Characterisations of two basic hybrid logics

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October 2, 2024

Abstract

We prove a van Benthem–Rosen-style characterisation theorem for two basic hybrid logics: modal logic with nominals, and modal logic with nominals and @. In each case, we show that over all Kripke models, and over all finite Kripke models, every first-order formula that is invariant under the appropriate bisimulations is equivalent to a hybrid formula, and we give optimal bounds on its modal depth in terms of the quantifier depth of the first-order formula.

We also show by example that the characterisation for modal logic with nominals does not extend to arbitrary bisimulation-closed classes of Kripke models.

Keywords: modal logic; nominals; van Benthem Rosen theorem; hybrid logic characterisation theorem; bisimulation; finite models.

MSC2020 classification: primary 03B45; secondary 03C13, 68Q85, 91A44.

1 Introduction

A basic fact about Kripke semantics for modal logic is van Benthem's theorem [6, 7] that up to logical equivalence, modal logic 'is' the bisimulation-invariant fragment of first-order logic — as far as formulas $\varphi(x)$ with at most one free variable are concerned, and in a signature comprising only unary and binary relation symbols. Modal logic is thus expressively complete for this fragment, and provides an effective syntax for it (the fragment itself is undecidable). Van Benthem's proof used the compactness theorem for first-order logic, and it applies to every elementary class of Kripke models.

This 'modal characterisation theorem' has attracted enormous interest, and a vast number of extensions have been found. Two kinds of extension are directly relevant to this note. On the one hand, Rosen [18] extended van Benthem's 'classical' result to *finite models,* showing that every first-order formula $\varphi(x)$ that is bisimulation invariant over finite Kripke models is equivalent to a modal formula over finite models. This does not follow from the classical result because some first-order formulas are bisimulation invariant over finite models but not over all models [16, 17]. Since its conclusion is stronger, the classical result is not an immediate consequence of the result in the finite either. One might ask if it follows with some extra effort, but Rosen rendered this question moot by providing a uniform argument for both the classical and finite cases, so reproving van Benthem's original result in a different way. Rosen's proof used Hanf locality rather than compactness, which fails in the finite. Otto [16, 17] gave an 'elementary' version of the proof, replacing Hanf locality by a direct application

*Thanks to Guillermo Badia and Rob Goldblatt for valuable comments and encouragement.

of Ehrenfeucht–Fraïssé games, and establishing an optimal bound $2^q - 1$ on the modal depth of an equivalent modal formula in terms of the quantifier depth q of $\varphi(x)$.

On the other hand, different notions of bisimulation have been given for various *hybrid logics,* and some characterisation theorems have been proved for them — see, e.g., [2, 10, 4, 14, 3, 5]. So far, results have been by and large classical, proved using compactness or ultraproducts, and do not cover finite models. But some do. For example, Abramsky and Marsden [1, theorem 11] characterised the temporal hybrid logic with $\downarrow$ and @ in terms of invariance under generated submodels and/or disjoint unions, again establishing the classical and finite cases uniformly. They state [1, §7] that the result still holds in the presence of nominals. Though not directly concerned with hybrid logic, [20] proves an immensely general coalgebraic characterisation theorem for a range of modal-like logics, again uniformly for all models and for finite models.

In this note, we prove a characterisation theorem for two basic hybrid logics, providing a uniform proof that works both classically and in the finite, as Rosen and Otto did, and giving optimal modal depth bounds as Otto did.

The first hybrid logic is simply modal logic with *nominals* — special propositional atoms that are true at precisely one point of each model. We could perhaps call it *'proto-hybrid logic'*. It is, to be sure, a minimal extension of modal logic, but still a 'far from negligible' one [9, p.49]. We use ordinary modal bisimulations here, showing that both classically and in the finite, proto-hybrid logic is the bisimulation-invariant fragment of first-order logic in signatures comprising constants as well as unary and binary relation symbols. Van Benthem's original theorem shows this in the classical case, since being a nominal is first-order definable and so the class of relevant models is elementary. But I have not found a characterisation theorem in the literature for proto-hybrid logic over finite models, nor any depth bounds.

The second hybrid logic is actually called *'basic hybrid logic'* [9, §6.2]. It adds the hybrid actuality operator @ to proto-hybrid logic. A classical characterisation theorem is known for this logic — see [2, theorem 6.1], recalled in [9, theorem 39] — but again, I am not aware of one for finite models, nor any depth bounds. The appropriate notion of bisimulation [9, definition 37] is now slightly stronger, and the bisimulation-invariant first-order fragment consequently slightly larger, but the difference is so slight that we can handle both basic hybrid and proto-hybrid logics here using much the same proof.

The proof itself follows standard lines. The key is to show that every bisimulation-invariant first-order formula $\varphi(x)$ is *'local'* — that is to say, invariant under passing to a 'local neighbourhood' that the hybrid logic can control. The neighbourhood is typically the set of points 'near' to x.

In a little more detail, we want to find a finite set $\mathcal{H}$ of hybrid formulas such that any two pointed Kripke models that agree on $\mathcal{H}$ also agree on φ. For φ will then be equivalent to a boolean combination of formulas in $\mathcal{H}$.

Locality, if we can establish it, lets us restrict each of the two models to a 'neighbourhood' without changing the value of φ. The two neighbourhoods should also agree on $\mathcal{H}$: and a sufficiently large $\mathcal{H}$ should control them well enough to ensure that they also agree on φ — for example, because they are bisimilar. We are done.

This trail was blazed by Rosen [18] and most later writers have followed it. What is perhaps novel here is the choice of neighbourhoods. *Modal unravellings* [9, §3.2] are often used to simplify neighbourhoods sufficiently for $\mathcal{H}$ to control. (Sometimes $\mathcal{H}$ is so powerful that this is unnecessary [1].) Unfortunately, unravellings involve duplicating points in a model. This is problematic with nominals, which must remain true at only one point. So we will use fairly obvious but perhaps new unravellings able to handle nominals. Notwithstanding this, neighbourhoods are more complicated in the presence of nominals, and nominals also interfere to a degree with *disjoint unions,* an ingredient of the proof of locality. We will therefore *interpret* the unravellings (in the model-theoretic sense) in simpler and better-behaved models.

These changes are not wholly trivial, because for basic hybrid logic, the optimal bound on the depth of the equivalent hybrid formula is larger than Otto's bound for modal logic. For proto-hybrid logic, it is larger still. Moreover, unlike for modal logic, the characterisation result for proto-hybrid logic does not extend to arbitrary bisimulation-closed classes of Kripke models: see example 8.1.

I tried to prove the characterisation theorems in this note because I wanted to know whether they were true in the finite. This is not a given. As a warning, while van Benthem's theorem shows that classically, modal logic is the bisimulation-invariant fragment of first-order logic over transitive models, this fails in the finite and additional modal connectives are needed [11]. Another (non-modal) warning example is furnished by two-variable first-order logic [16, 17]. More motivation for the results will be given in section 9 in the light of the greater context available at that point.

Layout. Sections 2–4 present background material and notation, increasingly specialised as we proceed, but with few surprises. Readers will most likely be familiar with this material, so the treatment is brief, but still it takes up around half the paper. Readers may of course skip it and refer back to it as needed. The real work begins in section 5, where we define and study the unravellings. The main characterisation theorem is in section 6. Section 7 gives examples to show optimality of modal depth bounds, and section 8 looks at possible extensions of the theorem to other classes, such as bisimulation-closed classes. The conclusion in section 9 has some comments, such as on possible further work.

2 Hybrid logics

This section presents basic definitions and notation, both for general matters and for the hybrid logics we consider.

We use standard (von Neumann) ordinals. Each ordinal α is the set of smaller ordinals, so the smallest infinite ordinal ω is $\{0, 1, \ldots\}$, and $n = \{0, 1, \ldots, n-1\}$ and $n+1 = n \cup \{n\}$ for $n < \omega$. Ordinal sum $\alpha + \beta$ is defined as usual (as the order type of α followed by β) — for example, $1+\omega = \omega < \omega+1$. We write $|S|$ for the cardinality of a set S, and use $\wp$ to denote the power-set operation. We write $\operatorname{dom} f$ (resp., $\operatorname{rng} f$) for the domain (resp., range) of a function f, and for $S \subseteq \operatorname{dom} f$ we write $f \restriction S$ for the restriction of f to S, and $f(S)$ for $\{f(s) : s \in S\}$ (this can be ambiguous but causes us no difficulties). For sets S, T, we let ${}^T S$ denote the set $\{f \mid f : T \to S\}$ of functions from T to S, and $S \uplus T$ denote the disjoint union of S and T. The latter can be defined formally as $S \times \{0\} \cup T \times \{1\}$, but we generally treat it informally. We generally write binary relations in infix form.

For hybrid logic, we broadly follow the notation in [9]. A *hybrid signature* is a set σ partitioned into two sets, PROP and NOM, where PROP denotes the set of *propositional atoms* (or *propositional variables*) and NOM denotes the set of *nominals.* For a hybrid signature $\sigma = \text{PROP} \uplus \text{NOM}$, we define the hybrid $\mathcal{L}^{\diamond @}(\sigma)$*-formulas* ψ, and their *modal depth* $d(\psi)$, as follows.

1. Each element of σ is an $\mathcal{L}^{\diamond @}(\sigma)$-formula, of modal depth 0.
2. $\top$ is an $\mathcal{L}^{\diamond @}(\sigma)$-formula, also of modal depth 0.
3. If ψ and θ are $\mathcal{L}^{\diamond @}(\sigma)$-formulas, then so are:

(a) $\neg\psi$, and $d(\neg\psi) = d(\psi)$,

(b) $\psi \wedge \theta$, and $d(\psi \wedge \theta) = \max(d(\psi), d(\theta))$,

(c) $\Diamond\psi$, and $d(\Diamond\psi) = 1 + d(\psi)$.

4. If ψ is an $\mathcal{L}^{\Diamond @}(\sigma)$-formula and $c \in \textsc{Nom}$, then $@_c\psi$ is an $\mathcal{L}^{\Diamond @}(\sigma)$-formula, and $d(@_c\psi) = d(\psi)$.

An $\mathcal{L}^{\Diamond}(\sigma)$*-formula* is a $\mathcal{L}^{\Diamond @}(\sigma)$-formula that does not involve any @ — that is, we drop clause 4 above. So $\mathcal{L}^{\Diamond}(\sigma)$-formulas are just modal formulas, except that they may involve nominals. We regard $\bot, \vee, \rightarrow, \leftrightarrow, \Box$ as the usual abbreviations (actually we hardly use them). For a nonempty finite set $S = \{\psi_1, \ldots, \psi_n\}$ of $\mathcal{L}^{\Diamond @}(\sigma)$-formulas, we write $\bigwedge S$ for $\psi_1 \wedge \ldots \wedge \psi_n$ and $\bigvee S$ for $\psi_1 \vee \ldots \vee \psi_n$; the order and bracketing of the ψ_i is immaterial (semantically). We let $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$.

A *Kripke model* (for σ) is a triple $M = (W, R^M, V)$, where $W \neq \emptyset$ is the set of 'worlds', also called the *domain* of M, $R^M \subseteq W \times W$ is the 'accessibility relation', and $V : \sigma \to \wp(W)$ is the 'valuation', satisfying $|V(c)| = 1$ for each $c \in \textsc{Nom}$ — we write c^M for the unique element of $V(c)$. We write $\mathrm{dom}(M)$, and more often just M, for its domain W. We say that M is *finite* if it has finite domain.

A *submodel of* M is a Kripke model of the form $N = (U, R^M \cap (U \times U), V_U)$, where $\emptyset \neq U \subseteq W$ and $V_U(p) = V(p) \cap U$ for $p \in \sigma$. (This is a well-defined Kripke model iff $c^M \in U$ for each $c \in \textsc{Nom}$.) We say that N is a *generated submodel of* M if $u \in U$, $w \in W$, and uR^Mw imply $w \in U$.

When we consider a hybrid signature $\tau \subseteq \sigma$, it will be implicit that the type (atom or nominal) of each symbol in τ is inherited from σ. We write $M \restriction \tau$ for the Kripke model $(W, R^M, V \restriction \tau)$.

We define the semantics of $\mathcal{L}^{\Diamond @}(\sigma)$-formulas in Kripke models M for σ as usual: for $w \in M$, we define $M, w \models p$ iff $w \in V(p)$, for $p \in \sigma$; $M, w \models \top$; $M, w \models \neg\psi$ iff $M, w \not\models \psi$; $M, w \models \psi \wedge \theta$ iff $M, w \models \psi$ and $M, w \models \theta$; $M, w \models \Diamond\psi$ iff $M, u \models \psi$ for some $u \in M$ with wR^Mu; and $M, w \models @_c\psi$ iff $M, c^M \models \psi$.

A *pointed Kripke model* (for σ) is a pair (M, w), where M is a Kripke model (for σ) and $w \in M$. We say that (M, w) is *finite* if M is finite. We say that pointed Kripke models (A, a) and (B, b) for σ *agree* on an $\mathcal{L}^{\Diamond @}(\sigma)$-formula ψ if $A, a \models \psi \iff B, b \models \psi$, and *agree* on a set S of $\mathcal{L}^{\Diamond @}(\sigma)$-formulas if they agree on every formula in S. 'Disagree' will mean 'do not agree'.

3 Bisimulations and games

Fix, for this section, a hybrid signature $\sigma = \textsc{Prop} \uplus \textsc{Nom}$. All Kripke models in this section are for σ.

3.1 Bisimulations

Much of this note is concerned with bisimulations. A $\Diamond$*-bisimulation* (generally called just a *bisimulation* in the literature) between Kripke models A and B is a binary relation $Z \subseteq A \times B$ such that for each $a \in A$ and $b \in B$ with aZb,

1. (A, a) and (B, b) agree on σ,

2. ('Forth') if $a' \in A$ and aR^Aa', then there is $b' \in B$ with bR^Bb' and $a'Zb'$,

3. ('Back') if $b' \in B$ and bR^Bb', then there is $a' \in A$ with aR^Aa' and $a'Zb'$.

Z is said to be a $\Diamond@$-*bisimulation* (called a 'bisimulation-with-names' in [9, §6.2]) if it also satisfies:

4. $c^A Z c^B$ for each $c \in \textsc{Nom}$.

The difference is that a plain $\Diamond$-bisimulation may not relate c^A to anything, nor c^B. It goes without saying that every $\Diamond@$-bisimulation is a $\Diamond$-bisimulation.

DEFINITION 3.1 For $\star \in \{\Diamond, \Diamond@\}$, we say that pointed Kripke models (A, a) and (B, b) are $\star$-*bisimilar,* and write $(A, a) \sim^\star (B, b)$, if there is a $\star$-bisimulation Z between A and B such that aZb.

EXAMPLE 3.2 If A, B are Kripke models and A is a generated submodel of B, then the inclusion map $\iota : A \hookrightarrow B$ is a $\Diamond@$-bisimulation, so $(A, a) \sim^{\Diamond@} (B, a)$ for every $a \in A$.

Perhaps we should say rather that the *graph* $\{(a, \iota(a)) : a \in A\}$ of ι is a bisimulation, but set-theoretically, a function is its graph.

FACT 3.3 For each $\star \in \{\Diamond, \Diamond@\}$, $\mathcal{L}^\star(\sigma)$-formulas are $\star$-*bisimulation invariant:* ie. if (A, a) and (B, b) are pointed Kripke models and $(A, a) \sim^\star (B, b)$, then (A, a) and (B, b) agree on all $\mathcal{L}^\star(\sigma)$-formulas. See, e.g., [9, lemmas 9 and 38].

3.2 Games

Bisimulations can be simulated by games. Let (A, a) and (B, b) be pointed Kripke models and let $\alpha \leq \omega$ be an ordinal. We define an α-round game $\mathrm{B}^{\Diamond}_{\alpha}(A, a, B, b)$ as follows. There are two players, $\forall$ and $\exists$. The successive rounds are numbered $0, 1, \ldots, t, \ldots$ for $t < \alpha$. The initial position, regarded as chosen by $\forall$, is defined by $a_0 = a$ and $b_0 = b$, and $\exists$ loses outright, before any rounds are played and even if $\alpha = 0$, if (A, a_0) and (B, b_0) disagree on σ.

At the start of each round $t < \alpha$, points $a_t \in A$ and $b_t \in B$ are already chosen. In the round, $\forall$ chooses some $a_{t+1} \in A$ with $a_t R^A a_{t+1}$, or some $b_{t+1} \in B$ with $b_t R^B b_{t+1}$. He loses if he can't do this. With full knowledge of his move, $\exists$ must respond with some $b_{t+1} \in B$ with $b_t R^B b_{t+1}$, or some $a_{t+1} \in A$ with $a_t R^A a_{t+1}$, respectively, and she loses if she can't. That completes the round, and $\exists$ loses the game at this point if (A, a_{t+1}) and (B, b_{t+1}) disagree on σ. $\exists$ wins if she never loses at any stage.

The game $\mathrm{B}^{\Diamond@}_{\alpha}(A, a, B, b)$ is the same, except that $\forall$ is allowed to choose the initial position (a_0, b_0) to be any pair in the set $\{(a, b), (c^A, c^B) : c \in \textsc{Nom}\}$.

A *strategy for* $\exists$ in any of the games in this note is a set of rules telling $\exists$ how to move in any position. A strategy is said to be *winning* if $\exists$ wins any play of the game in which she uses it.

DEFINITION 3.4 Let (A, a), (B, b) be pointed Kripke models, $\star \in \{\Diamond, \Diamond@\}$, and $\alpha \leq \omega$. We write $(A, a) \sim^\star_\alpha (B, b)$ if $\exists$ has a winning strategy in the game $\mathrm{B}^\star_\alpha(A, a, B, b)$.

The following is an elementary games lemma.

LEMMA 3.5 *Let (A, a), (B, b) be pointed Kripke models, $\star \in \{\Diamond, \Diamond@\}$, and $\alpha \leq \omega$.*

1. *$(A, a) \sim^{\Diamond@}_\alpha (B, b)$ iff $(A, a') \sim^\star_\alpha (B, b')$ for every $(a', b') \in \{(a, b), (c^A, c^B) : c \in \textsc{Nom}\}$. Hence, if $(A, a) \sim^{\Diamond@}_\alpha (B, b)$ then $(A, a) \sim^{\Diamond}_\alpha (B, b)$.*

2. *If $(A, a) \sim^\star_\alpha (B, b)$ then $(A, a) \sim^\star_\beta (B, b)$ for every $\beta < \alpha$.*

3. *If* $(A,a) \sim^\star_{1+\alpha} (B,b)$ *then the following all hold:*

   (a) $(A,a) \sim^\star_0 (B,b)$,

   (b) *for each* $a' \in A$ *with* $aR^A a'$, *there is some* $b' \in B$ *with* $bR^B b'$ *and* $(A,a') \sim^\star_\alpha (B,b')$,

   (c) *for each* $b' \in B$ *with* $bR^B b'$, *there is some* $a' \in A$ *with* $aR^A a'$ *and* $(A,a') \sim^\star_\alpha (B,b')$.

   The converse implication holds when $\star = \Diamond$.

4. $(A,a) \sim^\star_\omega (B,b)$ *iff* $(A,a) \sim^\star (B,b)$.

3.3 Games and formulas

In this subsection we assume that $\sigma = \text{PROP} \cup \text{NOM}$ is finite.

DEFINITION 3.6 We define, by induction on $k < \omega$, a finite set $\mathcal{F}_k$ (also written $\mathcal{F}_k^\Diamond$) of $\mathcal{L}^\Diamond(\sigma)$-formulas, and a finite set $\mathcal{F}_k^{\Diamond @}$ of $\mathcal{L}^{\Diamond @}(\sigma)$-formulas, as follows:

$$\begin{array}{rcl} \mathcal{F}_0 & = & \sigma, \\ \mathcal{F}_{k+1} & = & \sigma \cup \{\Diamond(\bigwedge S \wedge \neg \bigvee(\mathcal{F}_k \setminus S)) : S \subseteq \mathcal{F}_k\}, \\ \mathcal{F}_k^{\Diamond @} & = & \mathcal{F}_k \cup \{@_c\psi : c \in \text{NOM},\ \psi \in \mathcal{F}_k\}. \end{array}$$

The proof of the following lemma is quite standard, but we include a sketch to illustrate the games and show how @ is handled.

LEMMA 3.7 *Assuming* σ *finite, let* $(A,a), (B,b)$ *be pointed Kripke models,* $\star \in \{\Diamond, \Diamond @\}$, *and* $k < \omega$. *The following are equivalent:*

1. $(A,a) \sim^\star_k (B,b)$,

2. (A,a) *and* (B,b) *agree on all* $\mathcal{L}^\star(\sigma)$*-formulas of depth* $\leq k$,

3. (A,a) *and* (B,b) *agree on* $\mathcal{F}^\star_k$.

Proof. For $1 \Rightarrow 2$, by lemma 3.5(2) it suffices to prove by induction on $\mathcal{L}^\star(\sigma)$–formulas ψ that if $(A,a) \sim^\star_{d(\psi)} (B,b)$ then $A, a \models \psi$ iff $B, b \models \psi$. For $\psi \in \sigma \cup \{\top\}$ this is clear. Assume the result for ψ and θ inductively. The case $\neg\psi$ is very simple, the case $\psi \wedge \theta$ follows from lemma 3.5(2), and the case $\Diamond\psi$ from lemma 3.5(3). For the case $@_c\psi$ for a nominal c, suppose that $(A,a) \sim^{\Diamond @}_{d(@_c\psi)} (B,b)$ (the case $\star = \Diamond$ is of course impossible here). By lemma 3.5(1) and because $d(@_c\psi) = d(\psi)$, we have $(A, c^A) \sim^{\Diamond @}_{d(\psi)} (B, c^B)$, so inductively, $A, c^A \models \psi$ iff $B, c^B \models \psi$. So by semantics, $A, a \models @_c\psi$ iff $B, b \models @_c\psi$, as required.

Part 3 follows from part 2 since all formulas in $\mathcal{F}^\star_k$ have depth $\leq k$.

We first prove $3 \Rightarrow 1$ for $\star = \Diamond$. For a pointed Kripke model (M,m), write $\mathrm{tp}_k(M,m) = \{\psi \in \mathcal{F}_k : M, m \models \psi\}$. Then for $S \subseteq \mathcal{F}_k$ we have

$$\mathrm{tp}_k(M,m) = S \quad \text{iff} \quad M, m \models \bigwedge S \wedge \neg \bigvee(\mathcal{F}_k \setminus S). \tag{1}$$

We now show by induction on k that if (A,a) and (B,b) agree on $\mathcal{F}_k$ then $(A,a) \sim^\Diamond_k (B,b)$. For $k = 0$ it's clear. Assume the result for k and suppose that (A,a) and (B,b) agree on $\mathcal{F}_{k+1}$. We establish (a)–(c) of lemma 3.5(3). For (a), certainly $(A,a) \sim^\Diamond_0 (B,b)$ since $\sigma \subseteq \mathcal{F}_{k+1}$. For (b), take any $a' \in A$ with $aR^A a'$, and let $S = \mathrm{tp}_k(A, a')$ and $\tau = \bigwedge S \wedge \neg \bigvee(\mathcal{F}_k \setminus S)$. By (1), $A, a' \models \tau$, so by semantics, $A, a \models \Diamond\tau$. This formula is in $\mathcal{F}_{k+1}$, so $B, b \models \Diamond\tau$ as well. By semantics, there is $b' \in B$ with $bR^B b'$ and $B, b' \models \tau$ — and (1) gives $\mathrm{tp}_k(B, b') = S = \mathrm{tp}_k(A, a')$. So (A, a') and (B, b') agree on $\mathcal{F}_k$. Inductively, $(A, a') \sim_k^{\Diamond} (B, b')$. Similarly, we can prove that (c) for every $b' \in B$ with $bR^B b'$, there is $a' \in A$ with $aR^A a'$ and $(A, a') \sim_k^{\Diamond} (B, b')$. So by the 'converse implication' of lemma 3.5(3), $(A, a) \sim_{k+1}^{\Diamond} (B, b)$. This completes the induction and proves $3 \Rightarrow 1$ for $\star = \Diamond$.

Finally suppose that (A, a) and (B, b) agree on $\mathcal{F}_k^{\Diamond @}$. By definition of $\mathcal{F}_k^{\Diamond @}$ and semantics, (A, a') and (B, b') agree on $\mathcal{F}_k$ for every $(a', b') \in \{(a, b), (c^A, c^B) : c \in \textsc{Nom}\}$. By the $\Diamond$-case above, $(A, a') \sim_k^{\Diamond} (B, b')$ for every $(a', b') \in \{(a, b), (c^A, c^B) : c \in \textsc{Nom}\}$. By lemma 3.5(1), $(A, a) \sim_k^{\Diamond @} (B, b)$, as required. □

4 Classical logics

The purpose of this note is to compare basic hybrid logic with classical first-order logic, so we discuss the latter now. In fact, we go via infinitary logic, which we will use in interpretations below. For more information see, e.g., [13].

4.1 Classical infinitary logic

A (classical) *signature* is a set L of relation symbols with specified finite arities, and constants. *In this note, we do not need function symbols and will not consider them.* We say that L is *relational* if it contains no constants.

The *$L_{\infty\omega}$-formulas φ,* together with the (perhaps infinite) set $FV(\varphi)$ of free variables of φ and the (ordinal) *quantifier depth of φ,* are defined as in first-order logic with equality but allowing conjunctions and disjunctions of arbitrary sets of formulas. See, e.g., Hodges [13, §2.1]. Nearly all formulas that we consider will have finite quantifier depth. An $L_{\infty\omega}$-formula is said to be *atomic* if it has no proper subformulas, *quantifier-free* if it has no quantifiers, and *first-order,* or just an *L-formula,* if every conjunction and disjunction in it is over a finite set.

An *L-structure* M comprises a nonempty set $\mathrm{dom}(M)$, the *domain* of M, together with an *interpretation* s^M of each $s \in L$ as an n-ary relation on $\mathrm{dom}(M)$, if s is an n-ary relation symbol, and an element of $\mathrm{dom}(M)$ if s is a constant. We usually identify (notationally) M with its domain. We say that M is *finite* if its domain is.

For an L-structure M and an 'assignment' h mapping variables into $\mathrm{dom}(M)$, we define $M, h \models \varphi$ for each $L_{\infty\omega}$-formula φ in the usual way. For a formula φ, an index set I, and pairwise distinct variables x_i $(i \in I)$, we write $\varphi(x_i : i \in I)$ to indicate that $FV(\varphi) \subseteq \{x_i : i \in I\}$. As usual, whether $M, h \models \varphi$ or not depends only on $h \restriction FV(\varphi)$ (and on M and φ of course). So for a formula $\varphi(x_i : i \in I)$ and elements $a_i \in M$ $(i \in I)$, we can write $M \models \varphi(a_i : i \in I)$ if $M, h \models \varphi$, where $h(x_i) = a_i$ for each $i \in I$.

A *substructure* of M is an L-structure N with $\mathrm{dom}(N) \subseteq \mathrm{dom}(M)$ and $N \models \alpha(a_1, \ldots, a_n)$ iff $M \models \alpha(a_1, \ldots, a_n)$ for each atomic L-formula $\alpha(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in N$. The latter holds iff $R^N = R^M \cap \mathrm{dom}(N)^n$ for each n-ary relation symbol $R \in L$, and $c^N = c^M$ for each constant $c \in L$.

Let M, N be L-structures. A map $f : M \to N$ is said to be an *(L)-homomorphism* if for every atomic L-formula $\alpha(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$, we have $M \models \alpha(a_1, \ldots, a_n) \implies N \models \alpha(f(a_1), \ldots, f(a_n))$. A partial map $f : M \to N$ is said to be an *(L)-partial isomorphism* if

$M \models \alpha(a_1, \ldots, a_n) \iff N \models \alpha(f(a_1), \ldots, f(a_n))$ for every atomic L-formula $\alpha(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in \operatorname{dom} f$; if $\operatorname{dom} f = M$, then f is called an *(L)-embedding.*

If L is relational, the *disjoint union* $M + N$ of M and N is the L-structure A defined informally by $\operatorname{dom} A = \operatorname{dom} M \uplus \operatorname{dom} N$ and $R^A = R^M \uplus R^N$ for each relation symbol $R \in L$. It is finite if M, N are finite.

4.2 Correspondence

Central to this note is the correspondence between hybrid and classical logic. For a hybrid signature $\sigma = \textsc{Prop} \uplus \textsc{Nom}$, the classical 'correspondence' signature $L(\sigma)$ comprises a binary relation symbol R; a unary relation symbol P for each $p \in \textsc{Prop}$; and the elements of $\textsc{Nom}$, taken as constants.

A Kripke model $M = (W, R^M, V)$ for σ can be viewed as an $L(\sigma)$-structure as follows. The domain of this structure is W. We interpret R as the binary relation R^M on W, we interpret P (for $p \in \textsc{Prop}$) as the unary relation $V(p)$ on W, and we interpret c (for $c \in \textsc{Nom}$) as c^M as defined in §2. We denote the resulting $L(\sigma)$-structure also by M. Conversely, an $L(\sigma)$-structure can be construed as a Kripke model for σ in the obvious way. So we will regard a Kripke model for σ equally as an $L(\sigma)$-structure, making no distinction between them.

Let σ be a hybrid signature and $\mathcal{K}$ a class of pointed Kripke models for σ. Let $\varphi(x)$ be an $L(\sigma)$-formula, and ψ an $\mathcal{L}^{\Diamond @}(\sigma)$-formula. We say that φ and ψ are *equivalent over* $\mathcal{K}$ if $M \models \varphi(w)$ iff $M, w \models \psi$, for every $(M, w) \in \mathcal{K}$.

We can also in a sense view $\mathcal{L}^{\Diamond @}(\sigma)$-formulas as first-order $L(\sigma)$-formulas, via their *standard translations.* See, e.g., [2, proposition 3.1] and [4, proposition 11]. The standard translation of each $\mathcal{L}^{\Diamond @}(\sigma)$-formula ψ is an $L(\sigma)$-formula $\varphi(x)$ that is equivalent to ψ over every $\mathcal{K}$.

The converse question asks, for given $\mathcal{K}$ and $\star \in \{\Diamond, \Diamond @\}$, whether every $L(\sigma)$-formula $\varphi(x)$ is equivalent to some $\mathcal{L}^{\star}(\sigma)$-formula ψ over $\mathcal{K}$. By fact 3.3, $\mathcal{L}^{\star}(\sigma)$-formulas are $\star$-bisimulation invariant, so we restrict the question to those $\varphi(x)$ that are themselves *$\star$-bisimulation invariant over $\mathcal{K}$:* that is, $A \models \varphi(a)$ iff $B \models \varphi(b)$ whenever $(A, a), (B, b) \in \mathcal{K}$ and $(A, a) \sim^{\star} (B, b)$.

Assuming that $\varphi(x)$ is $\star$-bisimulation invariant over $\mathcal{K}$, we will answer the question affirmatively in theorem 6.1, both for $\mathcal{K}$ the class of all pointed Kripke models for σ, and the class of finite ones. The following lemma, showing robustness of bisimulation invariance, will be helpful in that theorem. The (easy) converse also holds, but we will not need it.

LEMMA 4.1 *Let $\tau \subseteq \sigma$ be hybrid signatures, let $\mathcal{K}_\tau$ (resp., $\mathcal{K}_\sigma$) be the class of all [or all finite] pointed Kripke models for τ (resp., σ), let $\varphi(x)$ be an $L(\tau)$-formula, and $\star \in \{\Diamond, \Diamond @\}$. If φ is $\star$-bisimulation invariant over $\mathcal{K}_\sigma$, then it is also $\star$-bisimulation invariant over $\mathcal{K}_\tau$.*

Proof. Suppose $(A, a), (B, b) \in \mathcal{K}_\tau$ and $(A, a) \sim^{\star} (B, b)$. We show that $A \models \varphi(a)$ iff $B \models \varphi(b)$. This is surprisingly tricky, mainly because σ might have nominals when τ does not.

Write $\sigma = \textsc{Prop} \uplus \textsc{Nom}$. The Kripke model A is for τ. It is easy to find a Kripke model A_1 for σ with $A_1 \restriction \tau = A$ (see §2 for the notation) and $P^{A_1} = \emptyset$ for each $p \in \textsc{Prop} \setminus \tau$. Plainly, $(A_1, a) \in \mathcal{K}_\sigma$, and $A \models \varphi(a)$ iff $A_1 \models \varphi(a)$ because A and A_1 agree on symbols in φ.

Define a second Kripke model A_2 for σ by adding to A_1 a new world $w \notin A \cup B$. Define each symbol in $L(\sigma)$ to have the exact same interpretation in A_2 as it does in A_1 (so w is an isolated world unrelated by R to any world). Then $(A_2, a) \in \mathcal{K}_\sigma$ as well. Plainly, A_1 is a generated submodel of A_2, so by example 3.2, the inclusion map $\iota : A_1 \hookrightarrow A_2$ is a $\star$-bisimulation. Since φ is assumed $\star$-bisimulation invariant over $\mathcal{K}_\sigma$, we obtain $A_1 \models \varphi(a)$ iff $A_2 \models \varphi(a)$.

Finally define a third Kripke model A_3 for σ. It is the same as A_2 except that each nominal in $\textsc{Nom} \setminus \tau$ is now interpreted as w. Then $A_2 \models \varphi(a)$ iff $A_3 \models \varphi(a)$, again because the two models agree on symbols in φ. Also, $(A_3, a) \in \mathcal{K}_\sigma$. Combining the three stages, we see that $A \models \varphi(a)$ iff $A_3 \models \varphi(a)$.

Now do the same for B, arriving at $(B_3, b) \in \mathcal{K}_\sigma$ with $B \models \varphi(b)$ iff $B_3 \models \varphi(b)$.

By assumption, $(A, a) \sim^\star (B, b)$. Let Z be a $\star$-bisimulation between A and B with aZb. It can be checked that $Z \cup \{(w, w)\}$ is a $\star$-bisimulation between A_3 and B_3. Since these models are in $\mathcal{K}_\sigma$, over which φ is assumed $\star$-bisimulation invariant, we obtain $A_3 \models \varphi(a)$ iff $B_3 \models \varphi(b)$. Putting all the steps together proves the lemma. □

4.3 Ehrenfeucht–Fraïssé games

Let L be a signature, let A, B be L-structures, let I be a possibly infinite index set, and let $a_i \in A$ and $b_i \in B$ for each $i \in I$. Write $\boldsymbol{a} = (a_i : i \in I)$ and $\boldsymbol{b} = (b_i : i \in I)$. (When I is a singleton $\{i\}$, we write $\boldsymbol{a}$ as simply a_i.) Let $q < \omega$ and suppose that $I \cap q = \emptyset$, to make things below well defined. The q-round Ehrenfeucht–Fraïssé game $\mathrm{EF}_q(A, \boldsymbol{a}, B, \boldsymbol{b})$ is played again by our players $\forall$ and $\exists$. The successive rounds are numbered $0, 1, \ldots, q-1$. In each round $t < q$, $\forall$ chooses a 'left element' $a_t \in A$, or a 'right element' $b_t \in B$.[1] Having seen $\forall$'s move, $\exists$ responds by choosing a right element $b_t \in B$ or a left element $a_t \in A$, respectively. That completes the round. At the end of play, $\exists$ wins if

$$A \models \alpha(a_{i_1}, \ldots, a_{i_n}) \quad \text{iff} \quad B \models \alpha(b_{i_1}, \ldots, b_{i_n}),$$

for every atomic L-formula $\alpha(x_1, \ldots, x_n)$ and $i_1, \ldots, i_n \in I \cup q$.

DEFINITION 4.2 For $q < \omega$, we write $(A, \boldsymbol{a}) \equiv_{\infty q} (B, \boldsymbol{b})$ if $A \models \varphi(a_i : i \in I)$ iff $B \models \varphi(b_i : i \in I)$ for every $L_{\infty\omega}$-formula $\varphi(x_i : i \in I)$ of quantifier depth $\leq q$.

LEMMA 4.3 *If $\exists$ has a winning strategy in $\mathrm{EF}_q(A, \boldsymbol{a}, B, \boldsymbol{b})$, then $(A, \boldsymbol{a}) \equiv_{\infty q} (B, \boldsymbol{b})$.*

Proof. A standard exercise by induction on φ (like $1 \Rightarrow 2$ in lemma 3.7); or see, e.g., the proof of [13, theorem 3.5.2]. □

The converse of the lemma also holds, but we will not need it.

4.4 Gaifman graph

Let L be a relational signature and A an L-structure. The *Gaifman graph* $\mathfrak{G}(A)$ of A is the undirected loopfree graph with $\mathrm{dom}(A)$ as its set of nodes, and with edges ab, for all distinct $a, b \in A$ such that for some relation symbol $R \in L$ of arity n, say, and some $a_1, \ldots, a_n \in A$, we have $A \models R(a_1, \ldots, a_n)$ and $a, b \in \{a_1, \ldots, a_n\}$.

For distinct $a, b \in A$, let $d^A(a, b)$ be the length of the shortest path from a to b in $\mathfrak{G}(A)$, and ∞ if there is no such path (we take the length of a path to be the number of edges on it). Put $d^A(a, a) = 0$ for all $a \in A$. We call the function d^A the *Gaifman metric* on A. It is an *extended metric* on A — a metric except that it may take value ∞ sometimes. It satisfies $d^A(a, b) = d^A(b, a) \geq 0$ and $d^A(a, c) \leq d^A(a, b) + d^A(b, c)$ (the *triangle inequality*), for every $a, b, c \in A$, where $\leq$ and $+$ are extended from ω to $\omega \cup \{\infty\}$ in the usual way.

For a family $\boldsymbol{a} = (a_i : i \in I)$ of elements of A, and $l < \omega$, we write

$$\mathcal{N}_l^A(\boldsymbol{a}) = \{a \in A : d^A(a, a_i) < l \text{ for some } i \in I\},$$

the open *Gaifman neighbourhood* of radius l of $\boldsymbol{a}$ in A.

[1] We need this 'left–right' nomenclature in case the game has the form $\mathrm{EF}_q(A, \boldsymbol{a}, A, \boldsymbol{b})$. We could rename the second A as B, but in proposition 4.4 we do not want to do this.

$$\begin{array}{ccccc} M & = & A & + & B \\ & & \uparrow{\scriptstyle \ell} & & \uparrow{\scriptstyle r} \\ & & A & \subseteq & B \end{array}$$

Figure 1: the structure M and embeddings

4.5 Locality

A key step in the modal characterisation theorems of Rosen and Otto was to show that every bisimulation-invariant first-order formula $\varphi(x)$ is *local:* invariant under restricting a model to a 'local neighbourhood' that, under the right circumstances, the modal logic can control. See [18, lemma 4], [16, theorem 3.1 step 1], [17, lemma 3.5], and [12, lemma 58]. The proofs were model-theoretic, via Hanf locality (Rosen) or Ehrenfeucht–Fraïssé games (Otto), and the method continues to be used to the present day — e.g., [20, theorem 27] and [1, 'workspace' lemma 13].

Proposition 4.4 below is a close relative of these results. It combines aspects of Rosen's and Otto's work and the proof is quite simple. The desired locality will follow from the proposition, but not as directly as in the cited references. We will obtain it in §5.3 via interpretations, discussed in §4.6.

To lay the groundwork for the proposition, let L be a relational signature and A, B be L-structures with A a substructure of B. Let $M = A + B$ (see §4.1). In hope of clarity, we will make use of the L-embeddings ℓ, r of A, B (respectively) into M, as shown in figure 1.

We write $\overline{\cdot} : M \to M$ for the partial map (a kind of conjugation) that exchanges the elements $\ell(a)$ and $r(a)$, for each $a \in A$: that is, $\overline{\ell(a)} = r(a)$ and $\overline{r(a)} = \ell(a)$ for $a \in A$, and $\overline{m}$ is undefined for $m \in r(B \setminus A)$. This is plainly a partial isomorphism of M (see §4.1), and an involution: if $\overline{m}$ is defined then so is $\overline{\overline{m}}$, and $\overline{\overline{m}} = m$.

We also write $\pi : M \to M$ for the projection onto $r(B)$, defined by $\pi(\ell(a)) = r(a)$ for $a \in A$, and $\pi(r(b)) = r(b)$ for $b \in B$. It is plainly a homomorphism (again see §4.1).

Finally, for $m, n \in M$ we define

$$d^\pi(m,n) = d^M(\pi(m), \pi(n)),$$

where d^M is the Gaifman metric on M (see §4.4). It follows from the triangle inequality for d^M that for each $x, y, z \in M$,

$$\left.\begin{array}{rcl} d^\pi(x,z) & = & d^M(\pi(x), \pi(z)) \\ & \leq & d^M(\pi(x), \pi(y)) + d^M(\pi(y), \pi(z)) \\ & = & d^\pi(x,y) + d^\pi(y,z) \end{array}\right\} \quad \text{triangle inequality for } d^\pi. \tag{2}$$

Now let I be an index set and $\boldsymbol{a} = (a_i : i \in I)$ a family of elements of A. For each $i \in I$, write $m_i = \ell(a_i)$ and $n_i = r(a_i)$. Then $\boldsymbol{m} = (m_i : i \in I)$ and $\boldsymbol{n} = (n_i : i \in I)$ are families of elements of M, and $\overline{m_i}$ is defined and is n_i for each $i \in I$.

PROPOSITION 4.4 *In this context, if $q < \omega$ and $\mathcal{N}^B_{2^q}(\boldsymbol{a}) \subseteq A$, then $(M, \boldsymbol{m}) \equiv_{\infty q} (M, \boldsymbol{n})$.*

Proof. By lemma 4.3, it suffices to show that $\exists$ has a winning strategy in $\mathrm{EF}_q(M, \boldsymbol{m}, M, \boldsymbol{n})$. For each ordinal $t < q$, we will write m_t and n_t for the 'left' and 'right' elements (respectively) chosen by the players in round t of this game. See §4.3 for the nomenclature. Notwithstanding the use of 'left' and 'right', each of m_t, n_t can of course be in either of the two summands of M.

Player $\exists$ will play in round t as follows, and also ensure that $R1_t$–$R4_t$ below hold at the start of round t and that $R1_{t+1}$–$R4_{t+1}$ hold at the end of round t (so also at the start of round $t+1$, when $t+1 < q$), where

$$d_t = 2^{q-t}.$$

We assume without loss of generality that $I \cap q = \emptyset$.

$R1_t$ Each element of $I \cup t$ is coloured either black or white.

$R2_t$ Suppose that $i \in I \cup t$ is white. Then $n_i = m_i$.

$R3_t$ Suppose that $j \in I \cup t$ is black. Then $\overline{m_j}$ is defined and $n_j = \overline{m_j}$. Moreover, if $m \in M$ and $d^\pi(m, m_j) < d_t$ then $\overline{m}$ is defined as well.

$R4_t$ Suppose that $i \in I \cup t$ is white and $j \in I \cup t$ black. Then $d^\pi(m_i, m_j) > d_t$.

These conditions imply that $\pi(m_i) = \pi(n_i)$ for each $i \in I \cup t$, and a little thought then shows that each m_i can be swapped with n_i in the conditions without changing their meaning. So R1–R4 are in fact left–right symmetric. To get this is why we use d^π rather than d^M.

For $t = 0$, $\exists$ colours each element of I black. Then $R1_0$ holds obviously, and $R2_0$ and $R4_0$ vacuously. $R3_0$ holds because $\overline{m_i}$ is defined and equal to n_i for each $i \in I$, and by the assumption that $\mathcal{N}^B_{d_0}(\boldsymbol{a}) \subseteq A$ (see §4.4 for the $\mathcal{N}$-notation).

Let $t < q$ and assume inductively that $R1_t$–$R4_t$ hold at the start of round t. Suppose in round t that $\forall$ chooses a left element $m_t \in M$, say (the argument when he chooses a right element $n_t \in M$ is similar because of the left–right symmetry of R1–R4). $\exists$ must select a right element n_t in response, and establish $R1_{t+1}$–$R4_{t+1}$. There are two cases.

Case 1: $d^\pi(m_t, m_j) \leq d_{t+1} = d_t/2$ *for some black* $j \in I \cup t$. Then $\exists$ extends the colouring of $I \cup t$ given by $R1_t$ to $I \cup (t+1)$, by colouring t black. Since $d^\pi(m_t, m_j) \leq d_{t+1} < d_t$, by $R3_t$ we see that $\overline{m_t}$ is defined. $\exists$ responds to $\forall$'s move with $n_t = \overline{m_t}$.

We check that $R1_{t+1}$–$R4_{t+1}$ hold. $R1_{t+1}$ and $R2_{t+1}$ are already clear. $R3_{t+1}$ for black $i \in I \cup t$ follows from $R3_t$, since $d_{t+1} \leq d_t$. The new case is t. We know that $\overline{m_t}$ is defined and $n_t = \overline{m_t}$. Let $m \in M$ with $d^\pi(m, m_t) < d_{t+1}$. By the triangle inequality (2) for d^π and the case assumption, $d^\pi(m, m_j) \leq d^\pi(m, m_t) + d^\pi(m_t, m_j) < 2d_{t+1} = d_t$, so by $R3_t$, $\overline{m}$ is defined, as required.

For $R4_{t+1}$, let $i \in I \cup t$ be white. We show that $d^\pi(m_i, m_t) > d_{t+1}$. If not, then as above, $d^\pi(m_i, m_j) \leq d^\pi(m_i, m_t) + d^\pi(m_t, m_j) \leq 2d_{t+1} = d_t$, contradicting $R4_t$. All other instances of $R4_{t+1}$ follow from $R4_t$.

Case 2: *otherwise.* This time, $\exists$ colours t white and sets $n_t = m_t$. So $R1_{t+1}$ and $R2_{t+1}$ obviously hold, and $R3_{t+1}$ follows from $R3_t$ as there are no new cases. The only new case to check in $R4_{t+1}$ is that $d^\pi(m_t, m_j) > d_{t+1}$ whenever $j \in I \cup t$ is black — and this is exactly the case assumption.

That completes the definition of $\exists$'s strategy. We check that it is winning. At the end of the game, $R1_q$–$R4_q$ hold, and $d_q = 2^0 = 1$. Let $\alpha(x_1, \ldots, x_k)$ be an atomic L-formula and $I' = \{i_1, \ldots, i_k\} \subseteq I \cup q$. We show that $M \models \alpha(m_{i_1}, \ldots, m_{i_k}) \leftrightarrow \alpha(n_{i_1}, \ldots, n_{i_k})$.

We can assume without loss of generality that $x_1, \ldots, x_k$ all occur in α. Suppose that $M \models \alpha(m_{i_1}, \ldots, m_{i_k})$. As π is a homomorphism, $M \models \alpha(\pi(m_{i_1}), \ldots, \pi(m_{i_k}))$. So by definition of d^π and Gaifman distance,

$$d^\pi(m_i, m_j) = d^M(\pi(m_i), \pi(m_j)) \leq 1 = d_q \quad \text{for each } i, j \in I'.$$

It follows from R4_q that $i_1, \ldots, i_k$ all have the same colour. If this is white, then by R2_q, $n_i = m_i$ for each $i \in I'$, so obviously $M \models \alpha(n_{i_1}, \ldots, n_{i_k})$. If it is black, then $n_i = \overline{m_i}$ for each $i \in I'$, by R3_q. Since $\overline{\cdot}$ is a partial isomorphism of M, we again obtain $M \models \alpha(n_{i_1}, \ldots, n_{i_k})$.

The converse is similar, again using left–right symmetry of R1_q–R4_q. So $\exists$ won. $\square$

4.6 Interpretations

We will use *interpretations* in §5.3 to extend the reach of proposition 4.4. We broadly follow Hodges [13, §5.3] for the definitions. We will need infinitary interpretations with parameters, but only one-dimensional quantifier-free unrelativised ones. (Results for more general interpretations can also be obtained.)

Let K, L be signatures (which here have no function symbols, recall), let I be an index set, and let v_i $(i \in I)$ be new pairwise distinct variables taken not to occur in $L_{\infty\omega}$-formulas. An *interpretation of L in K with parameters I* (more fully, *with parameters v_i $(i \in I)$*) is a map $\mathcal{I}$ that provides a quantifier-free $K_{\infty\omega}$-formula $\mathcal{I}(\alpha)(x_1, \ldots, x_n, v_i : i \in I)$ for each atomic L-formula $\alpha(x_1, \ldots, x_n)$. We extend $\mathcal{I}$ to all $L_{\infty\omega}$-formulas by induction in the obvious way: $\mathcal{I}(\neg\varphi) = \neg\mathcal{I}(\varphi)$, $\mathcal{I}(\bigwedge S) = \bigwedge\{\mathcal{I}(\varphi) : \varphi \in S\}$, similarly for $\bigvee S$, and $\mathcal{I}(\exists x\varphi) = \exists x\mathcal{I}(\varphi)$. Plainly, $\mathcal{I}(\varphi)$ always has the same quantifier depth as φ.

Now let A be a K-structure and $\boldsymbol{a} = (a_i : i \in I) \in {}^I A$. Let M be an L-structure with the same domain as A. We say that $\mathcal{I}$ *interprets M in $(A, \boldsymbol{a})$* if for each atomic L-formula $\alpha(x_1, \ldots, x_n)$ and $m_1, \ldots, m_n \in M$, we have

$$M \models \alpha(m_1, \ldots, m_n) \quad \text{iff} \quad A \models \mathcal{I}(\alpha)(m_1, \ldots, m_n, a_i : i \in I).$$

There is clearly at most one M that $\mathcal{I}$ interprets in $(A, \boldsymbol{a})$, so when there is one, we can write it as $\mathcal{I}(A, \boldsymbol{a})$.

LEMMA 4.5 *Let $K, L, I, \mathcal{I}, A, \boldsymbol{a}$ be as above, and suppose that $\mathcal{I}(A, \boldsymbol{a})$ exists. Also suppose that B is a K-structure, $\boldsymbol{b} = (b_i : i \in I) \in {}^I B$, and $\mathcal{I}(B, \boldsymbol{b})$ exists.*

1. *For each $L_{\infty\omega}$-formula $\varphi(x)$ and $a \in A$,*

$$\mathcal{I}(A, \boldsymbol{a}) \models \varphi(a) \quad \textit{iff} \quad A \models \mathcal{I}(\varphi)(a, a_i : i \in I).$$

2. *If $f : A \to B$ is a K-embedding and $f(a_i) = b_i$ for each $i \in I$, then $f : \mathcal{I}(A, \boldsymbol{a}) \to \mathcal{I}(B, \boldsymbol{b})$ is an L-embedding.*

3. *Let $j \in I$ and $q < \omega$. If $(A, \boldsymbol{a}) \equiv_{\infty q} (B, \boldsymbol{b})$, then $(\mathcal{I}(A, \boldsymbol{a}), a_j) \equiv_{\infty q} (\mathcal{I}(B, \boldsymbol{b}), b_j)$.*

Proof. (1) is straightforward by induction on φ, and follows from the 'reduction theorem' of [13, theorem 5.3.2]. For (2), for each atomic L-formula $\alpha(x_1, \ldots, x_n)$ and $\mathsf{a}_1, \ldots, \mathsf{a}_n \in A$,

$$\begin{array}{lll} & \mathcal{I}(A, \boldsymbol{a}) \models \alpha(\mathsf{a}_1, \ldots, \mathsf{a}_n) & \\ \text{iff} & A \models \mathcal{I}(\alpha)(\mathsf{a}_1, \ldots, \mathsf{a}_n, a_i : i \in I) & \text{by definition of } \mathcal{I}(A, \boldsymbol{a}), \\ \text{iff} & B \models \mathcal{I}(\alpha)(f(\mathsf{a}_1), \ldots, f(\mathsf{a}_n), b_i : i \in I) & \text{since } \mathcal{I}(\alpha) \text{ is quantifier-free} \\ & & \text{and } f \text{ a } K\text{-embedding}, \\ \text{iff} & \mathcal{I}(B, \boldsymbol{b}) \models \alpha(f(\mathsf{a}_1), \ldots, f(\mathsf{a}_n)) & \text{by definition of } \mathcal{I}(B, \boldsymbol{b}). \end{array}$$

For (3), let $\varphi(x)$ be any $L_{\infty\omega}$-formula of quantifier depth $\leq q$. Then the $K_{\infty\omega}$-formula $\mathcal{I}(\varphi)(x, v_i : i \in I)$ also has quantifier depth $\leq q$. So

$$\begin{array}{llll} \mathcal{I}(A, \boldsymbol{a}) \models \varphi(a_j) & \text{iff} & A \models \mathcal{I}(\varphi)(a_j, a_i : i \in I) & \text{by part 1,} \\ & \text{iff} & B \models \mathcal{I}(\varphi)(b_j, b_i : i \in I) & \text{since } (A, \boldsymbol{a}) \equiv_{\infty q} (B, \boldsymbol{b}), \\ & \text{iff} & \mathcal{I}(B, \boldsymbol{b}) \models \varphi(b_j) & \text{by part 1 for } B. \end{array}$$

□

5 Unravellings

The modal notion of 'unravelling' is well known: see, e.g., [9, §3.2]. Here, we introduce and study a modified unravelling that works in the presence of nominals. For the entire section, fix a hybrid signature $\sigma = \text{PROP} \uplus \text{NOM}$; all Kripke models will be for this signature. We write L for $L(\sigma)$ here. I apologise for the blizzard of structures coming up.

5.1 Unravelling a Kripke model

Until section 5.4, fix an arbitrary pointed Kripke model (A, a) and $q < \omega$, and let $l = 2^q$.

DEFINITION 5.1 We define an L-structure (or Kripke model for σ) A^l from A. It is our 'depth-l unravelling' of A, and is finite if A is finite. First, some preliminaries.

- Let $N = \{c^A : c \in \text{NOM}\} \subseteq A$.
- For $k < \omega$, a *path of length* k (in A) is a sequence $(a_0, a_1, \ldots, a_k) \in {}^{k+1}A$, where $a_0 \in A$, $a_1, \ldots, a_k \in A \setminus N$, and $A \models a_i R a_{i+1}$ for each $i < k$. Only the first element of a path can lie in N.
- For a path $t = (a_0, \ldots, a_k)$ and $a \in A \setminus N$ with $A \models a_k R a$, we write $t^\frown a$ for the path $(a_0, \ldots, a_k, a)$.
- For $a \in A$, we usually write $\hat{a}$ for the path (a) of length 0. This is more compact. For $S \subseteq A$ we put $\hat{S} = \{\hat{s} : s \in S\}$.
- For $k < \omega$ let $\text{Path}_{\leq k}(A)$ (resp., $\text{Path}_{<k}(A)$) be the set of paths of length $\leq k$ (resp., of length $< k$) in A.

We now define A^l. Its domain is $\text{Path}_{\leq l}(A)$. We read the symbols in σ according to the last elements of paths: so we define $A^l \models P((a_0, \ldots, a_k))$ iff $A, a_k \models p$ for each $p \in \text{PROP}$, and for each nominal $c \in \text{NOM}$ we put $c^{A^l} = (c^A)$, a path in $\text{Path}_{\leq l}(A)$ of length 0.

For the accessibility relation, let $t = (a_0, \ldots, a_k) \in \text{Path}_{\leq l}(A)$. Then we define:

- $A^l \models tRu$ for each $u \in \text{Path}_{\leq l}(A)$ of the form $t^\frown a$, where $a \in A \setminus N$.
- $A^l \models tR\hat{n}$ for each $n \in N$ with $A \models a_k R n$.
- If $k = l$, then $A^l \models tR\hat{a}$ for each $a \in A$ with $A \models a_l R a$.

These are the only instances of R. Finally let $\lambda : A^l \to A$ be the 'projection' function that maps each path $(a_0, \ldots, a_k)$ to its last element a_k.

The upshot of the definition of R is that 'R-arrows' in A^l can come into $\hat{N}$ from anywhere, and into $\hat{A} \setminus \hat{N}$ from paths of length l only. But each path of length $k \geq 1$ has a unique R-predecessor, namely, its initial segment (prefix) of length $k - 1$.

LEMMA 5.2 $(A, \mathsf{a}) \sim^{\Diamond @} (A^l, \hat{\mathsf{a}})$.

Proof. One can easily verify that $\lambda : A^l \to A$ is a $\Diamond @$-bisimulation and $\lambda(\hat{\mathsf{a}}) = \mathsf{a}$. We check only the 'Back' property. Suppose that $t = (a_0, \ldots, a_k) \in A^l$, so $k \leq l$ and $\lambda(t) = a_k$, and let $a \in A$ satisfy $A \models a_k R a$. We seek $u \in A^l$ with $A^l \models tRu$ and $\lambda(u) = a$. If $a \in N$ or $k = l$, take $u = \hat{a}$. If $a \in A \setminus N$ and $k < l$, take $u = t^\frown a$. $\square$

DEFINITION 5.3 We let $A^l_{<l}$ be the substructure of A^l with domain $\mathrm{Path}_{<l}(A)$. To reduce clutter, for λ as above, we write its restriction $\lambda \restriction A^l_{<l}$ as $\lambda : A^l_{<l} \to A$ as well.

So $A^l_{<l}$ is obtained by simply deleting from A^l all paths of length l. The result is nonempty (since $l \geq 1$) and contains all elements of A^l named by constants (nominals), so is an L-structure. It can be 'well controlled' by hybrid logic, as lemma 5.11 will show. Perhaps we had better point out that $A^l_{<l}$ is not A^{l-1}, since the two give different meanings to R on paths of length $l-1$.

The restriction $\lambda : A^l_{<l} \to A$ is an L-homomorphism (see §4.1) and preserves atoms and nominals both ways, but it is not in general a bisimulation, because paths of length l have been deleted, so the Back property may fail.

5.2 Invoking locality

DEFINITION 5.4 We introduce a new relational signature K, obtained from L by deleting each constant c ($c \in \textsc{Nom}$) and adding a new unary relation symbol P_a for each $a \in A$. So K comprises R, a unary relation symbol P for each $p \in \textsc{Prop}$, and the new symbols P_a. It depends on A and may be infinite.

We define an K-structure $A^{l:K}$ with the same domain and interpretations of atoms in $\textsc{Prop}$ as A^l, and with

- $A^{l:K} \models tRu$ iff $A^l \models tRu$ and $u \notin \hat{A}$. (So u is a path extending t by one.)
- $A^{l:K} \models P_a(t)$ iff $A^l \models tR\hat{a}$, for each $a \in A$. (So $a \in N$ or t is a path of length l.)

We let $A^{l:K}_{<l}$ be the substructure of $A^{l:K}$ with domain $\mathrm{Path}_{<l}(A)$.

In $A^{l:K}$, we have removed all R-arrows into $\hat{A}$, but the P_a ensure that they are not forgotten. We also removed the nominals: their values will be remembered 'by hand'.

We are now going to use A as an index set. Write $\boldsymbol{a} = (\hat{a} : a \in A)$. This is a family of elements of each of A^l, $A^l_{<l}$, $A^{l:K}$, and $A^{l:K}_{<l}$. It is clear from the definitions (see §4.4 for $\mathcal{N}$) that

$$\mathcal{N}_l^{A^{l:K}}(\boldsymbol{a}) = A^{l:K}_{<l}. \tag{3}$$

Let $M = A^{l:K} + A^{l:K}_{<l}$, and let $\ell : A^{l:K} \to M$ and $r : A^{l:K}_{<l} \to M$ be the respective K-embeddings, as in §4.5 but with 'A' and 'B' swapped. Note that if A is finite then so is M. Write $\boldsymbol{m} = (\ell(\hat{a}) : a \in A)$ and $\boldsymbol{n} = (r(\hat{a}) : a \in A)$. By proposition 4.4,

$$(M, \boldsymbol{m}) \equiv_{\infty q} (M, \boldsymbol{n}). \tag{4}$$

The proposition applies since K is relational, and $\mathcal{N}_l^{A^{l:K}}(\boldsymbol{a}) \subseteq A^{l:K}_{<l}$ by (3).

5.3 Invoking interpretations

To get back to L, we use an interpretation.

DEFINITION 5.5 We define an interpretation $\mathcal{I}$ of L in K with parameters A. It takes each atomic L-formula $\alpha(x_1, \ldots, x_n)$ to a quantifier-free $K_{\infty\omega}$-formula $\mathcal{I}(\alpha)(x_1, \ldots, x_n, v_a : a \in A)$. Here recall §4.6 — the index set I there is A here, and the pairwise distinct variables v_a $(a \in A)$ are taken not to occur in $L_{\infty\omega}$-formulas.

1. Let $\mathcal{I}(x = y)$ be $x = y$, and let $\mathcal{I}(P(x))$ be $P(x)$ for $p \in \textsc{Prop}$.
2. Let $\mathcal{I}(xRy)$ be $xRy \vee \bigvee_{a \in A}(P_a(x) \wedge y = v_a)$.
3. For an atomic L-formula $\alpha(x_1, \ldots, x_n, y_1, \ldots, y_m)$ not involving any constants, and constants (nominals) $c_1, \ldots, c_m \in L$, define $\mathcal{I}(\alpha(x_1, \ldots, x_n, c_1, \ldots, c_m))$ to be the result of substituting $v_{c_i^A}$ for y_i in the formula $\mathcal{I}(\alpha(x_1, \ldots, x_n, y_1, \ldots, y_m))$ defined above, for each $i = 1, \ldots, m$.

As an example, if $c \in \textsc{Nom}$ and $c^A = b \in A$, say, then $\mathcal{I}(x = c)$ is $x = v_b$; if $p \in \textsc{Prop}$ then $\mathcal{I}(P(c))$ is $P(v_b)$; and $\mathcal{I}(xRc)$ is $xRv_b \vee \bigvee_{a \in A}(P_a(x) \wedge v_b = v_a)$.

It should be clear that the L-structures $\mathcal{I}(A^{l:K}, \boldsymbol{a})$ and $\mathcal{I}(A^{l:K}_{<l}, \boldsymbol{a})$ exist and are A^l and $A^l_{<l}$, respectively. The L-structures $\mathcal{I}(M, \boldsymbol{m})$ and $\mathcal{I}(M, \boldsymbol{n})$ also exist. In contrast to M, they are not disjoint unions, because L is not relational (if $\textsc{Nom} \neq \emptyset$), and there may be '$R$-arrows' running between $\ell(A^{l:K})$ and $r(A^{l:K}_{<l})$. Nonetheless, we have the following:

LEMMA 5.6 $(A^l, \hat{\mathsf{a}}) \sim^{\Diamond @} (\mathcal{I}(M, \boldsymbol{m}), \ell(\hat{\mathsf{a}}))$ *and* $(A^l_{<l}, \hat{\mathsf{a}}) \sim^{\Diamond @} (\mathcal{I}(M, \boldsymbol{n}), r(\hat{\mathsf{a}}))$.

Proof. We show that $\ell : A^l \to \mathcal{I}(M, \boldsymbol{m})$ is a $\Diamond @$-bisimulation. Clearly, $\ell : A^{l:K} \to M$ is a K-embedding. By lemma 4.5(2), ℓ is also an L-embedding from $\mathcal{I}(A^{l:K}, \boldsymbol{a}) = A^l$ into $\mathcal{I}(M, \boldsymbol{m})$. It therefore preserves σ both ways, satisfies Forth, and is defined on all points named by nominals.

For Back, let $t \in A^l$ and $u \in M$ and suppose that $\mathcal{I}(M, \boldsymbol{m}) \models \ell(t)Ru$. We seek $t' \in A^l$ with $A^l \models tRt'$ and $\ell(t') = u$. We have $M \models \mathcal{I}(xRy)(\ell(t), u, m_a : a \in A)$ by definition of $\mathcal{I}(M, \boldsymbol{m})$. So by definition of $\mathcal{I}(xRy)$,

$$M \models \ell(t)Ru \vee \bigvee_{a \in A} (P_a(\ell(t)) \wedge u = m_a).$$

But each disjunct here implies $u \in \ell(A^l)$: the first by definition of M as a disjoint union, and the others since $m_a = \ell(\hat{a})$ for each $a \in A$. So let $t' = \ell^{-1}(u) \in A^l$. Then $\mathcal{I}(M, \boldsymbol{m}) \models \ell(t)R\ell(t')$. As ℓ is an L-embedding, $A^l \models tRt'$, as required. Essentially we proved that $\ell(A^l)$ is a generated submodel of $\mathcal{I}(M, \boldsymbol{m})$.

Similarly we can show that $r : A^l_{<l} \to \mathcal{I}(M, \boldsymbol{n})$ is a $\Diamond @$-bisimulation. $\square$

LEMMA 5.7 $(\mathcal{I}(M, \boldsymbol{m}), \ell(\hat{\mathsf{a}})) \equiv_{\infty q} (\mathcal{I}(M, \boldsymbol{n}), r(\hat{\mathsf{a}}))$.

Proof. By (4) and lemma 4.5(3), taking 'j' there to be a here, so the jth entry of the family $\boldsymbol{m} = (\ell(\hat{a}) : a \in A)$ is $\ell(\hat{\mathsf{a}})$, and the jth entry of $\boldsymbol{n} = (r(\hat{a}) : a \in A)$ is $r(\hat{\mathsf{a}})$. $\square$

5.4 Summary so far

We have proved the following:

PROPOSITION 5.8 *Let (A, a) be a pointed Kripke model, $q < \omega$, and $l = 2^q$. Then:*

$$(A, \mathsf{a}) \sim^{\Diamond @} (A^l, \hat{\mathsf{a}}) \sim^{\Diamond @} (\mathcal{I}(M, \boldsymbol{m}), \ell(\hat{\mathsf{a}})) \equiv_{\infty q} (\mathcal{I}(M, \boldsymbol{n}), r(\hat{\mathsf{a}})) \sim^{\Diamond @} (A^l_{<l}, \hat{\mathsf{a}}).$$

If A is finite then so are all the structures here.

Proof. By lemmas 5.2, 5.6, 5.7, and 5.6 again, respectively. The finiteness is obvious. □

5.5 Two connecting lemmas

So far, we have looked at unravellings of a single pointed Kripke model (A, a). Our final two lemmas draw out connections between the unravellings of two pointed Kripke models.

LEMMA 5.9 *Let $(A, \mathsf{a}), (B, \mathsf{b})$ be pointed Kripke models, $\star \in \{\Diamond, \Diamond @\}$, $\alpha \leq \omega$, and $1 \leq l < \omega$. Suppose that $(A, \mathsf{a}) \sim^\star_\alpha (B, \mathsf{b})$. Then $(A^l_{<l}, \hat{\mathsf{a}}) \sim^\star_\alpha (B^l_{<l}, \hat{\mathsf{b}})$.*

Proof. Recall (from just after definition 5.3) that $\lambda : A^l_{<l} \to A$ is an L-homomorphism (so preserves R forwards), and preserves atoms and nominals both ways. There is a similar map taking each path in $B^l_{<l}$ to its last element, which we also write as $\lambda : B^l_{<l} \to B$. They may not be bisimulations.

Assume that $\exists$ has a winning strategy in $\mathrm{B}^\star_\alpha(A, \mathsf{a}, B, \mathsf{b})$. She can use it in $\mathrm{B}^\star_\alpha(A^l_{<l}, \hat{\mathsf{a}}, B^l_{<l}, \hat{\mathsf{b}})$ as follows. In a play of this latter game, let the successive positions be $(t_0, u_0), \ldots, (t_s, u_s), \ldots$, say. $\exists$ will ensure that

(i) $t_s \in A^l_{<l}$ and $u_s \in B^l_{<l}$ are paths of equal length, for each s,

(ii) $(\lambda(t_0), \lambda(u_0)), \ldots, (\lambda(t_s), \lambda(u_s)), \ldots$ are successive positions in a play of $\mathrm{B}^\star_\alpha(A, \mathsf{a}, B, \mathsf{b})$ in which she is using her winning strategy.

If she can do this, then since her strategy is winning, $(A, \lambda(t_s))$ and $(B, \lambda(u_s))$ agree on σ, for each s. So by the remarks on λ above, $(A^l_{<l}, t_s)$ and $(B^l_{<l}, u_s)$ also agree on σ for each s, and $\exists$ will win.

We now explain how she can do it.

Suppose that $\forall$ chooses (t_0, u_0) as the initial position in $\mathrm{B}^\star_\alpha(A^l_{<l}, \hat{\mathsf{a}}, B^l_{<l}, \hat{\mathsf{b}})$. If $(t_0, u_0) = (\hat{\mathsf{a}}, \hat{\mathsf{b}})$, then clearly, t_0 and u_0 have equal length 0, $\lambda(t_0) = \mathsf{a}$, and $\lambda(u_0) = \mathsf{b}$. If $\star = \Diamond @$ and $(t_0, u_0) = (c^{A^l_{<l}}, c^{B^l_{<l}}) = ((c^A), (c^B))$ for some $c \in \textsc{Nom}$, then again t_0, u_0 have length 0, and $\lambda(t_0) = c^A$ and $\lambda(u_0) = c^B$. So in all cases it is legal for $\forall$ to choose $(\lambda(t_0), \lambda(u_0))$ for the initial position in $\mathrm{B}^\star_\alpha(A, \mathsf{a}, B, \mathsf{b})$. $\exists$ lets him do so. Then conditions (i) and (ii) above are met.

In round $s < \alpha$ of $\mathrm{B}^\star_\alpha(A^l_{<l}, \hat{\mathsf{a}}, B^l_{<l}, \hat{\mathsf{b}})$, assume that $\exists$ has kept the two conditions so far, and suppose that $\forall$ plays $t_{s+1} \in A^l_{<l}$ (the argument is similar if he plays in $B^l_{<l}$; and if he cannot move then $\exists$ wins at this point and we are done). By the game rules, $A^l_{<l} \models t_s R t_{s+1}$.

Write $a = \lambda(t_{s+1})$. As λ is a homomorphism, $A \models \lambda(t_s) R a$ too, and it is legal for $\forall$ to play a in round s of $\mathrm{B}^\star_\alpha(A, \mathsf{a}, B, \mathsf{b})$. $\exists$ lets him do it, and responds using her winning strategy with $b \in B$, say. As her strategy is winning, (†) $B \models \lambda(u_s) R b$ and (‡) (A, a) and (B, b) agree on σ.

We now define $\exists$'s response $u_{s+1} \in B^l_{<l}$ to $\forall$ in the main game $\mathrm{B}^\star_\alpha(A^l_{<l}, \hat{\mathsf{a}}, B^l_{<l}, \hat{\mathsf{b}})$. There are two cases. Suppose first that t_{s+1} is named in $A^l_{<l}$ by some nominal c. Then $t_{s+1} = (c^A)$, a path of length 0. Plainly, $a = c^A$. By (‡), $b = c^B$. Then $\exists$ lets $u_{s+1} = \hat{b} \in B^l_{<l}$, also a path of length 0.

Now suppose otherwise, and let the common length of t_s, u_s be n, say. Then $t_{s+1} = t_s{}^\frown a$ by definition of $A^l_{<l}$. This path has length $n+1$, and $n+1 < l$ because $t_{s+1} \in A^l_{<l}$. Because λ preserves nominals both ways, a is not named by a nominal. By (‡), neither is b, so $u_s{}^\frown b$ is a path in B. It has length $n+1$ as well, so is in $B^l_{<l}$ since $n+1 < l$. $\exists$ lets $u_{s+1} = u_s{}^\frown b$.

In each case, $\exists$ has found $u_{s+1} \in B^l_{<l}$ of the same path length (0 or $n+1$) as t_{s+1}, with $\lambda(u_{s+1}) = b$, and with $B^l_{<l} \models u_s R u_{s+1}$ (by (†) and the definition of $B^l_{<l}$). So $\exists$ can legally respond to $\forall$ with u_{s+1} in the main game $\mathrm{B}^\star_\alpha(A^l_{<l}, \hat{\mathsf{a}}, B^l_{<l}, \hat{\mathsf{b}})$, and in doing so, keep conditions (i) and (ii) above. Hence, we have described a winning strategy for her in $\mathrm{B}^\star_\alpha(A^l_{<l}, \hat{\mathsf{a}}, B^l_{<l}, \hat{\mathsf{b}})$. $\square$

DEFINITION 5.10 For $\star \in \{\Diamond, \Diamond @\}$, $1 \leq l < \omega$, and $n < \omega$, define

$$\mathsf{m}(\star, l, n) = \begin{cases} l-1, & \text{if } n = 0, \\ l, & \text{if } n > 0 \text{ and } \star = \Diamond @, \\ l \cdot (n+1), & \text{if } n > 0 \text{ and } \star = \Diamond. \end{cases}$$

LEMMA 5.11 *Suppose that* NOM *is finite, let* $\star \in \{\Diamond, \Diamond @\}$ *and* $1 \leq l < \omega$, *and write* $m = \mathsf{m}(\star, l, |\text{NOM}|)$. *Let* $(A, \mathsf{a}), (B, \mathsf{b})$ *be pointed Kripke models satisfying* $(A^l_{<l}, \hat{\mathsf{a}}) \sim^\star_m (B^l_{<l}, \hat{\mathsf{b}})$. *Then* $(A^l_{<l}, \hat{\mathsf{a}}) \sim^\star (B^l_{<l}, \hat{\mathsf{b}})$.

Proof. By an *R-chain of length $n < \omega$ in $A^l_{<l}$*, we will mean a sequence $t_0, \ldots, t_n$ of elements of $A^l_{<l}$ with $A^l_{<l} \models t_i R t_{i+1}$ for each $i < n$. (The word 'path' could be confusing here.) By construction of $A^l_{<l}$, every R-chain of length $\geq l$ contains a point named by a nominal. An *R-chain in $B^l_{<l}$* is defined similarly.

By lemma 3.5(4), it is enough to show that $(A^l_{<l}, \hat{\mathsf{a}}) \sim^\star_\omega (B^l_{<l}, \hat{\mathsf{b}})$. By assumption, $\exists$ has a winning strategy in $\mathrm{B}^\star_m(A^l_{<l}, \hat{\mathsf{a}}, B^l_{<l}, \hat{\mathsf{b}})$. The idea is for her to use it in $\mathrm{B}^\star_\omega(A^l_{<l}, \hat{\mathsf{a}}, B^l_{<l}, \hat{\mathsf{b}})$. Of course, it may run out, so she may have to reset it frequently.

There are three cases, according to how $m = \mathsf{m}(\star, l, |\text{NOM}|)$ is defined.

First take the case NOM $= \emptyset$, so $m = l - 1$. Then each play of $\mathrm{B}^\star_\omega(A^l_{<l}, \hat{\mathsf{a}}, B^l_{<l}, \hat{\mathsf{b}})$ comes to an end after at most m rounds, since no R-chain in $A^l_{<l}$ or $B^l_{<l}$ is longer than this when NOM $= \emptyset$. Hence, $\exists$ can just use her winning strategy in $\mathrm{B}^\star_m(A^l_{<l}, \hat{\mathsf{a}}, B^l_{<l}, \hat{\mathsf{b}})$. (Since NOM $= \emptyset$, the game does not depend on $\star$.)

Now take the case when NOM $\neq \emptyset$ and $\star = \Diamond @$, so $m = l$. By assumption and lemma 3.5(1), $\exists$ has winning strategies in $\mathrm{B}^\Diamond_l(A^l_{<l}, \hat{\mathsf{a}}, B^l_{<l}, \hat{\mathsf{b}})$ and $\mathrm{B}^\Diamond_l(A^l_{<l}, (c^A), B^l_{<l}, (c^B))$ for each $c \in$ NOM. She can use them repeatedly in a play of $\mathrm{B}^{\Diamond @}_\omega(A^l_{<l}, \hat{\mathsf{a}}, B^l_{<l}, \hat{\mathsf{b}})$, as follows. Initially, she chooses whichever strategy matches $\forall$'s initial move. As play of $\mathrm{B}^{\Diamond @}_\omega$ continues, consideration of the form of $A^l_{<l}$ and $B^l_{<l}$ shows that it will either end with a win for $\exists$ because $\forall$ can't move, or will arrive after $\leq l$ rounds at a position of the form $((c^A), (c^B))$ for some nominal c. $\exists$ can then pick up a winning strategy in $\mathrm{B}^\Diamond_l(A^l_{<l}, (c^A), B^l_{<l}, (c^B))$ lasting another l rounds. Continuing in this way, she will win. (Actually this argument works for infinite NOM so long as we still define $\mathsf{m}(\Diamond @, l, |\text{NOM}|) = l$.)

Finally take the case when NOM $\neq \emptyset$ and $\star = \Diamond$, so $m = l \cdot (|\text{NOM}| + 1)$. Let us say that elements $t \in A^l_{<l}$ and $u \in B^l_{<l}$ *match* if there is a finite R-chain in $A^l_{<l}$ running from $\hat{\mathsf{a}}$ to t, and for some nominal c we have $t = (c^A)$ and $u = (c^B)$.

Claim. If t, u match, then $\exists$ has a winning strategy in $\mathrm{B}^\Diamond_l(A^l_{<l}, t, B^l_{<l}, u)$.

Proof of claim. Take a shortest possible R-chain $t_0, \ldots, t_n$ in $A^l_{<l}$ from $\hat{\mathsf{a}}$ to t, so $t_0 = \hat{\mathsf{a}}$ and $t_n = t$. By minimality, $t_0, \ldots, t_n$ are pairwise distinct, and at most $|\text{NOM}|$ of them are named by a nominal.

But whenever $0 \le s < s+l \le n$, some point in $\{t_s, \dots, t_{s+l}\}$ is named by a nominal. It follows that $n \le l \cdot |\mathrm{NOM}|$.

By following the chain, playing in $A^l_{<l}$ in each round, $\forall$ can get from $\hat{\mathsf{a}}$ to t in $\le l \cdot |\mathrm{NOM}|$ rounds of a play of $\mathrm{B}^{\diamond}_m(A^l_{<l}, \hat{\mathsf{a}}, B^l_{<l}, \hat{\mathsf{b}})$. If $\exists$ uses her winning strategy in such a play, then as it preserves nominals, she will for sure arrive at u, and her winning strategy will still have $\ge m - l \cdot |\mathrm{NOM}| = l$ rounds left to run. So 'continue with the strategy in progress' is a winning strategy for her in $\mathrm{B}^{\diamond}_l(A^l_{<l}, t, B^l_{<l}, u)$. This proves the claim.

We finish as in the preceding case. Let $\forall$, $\exists$ play $\mathrm{B}^{\diamond}_\omega(A^l_{<l}, \hat{\mathsf{a}}, B^l_{<l}, \hat{\mathsf{b}})$, with $\exists$ initially using her winning strategy in $\mathrm{B}^{\diamond}_m(A^l_{<l}, \hat{\mathsf{a}}, B^l_{<l}, \hat{\mathsf{b}})$. Play will either end with a win for $\exists$ because $\forall$ can't move, or arrive after $\le l$ rounds at a position (t, u) of the form $((c^A), (c^B))$ for some nominal c. Plainly, t and u will then match. So by the claim, $\exists$ can then pick up a winning strategy lasting another l rounds. Continuing in this way forever, she will win. $\square$

6 Main theorem

THEOREM 6.1 *Let σ be a hybrid signature, let $\mathcal{K}$ be either the class of all pointed Kripke models for σ, or the class of all finite pointed Kripke models for σ, and let $\star \in \{\diamond, \diamond@\}$. Let $\varphi(x)$ be a first-order $L(\sigma)$-formula of quantifier depth q and involving n distinct constants (nominals) from $L(\sigma)$. Assume that φ is $\star$-bisimulation invariant over $\mathcal{K}$. Then φ is equivalent over $\mathcal{K}$ to some $\mathcal{L}^\star(\sigma)$-formula ψ of modal depth at most $m = \mathsf{m}(\star, 2^q, n)$.*

Proof. Write $\sigma = \mathrm{PROP} \uplus \mathrm{NOM}$. Let $\tau \subseteq \sigma$ be the $\subseteq$-least hybrid signature such that φ is an $L(\tau)$-formula. It simply collects all symbols of σ that actually occur in φ (after changing $p \in \mathrm{PROP}$ to $P \in L(\sigma)$). First we prove the theorem assuming that $\sigma = \tau$. (This may be enough for some.) So σ is finite and $|\mathrm{NOM}| = n$.

Claim 1. If $(A, \mathsf{a}), (B, \mathsf{b}) \in \mathcal{K}$ and $(A, \mathsf{a}) \sim^\star_m (B, \mathsf{b})$ then $A \models \varphi(\mathsf{a})$ iff $B \models \varphi(\mathsf{b})$.
Proof of claim. Write $l = 2^q$. As φ has quantifier depth q and is $\star$-bisimulation invariant over $\mathcal{K}$, by proposition 5.8 we have

$$A \models \varphi(\mathsf{a}) \iff A^l_{<l} \models \varphi(\hat{\mathsf{a}}) \tag{5}$$

— note here that since $(A, \mathsf{a}) \in \mathcal{K}$, all structures mentioned in the proposition are also in $\mathcal{K}$, and so each of its four steps preserves φ. Similarly, applying proposition 5.8 to (B, b) gives

$$B \models \varphi(\mathsf{b}) \iff B^l_{<l} \models \varphi(\hat{\mathsf{b}}). \tag{6}$$

By assumption, $(A, \mathsf{a}) \sim^\star_m (B, \mathsf{b})$, so by lemma 5.9, $(A^l_{<l}, \hat{\mathsf{a}}) \sim^\star_m (B^l_{<l}, \hat{\mathsf{b}})$, and thus $(A^l_{<l}, \hat{\mathsf{a}}) \sim^\star (B^l_{<l}, \hat{\mathsf{b}})$ by lemma 5.11 and since $m = \mathsf{m}(\star, l, |\mathrm{NOM}|)$. As already observed, these structures are in $\mathcal{K}$. As φ is $\star$-bisimulation invariant over $\mathcal{K}$, we obtain

$$A^l_{<l} \models \varphi(\hat{\mathsf{a}}) \iff B^l_{<l} \models \varphi(\hat{\mathsf{b}}). \tag{7}$$

Putting (5)–(7) together proves the claim.

The rest of the proof is quite standard. Since σ is finite, we can form the finite set $\mathcal{F}^\star_m$ of $\mathcal{L}^\star(\sigma)$-formulas of modal depth $\le m$ from definition 3.6. Define

$$\begin{aligned} \mathrm{tp}(A, \mathsf{a}) &= \{\psi \in \mathcal{F}^\star_m : A, \mathsf{a} \models \psi\} \cup \{\neg\psi : \psi \in \mathcal{F}^\star_m,\ A, \mathsf{a} \models \neg\psi\}, \quad \text{for } (A, \mathsf{a}) \in \mathcal{K}, \\ \psi &= \bigvee \big\{ \bigwedge \mathrm{tp}(B, \mathsf{b}) : (B, \mathsf{b}) \in \mathcal{K},\ B \models \varphi(\mathsf{b}) \big\}. \end{aligned}$$

Although $\mathcal{K}$ is a proper class, the class following the disjunction is finite since $\mathcal{F}^\star_m$ is finite, so ψ is an $\mathcal{L}^\star(\sigma)$-formula of modal depth $\leq m$.

Claim 2. φ is equivalent over $\mathcal{K}$ to ψ.
Proof of claim. Let $(A, \mathsf{a}) \in \mathcal{K}$. If $A \models \varphi(\mathsf{a})$ then $\bigwedge \mathrm{tp}(A, \mathsf{a})$ is a disjunct of ψ, and plainly $A, \mathsf{a} \models \bigwedge \mathrm{tp}(A, \mathsf{a})$, so $A, \mathsf{a} \models \psi$.

Conversely, assume that $A, \mathsf{a} \models \psi$. So there is some $(B, \mathsf{b}) \in \mathcal{K}$ with $B \models \varphi(\mathsf{b})$ and $A, \mathsf{a} \models \bigwedge \mathrm{tp}(B, \mathsf{b})$. It follows by definition of $\mathrm{tp}(B, \mathsf{b})$ that (A, a) and (B, b) agree on $\mathcal{F}^\star_m$. Since σ is finite, lemma 3.7 applies, giving

$$(A, \mathsf{a}) \sim^\star_m (B, \mathsf{b}). \tag{8}$$

Since $B \models \varphi(\mathsf{b})$, claim 1 and (8) yield $A \models \varphi(\mathsf{a})$, proving the claim, and the theorem when $\sigma = \tau$.

Now we prove the theorem without restrictions on σ. Let $\mathcal{K}_\tau$ be the class of all (finite, if the models in $\mathcal{K}$ are finite) pointed Kripke models for τ. As φ is assumed $\star$-bisimulation invariant over $\mathcal{K}$, by lemma 4.1 it is also $\star$-bisimulation invariant over $\mathcal{K}_\tau$. So by the case of the theorem already proved, φ is equivalent over $\mathcal{K}_\tau$ to an $\mathcal{L}^\star(\tau)$-formula ψ of modal depth $\leq m$. But of course, φ is equivalent to ψ over $\mathcal{K}$ as well. For let $(A, \mathsf{a}) \in \mathcal{K}$. Because they agree on symbols in φ, we have $A \models \varphi(\mathsf{a})$ iff $A \restriction \tau \models \varphi(\mathsf{a})$. As $A \restriction \tau \in \mathcal{K}_\tau$, this is iff $A \restriction \tau, \mathsf{a} \models \psi$. Because $A \restriction \tau$ and A agree on symbols in ψ, this is iff $A, \mathsf{a} \models \psi$, as required. $\square$

The case $\textsc{Nom} = \emptyset$ is just the modal case, and is well known, as made clear in the introduction. We include it to indicate how, and (see lemma 5.11) why, the bound on the modal depth of the equivalent formula varies with the choice of language.

7 Optimality of modal depth bounds

Theorem 6.1 showed that every first-order $L(\sigma)$-formula $\varphi(x)$ of quantifier depth q, written with n nominals, and $\star$-bisimulation-invariant over $\mathcal{K}$, is equivalent over $\mathcal{K}$ to a $\mathcal{L}^\star(\sigma)$-formula ψ of modal depth $\leq \mathsf{m}(\star, 2^q, n)$. Perhaps surprisingly for a model-theoretic method, but less so in the light of Otto's work, this bound is optimal. Of course, sometimes one can find a simpler ψ, but in the worst case one cannot. We now give examples to show this. Take $\sigma = \textsc{Prop} \cup \textsc{Nom}$ with $\textsc{Prop} = \{p\}$. It makes no difference which $\mathcal{K}$ in the theorem is chosen.

First consider the case $\textsc{Nom} = \emptyset$, when of course $\mathcal{L}^\Diamond(\sigma) = \mathcal{L}^{\Diamond @}(\sigma)$ is the ordinary modal language. This case was dealt with by Otto, who mentioned in [16, exercise 3.1] (also with Goranko in [12, p.283]), and showed in elegant detail in [17, corollary 3.6], that for each $q < \omega$, the modal formula $\psi = \bigvee_{i<2^q} \Diamond^i p$ is equivalent over $\mathcal{K}$ to a first-order $L(\sigma)$-formula $\varphi(x)$ of quantifier depth q. (As usual, $\Diamond^0 p = p$ and $\Diamond^{i+1} p = \Diamond\Diamond^i p$.) Clearly, ψ has modal depth $2^q - 1 = \mathsf{m}(\Diamond, 2^q, 0) = \mathsf{m}(\Diamond@, 2^q, 0)$. To paraphrase [17], $\varphi(x)$ is not invariant under $\sim^\Diamond_\ell$ for any $\ell < 2^q - 1$, hence not equivalent over $\mathcal{K}$ to any modal formula of depth less than $2^q - 1$.

To help with the other cases, for $q < \omega$ define a first-order $L(\sigma)$-formula '$xR^{2^q}y$' of quantifier depth q by induction: $xR^{2^0}y$ is xRy, and $xR^{2^{q+1}}y$ is $\exists z(xR^{2^q}z \wedge zR^{2^q}y)$.

EXAMPLE 7.1 Let $\textsc{Nom} = \{c\}$ and $\star = \Diamond@$. Let $q < \omega$ and $l = 2^q$, so $\mathsf{m}(\star, l, 1) = l$. Define $\varphi(x) = cR^l c$, an $L(\sigma)$-formula of quantifier depth q. Over $\mathcal{K}$, φ is expressible in $\mathcal{L}^{\Diamond @}(\sigma)$ by $@_c\Diamond^l c$, of modal depth l.

To show that l is optimal, define finite Kripke models $A_1 = (\{a, 0, \ldots, l-1\}, R_1, V_1)$ and $A_2 = (\{a, 0, \ldots, l-1\}, R_2, V_1)$, where $a \notin \{0, \ldots, l-1\}$, $R_1 = \{(i, i+1) : i < l-1\}$, $R_2 = R_1 \cup$

$\{(l-1,0)\}$, and $V_1(c) = \{0\}$. Then $(A_1, a), (A_2, a) \in \mathcal{K}$. The strategy 'copy $\forall$'s moves' is winning for $\exists$ in $\mathrm{B}^{\diamond @}_{l-1}(A_1, a, A_2, a)$, because the difference in the models is too far away from 0 to reach in $< l$ rounds. So by lemma 3.7, (A_1, a) and (A_2, a) agree on all $\mathcal{L}^{\diamond @}(\sigma)$-formulas of modal depth $< l$. But clearly $A_1 \models \neg\varphi(a)$ and $A_2 \models \varphi(a)$, so $\varphi(x)$ is not equivalent over $\mathcal{K}$ to any $\mathcal{L}^{\diamond @}(\sigma)$-formula of modal depth $< l$ — nor obviously to any $\mathcal{L}^{\diamond}(\sigma)$-formula without @, since $(A_1, a) \sim^{\diamond} (A_2, a)$, so $\varphi(x)$ is not $\diamond$-bisimulation invariant.

EXAMPLE 7.2 Finally let $\text{NOM} = \{c_1, \ldots, c_n\}$, where $c_1, \ldots, c_n$ are pairwise distinct, and $\star = \diamond$. Let $q < \omega$ and $l = 2^q$, so $\mathsf{m}(\star, l, n) = l(n+1) = m$, say. Define

$$\varphi(x) = xR^l c_1 \wedge c_1 R^l c_2 \wedge \ldots \wedge c_{n-1} R^l c_n \wedge c_n R^l c_1.$$

Again, φ has quantifier depth q. It is equivalent over $\mathcal{K}$ to the $\mathcal{L}^{\diamond}(\sigma)$-formula

$$\diamond^l (c_1 \wedge \diamond^l (c_2 \wedge \cdots \wedge \diamond^l (c_n \wedge \diamond^l c_1)) \cdots).$$

This has modal depth $l(n+1) = m$.

Define finite Kripke models $A_3 = (\{0, \ldots, m-1\}, R_3, V_3)$ and $A_4 = (\{0, \ldots, m-1\}, R_4, V_3)$, where $R_3 = \{(i, i+1) : i < m-1\}$, $R_4 = R_3 \cup \{(m-1, l)\}$, and $V_3(c_i) = \{li\}$ for $i = 1, \ldots, n$. Then $(A_3, 0), (A_4, 0) \in \mathcal{K}$, $A_3 \models \neg\varphi(0)$, $A_4 \models \varphi(0)$, and again $\exists$ has the winning strategy 'copy $\forall$'s moves' in $\mathrm{B}^{\diamond}_{m-1}(A_3, 0, A_4, 0)$, so lemma 3.7 yields that $(A_3, 0)$ and $(A_4, 0)$ agree on all $\mathcal{L}^{\diamond}(\sigma)$-formulas of modal depth $< m$. Hence, $\varphi(x)$ is not equivalent over $\mathcal{K}$ to any such formula.

Some may have been surprised when we defined $d(@_c\psi) = d(\psi)$ (rather than $1 + d(\psi)$) in the definition of modal depth in §2. So whilst we gave a bound on the nesting depth of $\diamond$s in the $\mathcal{L}^{\diamond @}(\sigma)$-formula ψ equivalent to $\varphi(x)$, perhaps ψ has @s nested to a much greater depth? The answer is 'no'. We obtained ψ as a boolean combination of formulas in $\mathcal{F}^{\diamond @}_m$ as in definition 3.6, and each formula in this set has at most one occurrence of @, so the '@-nesting depth' of ψ is at most 1.

8 Bisimulation-closed classes

Theorem 6.1 applies to the class of all pointed Kripke models for σ and the class of all finite ones, for a hybrid signature σ. Here we consider briefly whether it extends to other classes.

Van Benthem's classical proof can be used to extend the theorem to any elementary class $\mathcal{K}$ of pointed Kripke models for any σ, though the argument does not provide any modal depth bounds for ψ.

Following Otto [16], we now consider *bisimulation-closed classes.* For $\star \in \{\diamond, \diamond @\}$, a class $\mathcal{K}$ of pointed Kripke models (for some σ) is said to be closed under $\star$-bisimulation if $(A, a) \sim^{\star} (B, b) \in \mathcal{K}$ implies $(A, a) \in \mathcal{K}$.

For modal logic, Otto states the following in [16, corollary 4.1] (we paraphrase):

> Let $\mathcal{C}$ be a class of pointed Kripke models closed under bisimulation, and $\mathcal{C}_{\mathit{fin}}$ the class of finite structures within $\mathcal{C}$. Then a first-order formula $\varphi(x)$ of quantifer depth q is invariant under bisimulation over $\mathcal{C}$ [over $\mathcal{C}_{\mathit{fin}}$] iff $\varphi(x)$ is logically equivalent over $\mathcal{C}$ [over $\mathcal{C}_{\mathit{fin}}$] to a modal formula of modal depth $\leq 2^q - 1$.

It appears that this positive result generalises to basic hybrid logic (with 2^q replacing $2^q - 1$), but to give details would take us too far out of our way. However, it can fail for proto-hybrid logic, and we end with an example showing this.

EXAMPLE 8.1 All Kripke models here will be for the hybrid signature σ with just two nominals, c and d, and no propositional atoms. We call $\Diamond$-bisimulations simply 'bisimulations' and write $\sim^{\Diamond}$ as $\sim$. For each $n < \omega$, define finite Kripke models A_n and B_n as follows:

$$\begin{aligned} A_n &= (\{0,\ldots,n\}, \{(i,i+1) : i < n\}, V_n), \quad \text{where} \quad V_n(c) = V_n(d) = \{n\}, \\ B_n &= (\{0,\ldots,n+1\}, \{(i,i+1) : i \leq n\}, V'_n), \quad \text{where} \quad V'_n(c) = \{n\} \text{ and } V'_n(d) = \{n+1\}. \end{aligned}$$

Let $\mathcal{C}$ be the class of all pointed Kripke models (M,m) such that $(M,m) \sim (A_n,0)$ or $(M,m) \sim (B_n,0)$ for some $n < \omega$. Since $\sim$ is an equivalence relation, $\mathcal{C}$ is closed under bisimulation. Let $\mathcal{C}_{\mathit{fin}}$ be the class of finite structures within $\mathcal{C}$.

Now let $\varphi(x)$ be the $L(\sigma)$-formula $c = d$. We will show that φ is bisimulation invariant over $\mathcal{C}$ but not equivalent even over the subclass $\mathcal{C}_{\mathit{fin}}$ to any $\mathcal{L}^{\Diamond}(\sigma)$-formula.

First, observe that for each pointed Kripke model (M,m) and each $n < \omega$,

$$\begin{array}{l} \text{if } (M,m) \sim (A_n,0) \text{ then } M \models \varphi(m), \\ \text{if } (M,m) \sim (B_n,0) \text{ then } M \models \neg\varphi(m). \end{array} \tag{9}$$

For suppose that $(M,m) \sim (A_n,0)$. Let Z be a bisimulation between M and A_n with $mZ0$. By n successive applications of the Back property for Z, we see that there is $m' \in M$ with $m'Zn$, so (M,m') and (A_n,n) agree on c and d. Hence, $M,m' \models c \wedge d$ and $M \models \varphi(m)$. The argument when $(M,m) \sim (B_n,0)$ is similar: we end up with $M,m' \models c \wedge \neg d$ and so $M \models \neg\varphi(m)$. This proves (9).

Again since $\sim$ is an equivalence relation, it follows from (9) that φ is bisimulation invariant over $\mathcal{C}$ and so also over $\mathcal{C}_{\mathit{fin}}$. (It is obviously not bisimulation invariant in general.)

Suppose for contradiction that $\varphi(x)$ is equivalent over $\mathcal{C}_{\mathit{fin}}$ to some $\mathcal{L}^{\Diamond}(\sigma)$-formula ψ. Let $n = d(\psi) + 1$. It is plain that $(A_n,0) \sim^{\Diamond}_{d(\psi)} (B_n,0)$, so by lemma 3.7, $(A_n,0)$ and $(B_n,0)$ agree on ψ. But $A_n \models \varphi(0)$ and $B_n \models \neg\varphi(0)$. Since $(A_n,0),(B_n,0) \in \mathcal{C}_{\mathit{fin}}$, this is a contradiction.

So where does the proof of theorem 6.1 go wrong for $\mathcal{C}$ and $\mathcal{C}_{\mathit{fin}}$? The answer is of course that claim 1 in the proof fails. The quantifier depth q of φ above is zero. Let $l = 2^q = 1$ and $m = \mathsf{m}(\Diamond, l, |\textsc{Nom}|) = 3$. Consider $(A_4,0),(B_4,0) \in \mathcal{C}_{\mathit{fin}}$. As above, $(A_4,0) \sim^{\Diamond}_m (B_4,0)$, but $A_4 \models \varphi(0)$ and $B_4 \models \neg\varphi(0)$. So claim 1 fails for these two models.

Where does the proof of the claim fail? Write $A = (A_4)^l_{<l}$ and $B = (B_4)^l_{<l}$. They work out as

$$\begin{aligned} A &= (\{\hat{0},\ldots,\hat{4}\}, \{(\hat{3},\hat{4})\}, \hat{V}), && \text{where} \quad \hat{V}(c) = \hat{V}(d) = \{\hat{4}\}, \\ B &= (\{\hat{0},\ldots,\hat{5}\}, \{(\hat{3},\hat{4}),(\hat{4},\hat{5})\}, \hat{V}'), && \text{where} \quad \hat{V}'(c) = \{\hat{4}\} \text{ and } \hat{V}'(d) = \{\hat{5}\}. \end{aligned}$$

From this, $(A,\hat{0})$ and $(B,\hat{0})$ are indeed bisimilar, as the proof of the claim shows, but $A \models \varphi(\hat{0})$ and $B \models \neg\varphi(\hat{0})$, so they disagree on φ. Hence, (7) in the proof of the claim fails.

This does not contradict the argument that led to (7), because $(A,\hat{0})$ and $(B,\hat{0})$ are plainly not in $\mathcal{C}$. So even though they are bisimilar, we cannot deduce that they agree on φ, which is known to be bisimulation invariant only over $\mathcal{C}$.

9 Conclusion

We have proved characterisation theorems for proto-hybrid logic (modal logic with nominals) and for basic hybrid logic (modal logic with nominals and @), uniformly over arbitrary and finite Kripke models, and with optimal modal depth bounds. We also showed that the theorem for proto-hybrid logic does not extend to arbitrary bisimulation-closed classes.

I do not claim that these are major results at all. They are profoundly incremental, they fly in the face of the modern trend to generality in the subject (e.g., [3, 20, 5]), and the classical cases (though perhaps not the modal depth bounds) are already known. Still, the finite-models cases may fill a narrow but striking gap in knowledge. In particular, proto-hybrid logic is a rather small and obvious extension of modal logic — just add nominals. So whether it is, like modal logic, characterised by invariance under modal bisimulations in the finite seems a basic question to which modal and hybrid logicians ought to have an answer. Now they do.

Bisimulation characterisation theorems over finite models are still relatively rare. Whilst they do sometimes follow from very general results, such as [20], at other times they can be challenging or even impossible to achieve [11]. So it may be of some value to add two more for hybrid logics, as we have done here. And the methods we have used, in particular unravelling compatibly with nominals and the use of interpretations, may be helpful elsewhere.

One might ask about characterisation theorems in the finite for more powerful hybrid logics. The picture there may not be so rosy. Another problem is to prove characterisation theorems over particular classes of finite models. Results for modal logic over finite transitive models are known [11], and it may be interesting to extend them to hybrid logics. Finally, we have mentioned characterisation theorems that hold classically but not in the finite. It might be interesting to find characterisation theorems 'in the wild' that hold in the finite but not classically. Certainly there are trivial illustrative examples (not using bisimulations). Every first-order formula $\varphi(x)$ is equivalent to a propositional boolean formula over the class of pointed Kripke models based on finite irreflexive dense linear orders, because each such order is just a solitary irreflexive point. This fails for the class of all irreflexive dense linear orders.

One final remark: although the result proved in [20] is extremely general, apparently our results here cannot be derived from it, because as it stands, nominals are not admitted. However, coalgebraic semantics has been developed for hybrid logic [15, 19], and [20] may well be extended to hybrid logic in due course.

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