The following argument suggests that this may be inevitable.

Let $\alpha, \beta : \omega \to \omega$ be any functions (unrelated to the α and β in claims 1 and 2 earlier). Consider the properties:

(A) for each edge a_ia_j of G and each $x \in A_i$, there are $\geq \alpha(n)$ edges of A^* from x into A_j,

(B) there are $< \beta(n)$ edges of A^* lying in c-cycles.

Compare claims 1 and 3 above, where we had $\alpha(n) = 1$ and $\beta(n) = n$. If we can choose α, β so that the probabilities of (A) and (B) both go to 1 as $n \to \infty$, and

$$\alpha(n) \geq \beta(n) \quad \text{for large } n, \tag{14}$$

then we can delete all edges lying in c-cycles, and π will then be a bounded morphism.

Can we get this? First, let t be the probability that (A) fails. Combining the methods of claims 1 and 2, we find that $t \leq r^2 \cdot n \cdot n^{\alpha(n)-1} \cdot (1-p)^{n-\alpha(n)}$. Taking logs and using (9), we get

$$\begin{aligned} t &\leq \exp[2\ln r + \alpha(n)\ln n - (n-\alpha(n))n^{\delta-1}] \\ &= \exp[2\ln r + \alpha(n)\ln n + \alpha(n)n^{\delta-1} - n^{\delta}]. \end{aligned}$$

A necessary condition for this last expression to go to 0 is that

$$n^{\delta} > \alpha(n)\ln n \quad \text{for large } n. \tag{15}$$

On the (B)-side, the expected number E of c-cycles was found in claim 3 to satisfy $E \leq r^c \cdot n^{c\delta}$. The probability that $\geq \beta(n)$ edges of A^* lie in c-cycles is at most the probability that A^* has $\geq \beta(n)/c$ c-cycles, which by Markov's inequality is $\leq Ec/\beta(n) \leq r^c \cdot c \cdot n^{c\delta}/\beta(n)$. A necessary condition for this latter quantity to go to 0 is

$$\beta(n) > n^{c\delta} \quad \text{for large } n. \tag{16}$$

But recalling that $c \geq 1$ and (see (6)) $\delta > 0$, it follows from (15) and (16) that

$$\alpha(n) < \alpha(n)\ln n < n^{\delta} \leq n^{c\delta} < \beta(n) \quad \text{for large } n,$$

so we fail to get (14). Indeed, for large c and n, the third inequality is very strict.

This argument is perhaps suggestive but not conclusive, since the bounds we worked with might be improved, and the probability p in (7) or other aspects of the proof could perhaps be changed. (For example, we might delete just a single edge from each c-cycle, rather than all edges in c-cycles. This amounts to replacing $\beta(n)$ by $c \cdot \beta(n)$ in (B). Constant factors rarely make a difference, and indeed the reader can check that (14) still fails, so this idea does not seem to help.)

5. Conclusion

Theorem 4 yields a slightly simpler proof, based on Hell and Nešetřil's [13, theorem 4], of what we called the 'combinatorial heart' of the result in [20] that every first-order axiomatisation of RRA has infinitely many non-canonical sentences. Even those who doubt the degree of simplification achieved may still agree that it is good to have two (somewhat) different proofs of this theorem.

We end with some open problems. We were able to eliminate only the short *odd* cycles from the graph H in theorem 4. It may be interesting to find a way to avoid the difficulties of §4 and eliminate *all* short cycles, though this does not appear to have any bearing on bare canonicity.

Finite graphs with arbitrarily high chromatic number and odd girth can be constructed explicitly without invoking probability, using *shift graphs* [14, §2.5]. It might

be interesting to construct inverse systems of graphs and surjective bounded morphisms in such a way.

One may of course ask which other familiar canonical varieties in algebraic logic are barely canonical. Candidates include RA_n (shown canonical in [26, theorems 8, 9]), $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$ (for its canonicity see, e.g., [17, proposition 5.48]), $\mathsf{S}\mathfrak{Nr}_m\mathsf{CA}_n$ [15, 2.7.24], and ICrs_n [11, theorem 4.6] and [2, corollary 5.4], for $3 \leq m \leq n < \omega$.

By the way, the classes ID_n, and IG_n, for finite $n \geq 3$, were shown by Andréka and Thompson [3; 1] to be Sahlqvist-axiomatisable, hence canonical but not barely so. See also [2, corollary 5.4].

It would also be very worthwhile to extend the dichotomy of [24; 25] to wider classes of modal logics.

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