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Kruskal-Friedman Gap Embedding Theorems over Well-Quasi-Orderings

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by

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Abstract

We investigate new extensions of the Kruskal-Friedman theorems concerning well-quasi-ordering of finite trees with the gap condition. For two labelled trees s and t we say that s is embedded with gap into t if there is an injection from the vertices of s into t which maps each edge in s to a unique path in t with greater-or-equal labels. We show that finite trees are well-quasi-ordered with respect to the gap embedding when the labels are taken from an arbitrary well-quasi-ordering and each tree path can be partitioned into $k \in \mathbb{N}$ or less comparable sub-paths. This result generalizes both [Kří89] and [OT87], and is also optimal in the sense that unbounded partiality over tree paths yields a counter example.

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1 Introduction

Kruskal theorem, stating that finite trees are well-quasi-ordered under homeomorphic embedding, and its extensions, aside from being interesting as a combinatorial result by itself, has played an important role in both logic and computer science. In logic, and in particular proof theory, it was shown as independent of certain logical systems by exploiting its close relationship with ordinal notation systems, and in computer science it provides a common tool for proving the termination of many rewrite-systems.

The termination property is one of the most important properties of a rewrite system. Many times termination proofs amounts to showing that the rewrite relation is included in some well-founded ordering of the terms under consideration. The latter ordering is then called a *termination ordering*. A canonic such termination ordering is the *recursive path ordering* [Der82].

At the heart of proving the termination of the recursive path ordering is Kruskal's tree theorem which states that finite trees are well-quasi-ordered under homeomorphic embedding; meaning that there is a one-to-one mapping from s to t, such that vertices are mapped to vertices and edges to unique paths. However, the embeddability property inherent in the recursive path ordering is also responsible for its limitations.

A term ordering is said to have the *subterm property* if all terms are always bigger then their subterms. Termination orderings that have the subterm property are called *simplification orderings*. Any such simplification ordering is bound to include also the homeomorphic embedding relation. Nevertheless, it is sometimes necessary to prove termination of rewrite systems that are not simplifying.

Take for example the following rewrite rule: $ff \to fgf$. The rewrite relation induced by this rule is clearly terminating since the number of adjacent f symbols decreases in every application of the rule. In addition, the left term ff is homeomorphic embedded in fgf. Thus, for any simplifying termination ordering ff is smaller than fgf, which means that any such termination ordering fails to prove the termination of this rewrite rule. Indeed, in order to prove termination, we must orient the left term to be greater than the right term. Hence, it would be worthwhile to look for extensions of Kruskal theorem which uses embedding relation not possessing the subterm property.

One such embedding is the *embedding with gap condition*, which is at the center of investigation in this work. The first explicit introduction of the gap embedding is due to H. Friedman (see [Sim85]). Since its introduction, other variants of gap embeddings were introduced, most of which made use of a *total ordering* on the labels. The results in [Kří89], [Kří95] are of pure combinatorial nature. Those of [Gor89], [Gor90] are centered around proof theoretical matters. Other where applied to term graph rewriting (see [Oga95]).

Since, with regard to term rewriting, the tree labels ordering corresponds to the ordering of the function symbols pertaining to a certain signature, it would be beneficial, in regard to termination proofs, to use *partial* or *quasi* orderings on the labels, rather then a total one. Extensions of Kruskal theorem not possessing the subterm property are also believed to be of benefit for facilitating termination orderings for typed and higher order rewrite systems (see [JR98]).

The relationship between well-partial-ordered precedence relations on functions symbols and the order type of the induced termination ordering is investigated in [Wei92]. It is shown that many important order-theoretic properties of the precedence relation carry over to the induced termination ordering. This is done by defining a general framework for precedence-based termination orderings via a (so-called) *relativized ordinal notations*. Usually an ordinal notation system is given by a primitive-recursive set of terms T and a primitive-recursive relation $<$ on T. The terms of T are built by some constant symbol, say "0", and the tree constructors. Although there is no standard ordinal notation used in proof theory, [Wei92] uses a sufficiently strong ordinal notation system and the corresponding *relativized* system is defined such that "0" is replaced by the elements of Q, with minor changes applied to $\preceq$ in order to deal with all the symbols in Q, yielding the relative ordinal notation system $(T_Q, \prec)$. Based on few examples, it is further conjectured that every such application of a partial-order to an ordinal notation system, carries the order-theoretic properties of the partial-order to the relativized notation system. An example of such a construction, using Takeuti's ordinal diagrams, is introduced also in [OT87] by the name *quasi-ordinal-diagrams*. The definition of these diagrams is the only result known to us that deals with gap embedding of trees and *quasi* ordered labels. However, this result is limited in that the quasi ordering only resides on the *leaves* of trees, while interior vertices are bound to be labelled by some well ordering. Furthermore, the tree embedding is defined so that it forms a partial order over the fields of trees and not a quasi ordering. [OT87] also connects Friedman independence result for various formulations of Kruskal theorem with ordinal diagrams.

H. Friedman introduced the concept of embedding with gap in order to achieve orderings of large types, so that sufficient strong formal systems of arithmetic would be unable to prove the well foundedness of these orderings (see [Sim85]). The motivation was to produce a 'natural' mathematical statement, in contrast to the familiar metamathematical ones (e.g. Gödel's incompleteness proof), which is nevertheless independent of these formal systems (hence, the name 'natural independence'). There was criticism that although Kruskal theorem is a truly natural combinatorial result, the gap conditions are not, as they are cooked up in advance to yield the desired result. However the gap embedding was shown to be 'natural' due to its importance in proving the celebrated Graph Minor Theorem of Robertson and Seymour (see [FRS87], and [Rat94]; the latter contains a general discussion on the 'rewards of ordinal representation systems'). In this context Kruskal theorem, as a fundamental result of well-quasi-order theory is a result pertaining to combinatorics which is interesting enough by itself.

1.1 Results of this work

We show by a simple counter example (§2, prop. 2.7) that unbounded partiality over tree paths yields a bad sequence w.r.t. the gap embedding. We then prove it to be a canonic such structure for generating a bad sequence, by proving that bounding the partiality allowed on tree paths results in a wqo (§3, thm.3.1).

The proof of theorem 3.1 follows the road of [Kří89], technically however, it is somewhat more involved than the original. The main novelty here is the insight that as long as the label ordering is a wqo it is sufficient to maintain the totality only of the order induced by *each tree path*, instead of requiring that the label ordering on all trees should be total.

In section 4, based on the result of the preceding section we use a different approach to prove that if each tree path can be partitioned into some apriori $k \in \mathbb{N}$ comparable sub-paths, then the wqo property is preserved (§4, thm.4.2).

Section 2 sets up the basic terminology and results we relate to in this work. Conclusions and some open problems are discussed in Section 5.

2 Kruskal-Friedman Type Theorems

In this section we review certain facts about Kruskal-Friedman style theorems concerning well quasi orderings of finite trees.

2.1 Preliminaries

A *quasi ordering* is a set Q together with a reflexive and transitive relation $\preceq$. Given a quasi ordering $(Q, \preceq)$ and two elements $a, b \in Q$, we say that a and b are *comparable* if either $a \preceq b$ or $b \preceq a$, otherwise we say that they are *incomparable*. We denote by $\prec$ the strict part of $\preceq$, i.e., $a \prec b$ iff $a \preceq b$ and $b \not\preceq a$.

A quasi ordering $(Q, \preceq)$ is a *well-quasi-ordering* (wqo) if for all infinite sequences $(a_i)_{i\in\mathbb{N}} \subseteq Q$ there exist $i < j \in \mathbb{N}$ such that $a_i \preceq a_j$. A sequence $(a_i)_{i\in\mathbb{N}}$ s.t. for all $i < j$, $a_i \not\preceq a_j$ holds is called a *bad* sequence, otherwise it is called a *good* sequence. An infinite sequence $(a_i)_{i\in\mathbb{N}}$ is said to be an *antichain* if a_i is incomparable to a_j for all $i < j \in \mathbb{N}$.

We shall deal with infinite sequences of elements from some quasi order $(\mathcal{A}, \preceq)$. We define $\mathbb{N}^{(\omega)}$ to be the set of all infinite subsets of $\mathbb{N}$. Formally an infinite (ω-) sequence over $\mathcal{A}$ is a function $f : M \to \mathcal{A}$, where $M \in \mathbb{N}^{(\omega)}$. We denote the domain of f by $\mathrm{D}f$.

Greek letters $\alpha, \beta, \gamma, \ldots$ will use to denote ordinals, where an ordinal α is identified with the set $\{\beta \in On : \beta < \alpha\}$. We shall use the letters i, j and k to denote the natural numbers. We also identify sometimes the natural numbers in $\mathbb{N}$ with the ordinals $< \omega$.

A finite tree is defined to be a finite partial ordering $(t, \leq_t)$ such that the set of ancestors $\{v \in t : v \leq_t u\}$ of each vertex u in t, forms a linear ordering with a unique minimal element called the *root*. We assume also that the immediate successors of each vertex, i.e. its children, are *linearly ordered*. A *subtree* of a tree t, rooted at $u \in t$, is the upward closure $\{v \in t : u \leq_t v\}$ of u. We denote by $[u, v]$ the path beginning from u to v. For a given path $[u, v]$ we define naturally a sub-path to be $[u', v']$ where $u \leq_t u' \leq_t v' \leq_t v$. We also use $(u, v]$ etc. in the obvious way. We shall usually write simply $\leq$ instead of $\leq_t$, when it is clear from the context which ordering is used.

Let T_Q denote the set of all finite trees with labels from Q. We shall use T to denote the labelled trees over Q in sections 3 and 4. Formally the labels of a tree $t \in T_Q$ are determined by a labelling function $l_t : t \to Q$ and a labelled tree is a set of vertices (identified with t) combined with a partial ordering and a labelling function. We denote by $r(t)$ the root of t and by $pred(u)$ the *immediate* ancestor of a vertex $u \in t$ (i.e. its predecessor). When it is not ambiguous *we shall identify a vertex with its label*. Thus $v \leq u$ means that v is an ancestor of u in the tree ordering, while $v \preceq u$ means that v has a less-or-equal label than u. Let us denote also by $u \sqcap v$ the *greatest common ancestor* of a pair of vertices u, v.

2.2 Gap Embedding

Definition 2.1 ***(tree embedding)*** *For two trees s, t we say that s is* embedded *into t if:*
(1) there is an injection $f : s \to t$ such that $f(v \sqcap u) = f(v) \sqcap f(u)$ for all vertices v, u in s; and
(2) Let v be a vertex in s and v_1 and v_2 are distinct immediate successors of v such that v_1 precedes v_2 in the linear ordering of v's children. If $f : v \mapsto z$, then z_1 precedes z_2 in the linear ordering of z's children, where z_1 is the immediate successor of z on the path to $f(v_1)$ and z_2 is the immediate successor of z on the path to $f(v_2)$.

Remark.(i) The second condition of definition 2.1 is introduced so that for each vertex of s, the order of siblings in s is preserved in the embedding into t.
(ii) In the literature this kind of embedding appears also under the names *homeomorphic embedding* (cf. [DJ90]) or *topological minor relation* (cf. [RS84]).

Definition 2.2 ***(tree embedding with gap)*** *For two trees s, t we say that s is* embedded with gap *into t and write $s \hookrightarrow t$ if there is an embedding $f : s \to t$ for which the following conditions hold:*
(1) (label increasing) $\forall v \in s.\ v \preceq f(v)$ *;*
(2) (gap condition) *for all edges (u, v) in s and for all $w \in t$ s.t. $f(u) < w < f(v)$, $w \succeq v$;*
(3) (root gap condition) $u \succeq r(s)$ *for all vertices u in the access path $[r(t), f(r(s))]$ of t.*

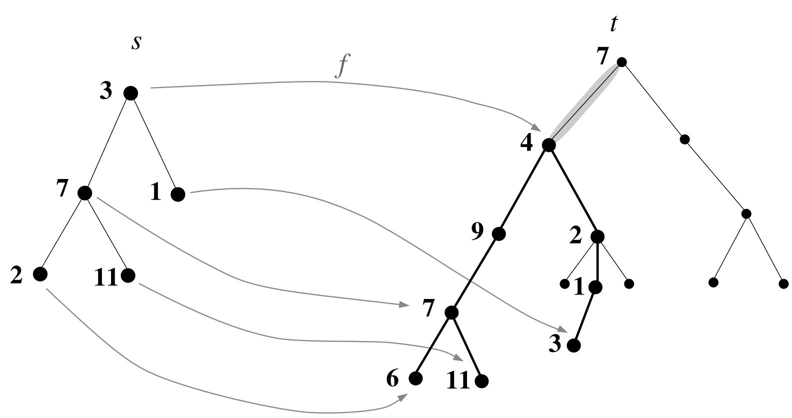


Figure 2.1: Gap embedding $s \hookrightarrow t$

The gap embedding forms a quasi-ordering over trees labelled by some quasi-ordering: reflexivity is obvious and transitivity stems from transitivity of tree embedding (without the gap conditions), and the transitivity of the labels ordering. A set of trees is well-quasi-ordered under gap embedding $\hookrightarrow$, if every infinite sequence of trees contains a pair s, t of trees, one preceding the other such that $s \hookrightarrow t$.

The following theorem was originally conjectured by H. Friedman and proved by I. Kříž in [Kří89].

Theorem (Kříž ['89]) 2.3 *For any well-order* $(W, \leq)$*,* $(T_W, \hookrightarrow)$ *is a wqo.*

2.3 Gap Embedding for Edge Labelled Trees

A different and more intuitive definition of the gap embedding can be given by trees with labels on the *edges* instead of the vertices. Let us define naturally for a tree $(t, \leq)$ the set of its edges by $E_t := \{\langle u, v\rangle \in t \times t \ : \ u = pred(v)\}$ and the labelling function $l_t : E_t \to Q$ for some quasi-order $(Q, \preceq)$. An edge $e \in E_t$ is said to be *in the path* $[u, v]$ if $e \in ([u, v] \times [u, v]) \cap E_t$. We have the following gap definition.

Definition 2.4 ***(tree embedding with gap second version)*** *Let s, t be two trees with edge labelling. We write $s \hookrightarrow_e t$ iff there is an embedding of s into t such that each edge of s is mapped to a path in t with $\preceq$ labels.*

Remark. Note that $\hookrightarrow_e$ lacks a root gap condition corresponding to the one in $\hookrightarrow$.

Let us denote by T'_Q the set of trees with labels on their edges, and let $\hat{0}$ be a new minimum element of Q s.t. $\forall q \in Q . \hat{0} \preceq q$. For a tree $(t', \leq') \in T'_Q$, we define its corresponding tree $(t, \leq) \in T_Q$, having labels on vertices instead of on the edges, by the rules:

(i) $(t, \leq) := (t', \leq')$;
(ii) $l_t(r(t)) := \hat{0}$;
(iii) for all edges (u, v) in E_t let $l_t(v) := l_{t'}(u, v)$.

That is, we simply put each edge label of (u, v), where $v \geq u$ to label the vertex v, and put $\hat{0}$ as the root label.

Proposition 2.5 *Let s', t' be two trees in T'_Q and s, t be their corresponding trees in $T_{Q \cup \{\hat{0}\}}$, achieved by following the above three rules, then $s' \hookrightarrow_e t'$ iff $s \hookrightarrow t$.*

Proof. By straightforward verification of the gap embedding conditions in definition (2.2). Assume $f : s' \hookrightarrow_e t'$, then this f is applicable also to s, t since the tree structure stays the same. Since $s' \hookrightarrow_e t'$ then for all vertices $u \in s$ excluding the root of s and for all edges $e \in [f(pred(u)), f(u)]$ in t' we have $l_{s'}(pred(u), u) \preceq l_{t'}(e)$. Hence for all vertices

$v \in (f(pred(u)), f(u)]$ in t we have $v \succeq u$. The root gap condition of $s \hookrightarrow t$ also holds since the new root label of s is $\hat{0}$ which is less or equal to all other labels and in particular to all the labels in the access path from the root of t to $f(r(s))$.

Conversely, if $s \hookrightarrow t$ (for which, by definition, $r(s)$ and $r(t)$ have $\hat{0}$ labels) then a similar verification process shows that indeed $s' \hookrightarrow_e t'$ in T'_Q. □

Consequently, if every infinite sequence in $T_{Q \cup \{\hat{0}\}}$ is good then every infinite sequence in T'_Q is good, since the existence of an infinite bad sequence $b \subseteq T'_Q$ would imply that the corresponding $T_{Q \cup \{\hat{0}\}}$ trees form a bad sequence either. Hence we have the following.

Corollary 2.6 *If for all wqo Q, $(T_Q, \hookrightarrow)$ is a wqo, then for all wqo Q, $(T'_Q, \hookrightarrow_e)$ is a wqo.*

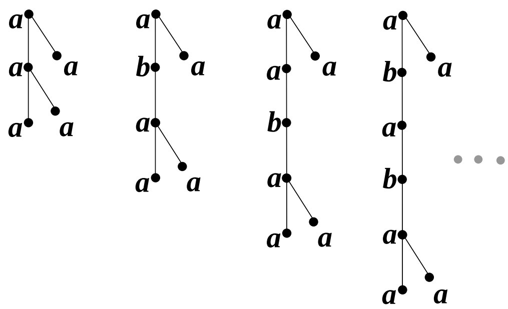
Therefore, it would be sufficient to prove that $(T_Q, \hookrightarrow)$ is wqo in order to show that both embeddings are. From now on we shall deal only with the first gap embedding definition.

2.4 Quasi Ordered Labels

It seems natural to extend Theorem 2.3, stating that finite trees with well-ordered labels are well-quasi-ordered under gap embedding, to some arbitrary well-quasi-ordered labels. Indeed, finite trees ordered by embeddability (without the gap condition) with wqo labels is the result proven originally by Kruskal [Kru60]. As it turns out, however, wqo is not closed under embedding with gap. Even if Q has only one pair of disjoint elements then there is a counter example.

Proposition 2.7 *If $(Q, \preceq)$ is not total then $(T_Q, \hookrightarrow)$ is not a wqo.*

Proof. Since Q is not total then there exist an incomparable pair of elements a and b in Q. We have the following antichain:



Note that in any embedding of two trees in this sequence, roots ought to be mapped to roots and the same thing happens with the immediate predecessors of leaves, since these are the only vertices having two children. □

In this work we show that this counter example is the *canonic* one.

Definition 2.8 *Given a tree path* $[u, v]$*, we say that the path is* comparable *if all the vertices in it have comparable labels, that is,* $\forall x, y \in [u, v].\, x \preceq y \vee y \preceq x$.

Let Q be a wqo and let T^k be the set of all trees labelled from Q such that each path in a tree, beginning in the root, can be partitioned into some fixed $k \in \mathbb{N}$ or less comparable sub-paths (surely, any path of length n can also be partitioned into n comparable sub-paths, each sub-path contains only one vertex). The main result of this work is the following.

Main Theorem 2.9 T^k *is wqo under gap embedding.*

The proof is the corollary of the next two sections.

3 Comparable Path Trees

3.1 Definitions and Terminology

Let us denote by $\widehat{T}$ the set of trees labelled by Q such that *each path consists of only comparable labels from* Q. Note that siblings might be disjoint that way. The following is the main theorem of this section.

Theorem 3.1 *$\widehat{T}$ is well-quasi-ordered by gap embedding for all wqo Q.*

In order to prove the theorem we first prove a *minimal bad sequence theorem* which is a variant of the main theorem in [Kří89] accommodated to our settings. Before we do this we need yet some more definitions.

Definition 3.2 ***(gap subtree)*** *For two labelled trees s, t we say that t is a gap subtree of s and write $t \trianglelefteq s$ iff t is a subtree of s and the access path $[r(s), r(t)]$ from the root of s to the root of t, keeps the following gap condition:*

$$\min_{\preceq}(r(s), \ldots, r(t)) \in \{r(s), r(t)\}\,.$$

Accordingly, we write $t \triangleleft s$ if t is a proper subtree of s and the above gap condition holds.

We shall use also the notations $\trianglerighteq$ and $\triangleright$ to denote the inverse of $\trianglelefteq$ and $\triangleleft$, respectively. We have the following three properties:

$$s \trianglerighteq t \trianglerighteq u \wedge r(t) \succeq r(u) \quad \Rightarrow \quad s \trianglerighteq u \tag{3.1}$$

$$s \trianglerighteq t \trianglerighteq u \wedge r(s) \preceq r(t) \quad \Rightarrow \quad s \trianglerighteq u \tag{3.2}$$

$$s \hookrightarrow t \trianglelefteq u \wedge r(t) \preceq r(u) \quad \Rightarrow \quad s \hookrightarrow u \tag{3.3}$$

Note that in contrast to the usual subtree relation, the gap subtree relation $\trianglelefteq$ is *not* transitive. For that reason we introduce two more subtree relations for which transitivity does hold.

Definition 3.3 *For two labelled trees s and t, such that t is a subtree of s we define*
(i) $t \triangleleft_{\succeq} s \quad \Leftrightarrow \quad t \triangleleft s \ \wedge \ r(t) \succeq r(s)$*;*
(ii) $t \triangleleft_{\prec} s \quad \Leftrightarrow \quad t \triangleleft s \ \wedge \ r(t) \prec r(s)$*.*

The following is a key observation of our proof, which stems from the condition that paths are comparable in $\widehat{T}$.

Observation. Both $\triangleleft_{\succeq}$ and $\triangleleft_{\prec}$ relations are transitive as relations on $\widehat{T}$.

For simplicity we shall denote by Seq the set of all infinite sequences over $\widehat{T}$ and by Bad the set of all infinite bad sequences from Seq.

We now define several relations on the set Seq that correspond to the basic gap-subtree relations above.

Definition 3.4 *For two sequences* $a, b \in Seq$ *s.t.* $\mathrm{D}b \subseteq \mathrm{D}a$ *we write*
(1) $a \rhd b$ *iff* $\forall i \in \mathrm{D}b.\ a(i) \rhd b(i)$
(2) $a \trianglerighteq b$ *iff* $\forall i \in \mathrm{D}b.\ a(i) \trianglerighteq b(i)$ *;*
(3) $a \rhd_{\preceq} b$ *iff* $\forall i \in \mathrm{D}b.\ a(i) \rhd_{\preceq} b(i)$ *;*
(4) $a \rhd_{\succ} b$ *iff* $\forall i \in \mathrm{D}b.\ a(i) \rhd_{\succ} b(i)$ *;*

Remark. Note that the relations defined in (3.4) are all *element-wise* relations, but they ignore positions at which one or both are undefined. Also notice that $(\rhd_{\preceq})^{-1} = \lhd_{\succeq}$ and $(\rhd_{\succ})^{-1} = \lhd_{\prec}$.

The following observation is essential to the proof, and is the counterpart of the previous observation.

Observation. Both $\lhd_{\succeq}$ and $\lhd_{\prec}$ relations are transitive as relations on Seq.

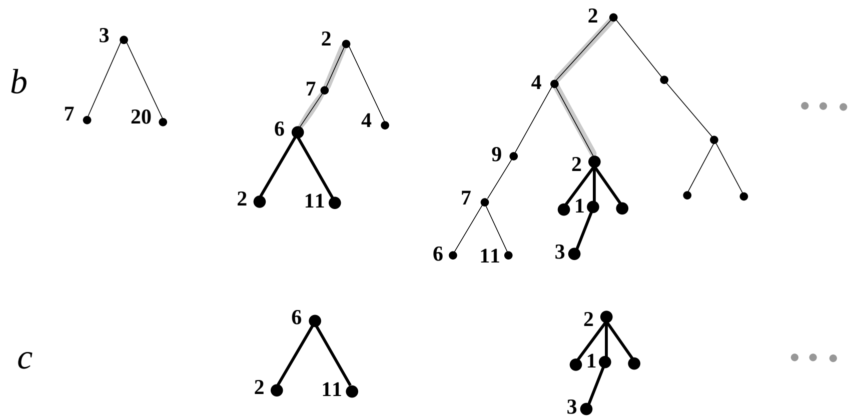


Figure 3.1: Two sequences of trees labelled from $\mathbb{N}$ such that $c \lhd_{\succeq} b$.

For a given sequence s and a constant $k \in \mathbb{N}$ we write $s|_{<k}$ to denote $u \restriction (\mathrm{D}u \cap k)$; appropriately $s|_{\leq k}$ denotes $u \restriction (\mathrm{D}u \cap (k+1))$. We shall use $\otimes$ to denote the concatenation of two sequences defined as follows.

Definition 3.5 *Given two sequences* $h, g \in Seq$ *such that* $i := \min \mathrm{D}g$*, we define the* concatenation of h to g *by* $h \otimes g := h|_{<i} \cup g$.

We now introduce a minimization relation. The proof of the theorem is based on the existence of such relation for which certain closure properties are preserved. Consequently, we would always be able to define a new 'minimal' sequence w.r.t. this relation.

Definition 3.6 *For two sequences* $u, v \in \mathrm{Seq}(Q)$*, we write* $u <_{lex} v$ *iff either* $\exists k \in \mathrm{D}u.\ u|_{<k} = v|_{<k}$ *and* $u(k) \prec v(k)$ *or* $\min(\mathrm{D}u) < \min(\mathrm{D}v)$.

For two sequences $g, g' \in Seq$ we write $g <_{lex} g'$ to mean that the sequence of *roots* from Q induced by the trees of g is $<_{lex}$ than that induced by g'. We call a sequence $a_1, a_2, \ldots$ from Q *increasing* if $i < j \in \mathbb{N}$ implies $a_i \preceq a_j$. We shall call also a sequence of *trees* $g \in Seq$ *increasing* if its roots sequence induces an increasing sequence. By $Bad\uparrow$ we denote the set of all sequences of Bad that are also increasing.

In what follows we shall use the fact that since Q is wqo each infinite sequence has an *increasing infinite subsequence* by Ramsey. For a sequence $g \in Seq$ denote by $g\uparrow$ some infinite *increasing* subsequence of g.

Notice that by the pigeonhole principle, for two sequences $a, b \in Seq$ s.t. $a \lhd b$ either there exists an infinite subsequence a' of a s.t. $a' \lhd_{\succeq} b$ or an a' s.t. $a' \lhd_{\prec} b$.

In the next sections we concentrate on proving the following minimal bad sequence theorem.

Theorem (minimal bad sequence) 3.7 *Let* Q *be some wqo. If there is a bad infinite sequence from* $\widehat{T}$ *then there exists a minimal bad increasing sequence* m *s.t. there is no infinite bad sequence* f *with* $f \lhd m$.

3.2 The Minimal Bad Sequence Theorem

Let us restate more succinctly the *minimal bad sequence* theorem.

Theorem 3.7 $\quad Bad \neq \emptyset \implies \exists m \in Bad\uparrow .\ m \in \min_{\lhd} Bad\,.$

3.2.1 The Construction

In order to prove the theorem we assume it is false and build a construction yielding a contradiction via a cardinality argument. Thus, we assume by a way of contradiction the following hypothesis.

$$\text{(hyp)} \qquad Bad \neq \emptyset \ \wedge\ \min_{\lhd} Bad = \emptyset\,.$$

Note that this hypothesis is indeed the negation of theorem 3.7 since $Bad \neq \emptyset \Rightarrow Bad\!\uparrow \neq \emptyset$ by Ramsey. Under this hypothesis we build by transfinite induction a sequence $\langle h_\alpha \,|\, \alpha < \omega_1 \rangle$ of *distinct* increasing bad sequences with order type ω_1:

$$h_0 \trianglerighteq h_1 \trianglerighteq h_2 \ldots \trianglerighteq h_\omega \ldots \trianglerighteq h_\alpha \ldots \qquad (\alpha < \omega_1) \, ,$$

$$h_\alpha \in Bad\!\uparrow \qquad \text{and} \qquad \forall \beta < \alpha . \mathrm{D}h_\beta \supseteq \mathrm{D}h_\alpha \, .$$

Let $g'_0 := z$ for some $z \in \min_{<_{lex}} Bad$. Then we put

$$h_0 := g'_0\!\uparrow$$

Having built h_α already, in order to build $h_{\alpha+1}$ we do the following. Define

$$g'_{\alpha+1} := \min_{<_{lex}} \{ s \in Bad \;:\; s \lhd h_\alpha \} \, .$$

We will show that $g'_{\alpha+1}$ exists in §3.2.5(1),(2). Then, take $g_{\alpha+1}$ to be a subsequence of $g'_{\alpha+1}$, such that

$$h_\alpha \rhd_{\preceq} g_{\alpha+1} \, . \tag{3.4}$$

Such a $g_{\alpha+1}$ exists by Lemma 3.11 (p. 21). Now define

$$h_{\alpha+1} := h_\alpha \otimes (g_{\alpha+1}\!\uparrow) \, . \tag{3.5}$$

Note that the sequences h_α are indeed distinct from each other since for all $\alpha + 1 < \omega_1$ we have that $h_\alpha \rhd_{\preceq} g_{\alpha+1} \subseteq h_{\alpha+1}$.

For a limit ordinal λ we define h_λ as follows.

$$\begin{aligned} \mathrm{D}g_\lambda &:= \bigcap \{ \mathrm{D}h_\alpha | \alpha < \lambda \} \\ g_\lambda(i) &:= \lim_{\alpha \to \lambda} h_\alpha(i) \quad \text{ for } i \in \mathrm{D}g_\lambda \\ f &:= \min_{<_{lex}} \{ s \in Bad \,|\, s \lhd g_\lambda \} \\ h_\lambda &:= f\!\uparrow \end{aligned}$$

We shall show in the sequel that $\lim_{\alpha \to \lambda} h_\alpha(i)$ is converging to some *fixed* tree for any limit $\alpha < \omega_1$ and $i \in \mathrm{D}g_\lambda$. Further, g_λ will be shown bad, hence f exists by (hyp) and §3.2.5(1),(2).

Consequently, we have the following two *invariants* of the construction.

$$\forall n \in \mathbb{N} \, \forall \alpha < \omega_1 . \; h_\alpha \trianglerighteq_{\preceq} h_{\alpha+n} \tag{3.6}$$

For all limit $\lambda \neq 0$ we have

$$\forall i \in \mathrm{D}g_\lambda \exists \alpha < \lambda . \; g_\lambda(i) = h_\alpha(i) \tag{3.7}$$

Figure 3.2: THE INDUCTIVE CONSTRUCTION. Illustration of an initial segment of the sequence $\langle h_\alpha \mid \alpha < \omega_1 \rangle$ up to $h_{\omega 2}$. For each bad sequence h_α, the bright trees are the fixed (limit) trees taken from the previous bad sequence, while the dark ones are those in $\mathrm{D}g_\alpha$. The shaded background represents the domains $\mathrm{D}g_\alpha$. At the limits, these domains vanish, and the limit sequences contain only the limit trees. Note that a fixed tree in h_n for $n \in \mathbb{N}$ is fixed only up to ω, after which it can be reduced again by some of its proper subtrees. This rule applies to all limits less then ω_1.

Invariant 3.6 holds by 3.4 and the definition of $\trianglerighteq_{\preceq}$ (recall that the relation $\trianglerighteq_{\preceq}$ over *trees* is transitive, hence the corresponding element-wise relation $\trianglerighteq_{\preceq}$ over *sequences* is also transitive). Invariant 3.7 holds by definition of g_λ.

3.2.2 Correctness of the Construction

We show now that the construction maintains the fact that $h_{\alpha+1}$ is indeed an increasing bad sequence for any ordinal α, and that the limit sequences defined for limit ordinals, are indeed infinite bad.

Lemma 3.8 *For all $\alpha < \omega_1$ we have that $h_{\alpha+1} \in Bad{\uparrow}$.*

Proof. We show first that $h_{\alpha+1}$ is increasing.

$$g_{\alpha+1} \lhd_{\succeq} h_\alpha \implies \forall k \in \mathrm{D}g_{\alpha+1}.\ r(g_{\alpha+1}(k)) \succeq r(h_\alpha(k))\ . \tag{3.8}$$

Define

$$i := \min \mathrm{D}g_{\alpha+1} \quad \text{and} \quad j := \max \mathrm{D}h_\alpha|_{<i}\ .$$

If $j = \emptyset$ then the claim is obvious, otherwise we have the following:

$$\begin{array}{rrcl} \text{(by definition of } h_{\alpha+1}) & h_{\alpha+1}(j) & = & h_\alpha(j) \\ \text{(since } h_\alpha \text{ is increasing)} & & \preceq & h_\alpha(i) \\ \text{(by (3.8))} & & \preceq & g_{\alpha+1}(i) \\ \text{(again, by definition of } h_{\alpha+1}) & & = & h_{\alpha+1}(i), \end{array}$$

Since by the definition of the concatenation operation $\otimes$, $h_{\alpha+1}(i)$ immediately succeeds $h_{\alpha+1}(j)$ in the sequence $h_{\alpha+1}$, we conclude that $h_{\alpha+1}$ is increasing .

To show that $h_{\alpha+1} \in Bad$, assume otherwise. Hence there are $s \in h_{\alpha+1}|_{<i}$ and $t \in g_{\alpha+1}$ for which $s \hookrightarrow t$. Let w be the corresponding super-tree of t in h_α. Since $r(t) \succeq r(w)$ and $t \lhd w$ then $r(w)$ holds the minimum label on the access path to t in w. By increasingness of h_α we have that $r(s) \preceq r(w)$ so the root gap condition holds for t in w hence $s \hookrightarrow w$ which contradicts the badness of h_α. □

Lemma 3.9 *For all limit $\lambda < \omega_1$ we have that $g_\lambda \in Bad$.*

Proof. That the limit sequence g_λ is bad is obvious from the definition, since any pair of trees, one embedded into the other in g_λ, would also imply this for h_γ for some $\gamma < \lambda$ by invariant (3.7). Therefore it is sufficient to show that the sequences h_γ for $\gamma < \lambda$ converge to some infinite sequence.

Intuitively speaking, this is indeed the case since the trees are finite, hence for each tree in the initial sequence h_0, taking a subtree can occur only finitely many times throughout the construction.

Formally, since all trees are finite and λ is a limit, then for any index $i \in \mathbb{N}$ there is a maximal $\beta < \lambda$ beyond which there are no more gap subtrees taken from $h_\beta(i)$. Hence we have

$$\forall i \in \mathbb{N}\, \exists \beta < \lambda\, \forall \gamma.\ \beta < \gamma < \lambda \to i \notin \mathrm{D}g_\gamma\ .$$

This means that starting from h_β up until g_λ the value in index i is *fixed* (might be $\emptyset$). Thus, the trees converge for all indices $i \in \mathbb{N}$ so that the following holds

$$\lim_{\gamma \to \lambda} \inf(\min \mathrm{D} g_\gamma) = \omega. \tag{3.9}$$

It remains to show that the limit sequence g_λ converges into an *infinite* sequence. Therefore, note that when we build $h_{\beta+1}$, if the first index of $\mathrm{D}g_\beta$, denoted by i, is less then that of $\mathrm{D}g_{\beta+1}$ then $h_\beta(i)$ $(= g_\beta(i))$ gets into $h_{\beta+1}$. Now, if *every* such first index of $\mathrm{D}g_\gamma$ is greater than i for all $\beta < \gamma < \lambda$ then $h_\beta(i)$ remains a tree in every sequence h_γ, i.e. $h_\gamma(i) = h_\beta(i)$ for all $\beta < \gamma < \lambda$. By 3.9 we know that this must be the case for infinitely many $\mathrm{D}g_\beta$ where $\beta < \lambda$. Formally we have

$$\left\{ \inf_{\beta \leq \gamma < \lambda} (\min \ \mathrm{D}g_\gamma) \mid \beta < \lambda \right\} \subseteq \bigcap \{\mathrm{D}h_\gamma \mid \gamma < \lambda\} \ ,$$

with the left hand side infinite. Thus we can build g_λ by taking the limit trees in the domain $\bigcap \{\mathrm{D}h_\gamma \mid \gamma < \lambda\}$. □

3.2.3 Existence Conditions For The Construction

Let us repeat, and restate, the way h_α was defined in §3.2.1. To find $g_{\alpha+1}$ we work through the following steps.

(i) Define $\mathcal{K}_0 := Bad$ and put $g'_0 := z$ for some $z \in \min_{<_{lex}} \mathcal{K}_0$. Then we define h_0 to be some infinitely increasing subsequence $g'_0\!\uparrow$ of g'_0.

(ii) Define

$$\mathcal{K}_{\alpha+1} := \{s \in Bad \ : \ s \lhd h_\alpha\} \ ,$$

and put

$$g'_{\alpha+1} := \min_{<_{lex}} \mathcal{K}_{\alpha+1} \ .$$

(iii) We shall show that although we know by (hyp) only that there exists some $\lhd h_\alpha$ sequence, we actually have

$$\forall \alpha < \omega_1 \, \neg \exists g \in Bad. \ g \lhd_{\prec} h_\alpha \tag{3.10}$$

Therefore, we can let $g_{\alpha+1} \subseteq g'_{\alpha+1}$ be an infinite sequence s.t. $g_{\alpha+1} \lhd_{\succeq} h_\alpha$. Let $\mathcal{J}_{\alpha+1} := \{s \in Bad \ : \ s \lhd_{\prec} h_\alpha\}$, so (3.10) becomes

$$\forall \alpha < \omega_1 . \, \mathcal{J}_{\alpha+1} = \emptyset \tag{3.11}$$

We have the following induction invariant which will be used throughout the next sections.

Corollary 3.10 *Minimality w.r.t.* $<_{lex}$*:*

$$\forall \beta < \alpha \neg \exists s.\, s \in \mathcal{K}_{\beta+1} \wedge s <_{lex} g'_{\beta+1} \,.$$

It remains to prove (3.11) and that $\mathcal{K}_{\alpha+1}$ is closed under $<_{lex}$, that is:

$$\min_{<_{lex}} \mathcal{K}_{\alpha+1} \neq \emptyset \tag{3.12}$$

3.2.4 Outline of the Existence Proof

For some $\mathcal{I} \subseteq Seq$ let us denote by $\mathrm{CL}(\mathcal{I})$ that $\mathcal{I}$ is closed, that is,

$$\mathrm{CL}(\mathcal{I}) \quad \Leftrightarrow \quad \forall z \in Seq.\, (\forall i \in \mathbb{N}\, \exists z' \in \mathcal{I}.\, z'|_{\leq i} = z|_{\leq i}) \rightarrow z \in \mathcal{I} \,,$$

Let us denote by i.h.(α) that induction hypothesis $\mathcal{J}_{\gamma+1} = \emptyset$ holds for all $\gamma + 1 \leq \alpha$, hence the construction exists (and its invariants hold) up to h_α, by the previous section. Note that if α is a limit then it is sufficient to have for all $\gamma < \alpha$ that $\mathcal{J}_\gamma = \emptyset$ in order to know that the construction invariants hold up to α *including* α *itself*.

The following is an outline of what we prove next.

1. $\forall \mathcal{S} \subseteq Seq.\, \mathrm{CL}(\mathcal{S}) \Rightarrow (\mathcal{S} \neq \emptyset \rightarrow \min_{<_{lex}} \mathcal{S} \neq \emptyset)\,$;
2. $\mathrm{CL}(\mathcal{K}_{\alpha+1})$ for all $\alpha < \omega_1$;
3. (i) i.h.(0): $\min_{<_{lex}} \mathcal{K}_0 \neq \emptyset\,$;
   (ii) i.h.$(\alpha) \Rightarrow \mathcal{J}_{\alpha+1} = \emptyset\,$.

By (hyp) we already know that $\mathcal{K}_{\alpha+1} \neq \emptyset$, hence by (2) and (1) we get (3.12). (3i) is the base case of the induction, combined with (3ii) we get (3.11).

3.2.5 The Existence Proof

(1) $\forall \mathcal{S} \subseteq Seq.\, \mathrm{CL}(\mathcal{S}) \Rightarrow (\mathcal{S} \neq \emptyset \rightarrow \min_{<_{lex}} \mathcal{S} \neq \emptyset)$.

Remark. Note that this is actually the original Nash-Williams' sense of a minimal bad sequence, except that we need here to skip empty 'slots' in the domains of sequences (i.e., natural numbers absent from the domains), and we order elements by the relation $\preceq$ on their roots, instead of by their size.

Proof. We simply build a 'minimal till i' sequence by induction for every finite i, and by assumption conclude that the corresponding limit sequence exists and therefore is a minimal sequence.

Formally, assume that $\mathcal{S} \neq \emptyset$ then let us build by induction on $n \in \mathbb{N}$ a minimal sequence f_n of length n, with no $g \in \mathcal{S}$ s.t. $g|_{\leq i_n} <_{lex} f_n|_{\leq i_n}$. For the base case let $i_1 := \min\{i \in \mathbb{N} : \exists g{\in}\mathcal{S}.\, card(\mathrm{D}g|_{\leq i}) = 1\}$, i.e. i_1 is the minimal index of the first elements in all sequences in $\mathcal{S}$. Since Q is well founded and $\mathcal{S} \neq \emptyset$ then there is a (not necessarily unique) minimal sequence g of length 1 that starts at i_1, thus let $f_1 := g|_{\leq i_1}$.

Now, having built f_n, in order to build f_{n+1} we let $S := \{g \in \mathcal{S} : g|_{\leq i_n} = f_n\}$ and put $i_{n+1} := \min\{i \in \mathbb{N} : \exists g{\in}S.\, card(\mathrm{D}g|_{\leq i}) = n+1\}$. Again, since Q is well founded we have a $<_{lex}$-minimal sequence g of length $n+1$ from S, then let $f_{n+1} := g|_{\leq i_{n+1}}$.

Let f be the limit sequence $f := \lim_{n\to\omega} f_n$. By closure assumption for $\mathcal{S}$ we have $f \in \mathcal{S}$. Further, f is clearly $<_{lex}$-minimal in $\mathcal{S}$, otherwise it would have contradicted our choice of f_n for some $n \in \mathbb{N}$. □

(2) CL$(\mathcal{K}_{\alpha+1})$ for all $\alpha < \omega_1$.

If every finite initial segment of some sequence $s \in Seq$ is bad, then clearly $s \in Bad$. Furthermore, since the $\lhd$ relation is element-wise then again if for every initial segment of $s \in Seq$ that ends at $j \in \mathbb{N}$, we have $\forall i \in \mathrm{D}s.\, i \leq j \to s(i) \lhd h_\alpha(i)$, then obviously $s \lhd h_\alpha$.

(3i) $\min_{<_{lex}} \mathcal{K}_0 \neq \emptyset$.

Proof. Similarly to (2), $\mathcal{K}_0 = Bad$ is closed since if a sequence $z \in Seq$ is not bad then there is a *good* finite initial segment $z|_{\leq j}$ for some $j \in \mathbb{N}$, which implies that not all finite initial segments of z could be an initial segments of some sequence in Bad.

By assumption $Bad \neq \emptyset$ then (1) above implies that $\min_{<_{lex}} \mathcal{K}_0 \neq \emptyset$. □

(3ii) This statement, $\mathcal{J}_{\alpha+1} = \emptyset$, is at the heart of the proof. Equivalently, we have the following lemma.

Lemma 3.11 *If i.h.*(β) *holds then there is no* $g \in Bad$ *s.t.* $g \lhd_{\prec} h_\beta$.

Proof. Assume that the lemma is false and let $g \in Bad$ be such that $g \lhd_{\prec} h_\beta$.

We have to deal with three different cases: (i) $\beta = 0$; (ii) $\beta = \gamma + 1$ for some $\gamma < \omega_1$; (iii) $\beta = \lambda$ is a limit ordinal $< \omega_1$.

(i) $\beta = 0$.
Define

$$l := g'_0 \otimes (g|_{Dg'_0}) \ .$$

We need to show that $l \in Bad$ and $l <_{lex} g'_0$ in contrast to the base case in which we took a $<_{lex}$-minimal sequence $g'_0 \in Bad$.

By definition l and g'_0 are identical up to $\min(\mathrm{D}g)$. We have

$$g \lhd_{\prec} h_0 \subseteq g'_0 \ \Rightarrow \ \mathrm{D}g \subseteq \mathrm{D}g'_0 \ ,$$

and

$$g \lhd_{\prec} h_0 \ \Leftrightarrow \ \forall i \in \mathrm{D}g.\, g(i) \lhd_{\prec} h_0(i) = g'_0(i) \, .$$

Consequently, $g \lhd_{\prec} g'_0$ which means that both $g(j)$ and $g'_0(j)$ are defined and $g(j) \lhd_{\prec} g'_0(j)$, hence $r(g(j)) \prec r(g'_0(j))$. Therefore, by $<_{lex}$ definition we have $l <_{lex} g'_0$.

Now, assume by a way of contradiction that l is good. Hence there are $y \hookrightarrow t$, where $y \in g'_0|_{\min(\mathrm{D}g)}$ and $t \in g$ (if $g'_0|_{<\min \mathrm{D}g} = \emptyset$ then the claim is trivial). Let w be the corresponding supertree of t in h_0 Since $g \lhd_{\prec} h_0$ then $r(t) \prec r(w)$ and $r(t)$ has the minimum label on the access path to t in w. Therefore $y \hookrightarrow w$, a contradiction to the badness of h_0.

(ii) $\beta = \gamma + 1$ for some $\gamma < \omega_1$.

We abuse the notation slightly and write $\beta{-}1$ for γ. We have $g \lhd_{\prec} h_\beta \unlhd h_{\beta-1}$, thus by property (3.1) (see on p.13) we get

$$g \lhd h_{\beta-1} \, . \tag{3.13}$$

Define

$$l := g'_\beta \otimes (g|_{\mathrm{D}g'_\beta}) \, .$$

By (3.13) we have that

$$l \lhd h_{\beta-1} \, . \tag{3.14}$$

By the construction of h_β we have $g \lhd_{\prec} h_\beta = h_{\beta-1} \otimes g_\beta$. This means also that $g \lhd_{\prec} g'_\beta \supseteq g_\beta$ (recall that the relation $\lhd_{\prec}$ on Seq is element-wise and ignores undefined places). Similarly to the previous case then, $g \lhd_{\prec} g'_\beta$ implies

$$l <_{lex} g'_\beta \, . \tag{3.15}$$

We need to show now that l is bad, which, combined with (3.14) and (3.15), contradicts corollary (3.10). Assume that l is good. Hence there are $s \hookrightarrow t$ for $s \in g'_\beta|_{<\min \mathrm{D}g}$ and $t \in g$ (if $g'_\beta|_{<\min \mathrm{D}g} = \emptyset$ then the claim is trivial). Let w be the corresponding super-tree of t in g_β. Since $g \lhd_{\prec} h_\beta \supseteq g_\beta$ then $r(t) \prec r(w)$ and so $r(t)$ is the minimal label on the access path to t in w. Since $r(s) \preceq r(t)$, by the gap embedding definition, then $\forall x \in [r(w), r(t)].\, r(s) \preceq r(x)$. Thus, $s \hookrightarrow w$ which contradicts the badness of g_β (and hence of $g'_\beta \supseteq g_\beta$ too).

(iii) $\beta = \lambda$ is a limit ordinal.

We have

$$g \lhd_{\prec} h_\lambda \,. \tag{3.16}$$

Let g_λ be the limit sequence $\lim_{\alpha \to \lambda} h_\alpha$ and $f := \min_{<_{lex}} \{s \in Bad \,|\, s \lhd g_\lambda\}$ as defined in §3.2.1,.

We shall show that the sequence defined by $f \otimes g$ yields a contradiction to the definition of f itself.

By definition $h_\lambda = f\!\uparrow$, thus as before $g \lhd_{\prec} h_\lambda$ implies

$$f \otimes g <_{lex} f \,. \tag{3.17}$$

Let u be any tree in g, w its supertree in f and z the supertree of w in g_λ. By (3.16) we have $r(u) \prec r(w)$ which implies that $\min\{r(u), r(w), r(z)\} \in \{r(u), r(z)\}$, thus $u \lhd z$. We have then

$$f \otimes g \lhd g_\lambda \,. \tag{3.18}$$

Since $r(u) \prec r(w)$ then for all $v \in f|_{<\min \mathrm{D}g}$:

$$v \hookrightarrow u \Rightarrow v \hookrightarrow w$$

in contrast to f badness, hence

$$f \otimes g \in Bad \,. \tag{3.19}$$

By (3.17), (3.18) and (3.19) we reach a contradiction to the minimality of f w.r.t. $<_{lex}$.

□

3.2.6 Concluding the Minimal Bad Sequence Theorem 3.7

For any $\alpha < \omega_1$ we had built a bad increasing infinite sequence. These sequences form a sequence of distinct increasing bad sequences $\langle h_\alpha | \alpha < \omega_1 \rangle$ of length ω_1, such that $h_\beta \trianglerighteq h_\alpha$ for all $\beta < \alpha$.

In any sequence $h_{\alpha+1}$ we replace some tree, say $h_\alpha(i)$, by its *proper* subtree $h_{\alpha+1}(i)$. This process is obviously finite for every finite tree $h_0(i)$ where $i \in \mathbb{N}$. Consequently, the amount of such replacements, and hence the amount of such distinct sequences h_α, is bounded from above by $\sum_{i<\omega} |h_0(i)|$. Let λ be the supremum of the sizes of trees in h_0 then we have

$$\sum_{i<\omega} |h_0(i)| \leq \sum_{i<\omega} \lambda \leq \aleph_0 \cdot \aleph_0 = \aleph_0 \,.$$

Therefore we can have only $\aleph_0$ sequences h_α in $\langle h_\alpha | \alpha < \omega_1 \rangle$, which yields a contradiction.

□

3.3 Concluding Theorem 3.1

Let us now restate the main theorem we were intending to prove in this section.

Theorem 3.1 *$\widehat{T}$ is well quasi ordered by $\hookrightarrow$ for any wqo Q.*

Proof. We apply here the usual Nash-Williams' method [NW63]. Assume by a way of contradiction that the theorem is false. We take $m := \min_{\lhd} Bad{\uparrow}$. By theorem (3.7) such an m exists. Let S be the set consisting of *all immediate subtrees* of trees in m, that is, trees rooted by immediate children of trees in m. Since Q is a wqo then there are at most finitely many trees of one vertex in m, therefore S is infinite.

Let t be the i$^{\text{th}}$ tree of m, that is, $t = m(i)$. We let $\langle t_1, \dots t_{n_i}\rangle$ denote the finite ordered sequence consisting of the immediate subtrees of t, in the order they occur as children of $r(t)$. Hence $t = r(t)(t_1, \dots t_{n_i})$, represented as a term.

Now, if S is a wqo then let $(s_i)_{i\in \mathrm{D}m}$ be an infinite sequence defined s.t.

$$\forall i \in \mathrm{D}m.\; s_i := \langle m(i)_1, \dots m(i)_{n_i}\rangle\,.$$

Since S is a wqo then by Higman lemma [Hig52], $(s_i)_{i\in \mathrm{D}m}$ is a good sequence w.r.t. the embedding relation on finite sequences of trees from $\widehat{T}$ defined by

$$\langle s_1, \dots s_k\rangle \hookrightarrow \langle t_1, \dots, t_l\rangle \;\Leftrightarrow$$
$$\exists f : \{1,\dots,k\} \to \{1, \dots, l\} \wedge f \text{ is strictly monotone} \wedge \forall j\, (1 \le j \le k).\, s_j \hookrightarrow t_{f(j)}\,.$$

Therefore, since m is *increasing* there exists a pair of *trees* s, t in m, s.t. s precedes t and $s = r(s)(s_1, \dots s_k) \hookrightarrow r(t)(t_1, \dots, t_l) = t$ (for s and t represented as terms), where the root is mapped to the root and the immediate subtrees of s are mapped to those of t, according to the Higman embedding. Hence we arrive at a contradiction to the badness of m.

In case S is not a wqo then take a bad infinite sequence $b \subseteq S$. Since for each tree in m, the number of children pertaining to the root is finite, then we can assume that b contains at *most* one subtree for each tree in m. Therefore, $b \lhd m$ in contradiction to the minimality of m. □

4 Comparable Sub-Paths Trees

Let us denote by $\widehat{T}(A)$ the set of all trees for which internal vertices are labelled by Q and leaves labelled by $A \cup Q$ for some two independent quasi-orderings Q and A. Let us call a leaf labelled by A an A-*leaf*. All other vertices (including possible leaves) are called Q-*vertices*.

The gap embedding for $\widehat{T}(A)$, denoted by $\hookrightarrow'$, is defined the same as before except for leaves labelled by A, for which the gap condition is not applicable.

Definition 4.1 *(*$\hookrightarrow'$ ***tree embedding****) For two trees* $s, t \in \widehat{T}(A)$*, we write* $s \hookrightarrow' t$ *iff there is an embedding* $f : s \to t$ *with the following properties:*

1. *for all* Q*-vertices of* $v \in s$ *we have* $v \preceq f(v)$ *;*
2. *If* $u \in s$ *is an* A*-leaf, then* $f(u) \in t$ *is a leaf too and* $u \preceq f(u)$ *;*
3. (root gap condition) *If the root of* s*,* $r(s)$*, is not an* A*-vertex then for all vertices* u *in the access path from* $r(t)$ *to* $f(r(s))$ *we have* $u \succeq r(s)$ *;*
4. (gap condition) *For all edges* (u, v) *in* s *where* v *is not an* A*-vertex and for all* $w \in t$ *such that* $f(u) < w < f(v)$ *we have* $w \succeq v$ *.*

Also note that if there are no A-leaves in s then the gap embedding $s \hookrightarrow' t$ is the same as $\hookrightarrow$.

The main theorem of this section is the following (see also page 12).

Main Theorem 4.2 *Let* Q *be a wqo and let* T^k *be the set of all finite trees such that each path in a tree can be partitioned into* $k \in \mathbb{N}$ *or less comparable sub-paths then* T^k *is a wqo under gap embedding* $\hookrightarrow$*.*

We prove the main theorem in two steps. First we show that putting an arbitrary well-quasi-ordering on leaves from $\widehat{T}$ yields a wqo under $\hookrightarrow'$. And then show how this construction can be applied by induction k times.

Let $Seq(A)$ be the set of all infinite sequences over $\widehat{T}(A)$, $Bad(A)$ the set of bad sequences from $Seq(A)$ and $Bad\!\uparrow\!(A)$ the set of increasing sequences from $Bad(A)$. All other definitions of the previous section concerning $\widehat{T}$ remain the same when applied to $\widehat{T}(A)$.

Lemma 4.3 $\widehat{T}(A)$ *is well quasi ordered by* $\hookrightarrow$ *for any two well quasi orderings* Q *and* A*.*

To prove this we need the following variant of the minimal bad sequence theorem.

Theorem (minimal bad sequence for $\widehat{T}(A)$) 4.4 *If there is a bad infinite sequence from* $\widehat{T}(A)$ *then there exists a minimal* $m \in Bad\!\uparrow\!(A)$ *s.t. there is no infinite bad sequence* f *with* $f \lhd m$*.*

Proof. We verify that the same proof of 3.7 applies here too. We need to prove

$$Bad(A) \neq \emptyset \implies \exists m \in Bad{\uparrow}(A).\, m \in \min_{\lhd} Bad(A)\,.$$

The contradiction hypothesis (hyp) now becomes $Bad(A) \neq \emptyset \ \wedge \ \min_{\lhd} Bad(A) = \emptyset$. Consequently, for all $h \in Bad(A)$ we have some subtree subsequence $h' \in Bad(A)$ s.t. $h' \lhd h$. However, since A is a wqo then there are only finitely many trees in h' that have a root from A (these are trees of only one vertex) or else h' was good. Therefore, we can ignore all of these trees. This means that $Bad(A)$ and $Bad{\uparrow}(A)$ contain only sequences *whose roots are entirely from* Q (which are infinite, since we ignore only finitely many elements in each sequence). Consequently, since $Bad(A) = Bad$ and $Bad \uparrow (A) = Bad{\uparrow}$ we can repeat each step in the proof of 3.7. The construction and its existence proof thus remain the same. □

Proof of 4.3. Similar to the proof of (3.1). We take $m := \min_{\lhd} Bad{\uparrow}(A)$. By 4.4 such an m exists. Let S be the set consisting of *all immediate subtrees* of trees in m.

If S is a wqo then let $(s_i)_{i \in \mathrm{D}m}$ be defined s.t. $\forall i \in \mathrm{D}m.\ s_i := \langle m(i)_1, \ldots m(i)_{n_i} \rangle$, where $m(i)_k$ is the k^{th} immediate subtree of $m(i)$ (cf. p. 24). Since S is wqo then by Higman lemma $(s_i)_{i \in \mathrm{D}m}$ is good with respect to the $\hookrightarrow'$ relation extended to finite sequences of trees. Therefore, since m is *increasing* we have a contradiction as m turns out to be good too.

In case S is not a wqo, then take a bad infinite sequence $b \subseteq S$. Again we can assume w.l.g. that b contains at most one subtree for each tree in m. As A is a wqo then b has only *finitely many* trees whose roots are from A. Hence we can discard these trees from b, and get a bad infinite sequence b' such that $b' \lhd m$ in contradiction to the minimality of m. □

We saw that every infinite sequence from $\widehat{T}(A)$ has a pair of trees $s \hookrightarrow' t$ where s precedes t. Assume that A itself is a set of trees ordered by the embeddability relation $\hookrightarrow'$ and 'unfold' the leaves of s and t s.t. all leaves $u \in s$ and $v \in t$ labelled from A become the corresponding *subtrees* of s, and the upward closure of $u(v)$ becomes the tree from A that was labelling u (respectively, v). We denote these leaf-unfolded trees by s^* and t^*, respectively. In order to show that the set of such unfolded trees is a wqo under $\hookrightarrow$, theorem 4.3 is not sufficient since we have not guaranteed the gap condition for leaves in $\widehat{T}(A)$. Indeed we can see in figure 4 that the leaf $u \in s$ is embedded with gap into $v \in t$ but since the gap condition for u does not hold, then $f : s^* \not\hookrightarrow t^*$.

Consequently, in order to keep this gap condition we should take care that *immediate predecessors of leaves are mapped to each other*, and not to other internal vertices. This is what we do next.

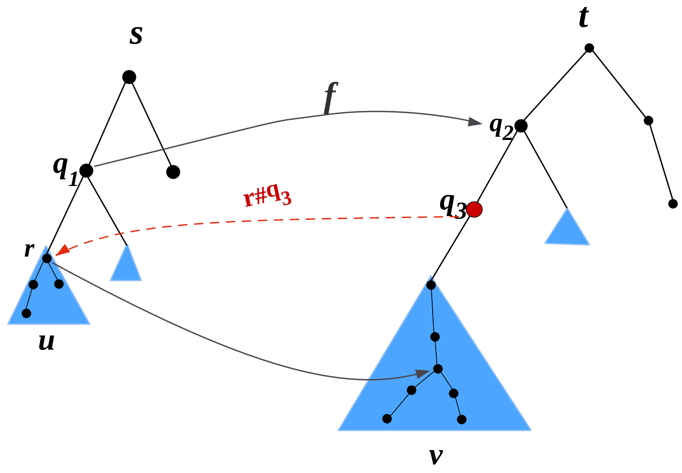


Figure 4.1: $\hookrightarrow'$ embedding between two trees s, t in $\widehat{T}(A \cup Q)$, does not yield necessarily $s \hookrightarrow t$.

Let $\widehat{T}$ be the set of path comparable trees over Q and let A be some other set of labelled trees. We shall denote by $\widehat{T}*(A)$ the set of trees from $\widehat{T}(A)$ s.t. each leaf from A is unfolded. Accordingly, $\widehat{T}^*(\widehat{T})$ is the set of all trees labelled from Q s.t. each tree path can be partitioned into two or less comparable sub-paths.

For two trees $s, t \in \widehat{T}(\widehat{T})$ such that $f : s \hookrightarrow' t$, let $\mathrm{rng}(f) \subseteq t$ denote the range of f, and let $s^*, t^* \in \widehat{T}^*(\widehat{T})$ be the corresponding leaf-unfolded trees for s, t, respectively.

Observation. For all $s, t \in \widehat{T}(\widehat{T})$ such that $f : s \hookrightarrow' t$ we have

$$\forall u \in \mathrm{rng}(f).\, leaf(u) \to pred(u) \in \mathrm{rng}(f) \qquad (4.1)$$
$$\implies \quad s^* \hookrightarrow t^*$$

Let ∞ be a new maximum element of Q such that $\infty \succ x$ for all $x \in Q$. For all leaves v of trees in $\widehat{T}(\widehat{T})$ we shall break the edge (u, v) into two edges, (u, x) and (x, v), where x is a new vertex labelled by ∞ and $u \leq x \leq v$ for the tree order $\leq$. Denote this set by T_∞. Since ∞ is comparable to all elements of Q then all the paths of T_∞ (excluding perhaps the leaves) are comparable. Further, since $\widehat{T}$ is a wqo, then by 4.3 T_∞ is a wqo under $\hookrightarrow'$.

Lemma 4.5 *For two trees $s_\infty, t_\infty \in T_\infty$, let $s, t \in \widehat{T}(\widehat{T})$ be their original corresponding trees, achieved by deleting all ∞ vertices, respectively. Then $s_\infty \hookrightarrow' t_\infty$ implies that $s^* \hookrightarrow t^*$.*

Proof. Let f be the gap embedding function for $s_\infty \hookrightarrow' t_\infty$. Each ∞ leaf in s_∞ ought to be mapped to some ∞ leaf in t_∞. We shall show that whenever $v \in \operatorname{rng}(f)$ for a leaf $v \in t_\infty$, then its grandfather $u = pred(pred(v))$ is in $\operatorname{rng}(f)$ too.

That there is a $\hookrightarrow'$ embedding with gap between s and t is immediate (we just contract the ∞ edges. Recall that this embedding does not necessarily respects the gap condition for leaves). Now, take a leaf v in t such that $v \in \operatorname{rng}(f)$ and let $w \in s_\infty$ be its source leaf in the embedding f, that is, $f : w \mapsto v$. Thus, the vertex $pred(w) \in s_\infty$ is an ∞-vertex and so, by the embedding definition, it must be mapped to some ancestor of $v \in t_\infty$, which is also an ∞-vertex in t_∞. The only such ancestor is $pred(v)$. Thus we have $f : pred(w) \mapsto pred(v)$.

Let $u := pred(pred(v)) \in t_\infty$ and assume by a way of contradiction that $u \notin \operatorname{rng}(f)$. Hence, we have that $pred(pred(w))$ is mapped to some *ancestor of* u, thus by the gap condition $u \succeq pred(w) = \infty$, a contradiction, since $u \prec \infty$. Therefore we have that $s \hookrightarrow' t$ and condition (4.1) holds, which yields, by the above observation, that $s^* \hookrightarrow t^*$. □

Corollary 4.6 *$\widehat{T}^*(\widehat{T})$ is a wqo under $\hookrightarrow$.*

Proof. By lemma 4.5, the existence of a bad sequence in $(\widehat{T}^*(\widehat{T}), \hookrightarrow)$ implies a corresponding bad sequence in the wqo $(T_\infty, \hookrightarrow')$, a contradiction. (For each $t^* \in \widehat{T}^*(\widehat{T})$ let $t \in \widehat{T}(\widehat{T})$ be its corresponding tree; and t_∞ is the corresponding tree in T_∞, achieved by putting ∞ vertices as predecessors of leaves). □

We now restate the main theorem.

Main Theorem 4.7 *Let Q be a wqo and let T^k be the set of all trees labelled from Q such that each path in a tree, beginning from the root, can be partitioned into $k \in \mathbb{N}$ or less comparable sub-paths, then T^k is a wqo under gap embedding $\hookrightarrow$.*

Proof. The proof is a simple corollary of 4.6. By 4.6 the theorem holds for $k = 2$. Hence by 4.3, $\widehat{T}(T^2)$ is also a wqo under $\hookrightarrow'$, which implies that $\widehat{T}^*(T^2) = T^3$ is a wqo under $\hookrightarrow$ too. (Note that in the proof of 4.5, we only use the fact that the labels of the leaves form a wqo, therefore any wqo labelling of leaves (e.g. T^2) is applicable to it and hence also to corollary 4.6.) Hence by induction the theorem holds for k. □

5 Conclusions and Directions of Further Research

We have showed how to extend the gap embedding to trees with wqo labels. We also showed that bounding the amount of comparable sub-paths in each tree path is not only sufficient but necessary in order to obtain a well-quasi-ordering.

Comparing our minimal bad sequence theorem (MBS) (3.7) and that of [Kří89], we notice that the conditions for the abstract framework introduced by Kříž for a *quasi-ordering with gap condition* are all satisfied (Ibid. definition 1.4). Hence we obtain a similar result to the main theorem of [Kří89], that is, *an abstract formulation of MBS which holds for an arbitrary wqo* (no concept of bounding comparable paths is relevant here, since we only introduce this concept so that the set of finite labelled trees would comply with the conditions of the abstract formulation of the MBS). This can be proved by simple verification that each of the conditions in section 1.4 [Kří89] holds also in the proof of (3.7) in this work.

A natural application of our main result should be a corresponding extension of Okada-Takeuti q.o.d. [OT87], so that quasi ordered label would not be restricted to leaves only. In this context, Okada showed in [Oka88] how to extend Buchholz's variant of the Hydra game [Buc87] from ordinals less then $\omega + 1$ labels to ordinals below ϵ_0. The termination of the Hydra game, as defined by Okada, is proved by exploiting its relationship with ordinal-diagrams. Consequently, we can further extend this result to a game on trees labelled by some wqo Q, with a fixed bound k on their comparable sub-paths partitions (i.e. what we denote by T^k), where Q has no chain of length greater than ϵ_0.

As mentioned in the introduction, [Wei92] speaks of the interrelations between a precedence relation on function symbols and the induced termination ordering. Precedence orderings seem to transfer elementary properties to the corresponding termination orderings.

It remains to find the best way to use our extension of Kruskal theorem or that of Kříž, in order to construct a new recursive path ordering which can orient non-simplifying rewrite systems (i.e. systems that lack the subterm property). We have done a preliminary attempt in this direction. As mentioned in the introduction, some applications of the gap embedding (with a somewhat different definition) for proving termination of term graph rewriting was introduced in [Oga95].

With regard to pure proof theoretical matters, the strength of theorem (2.3) which states that $(T_{On}, \hookrightarrow)$ is a wqo, i.e., its proof theoretic ordinal, is not known yet [Sim99]. Furthermore, it might be that the strength of the universal statement "for all natural k, T^k is a wqo" is bigger than that of 2.3.

References

[Buc87] Wilfried Buchholz. An independence result for $(\Pi^1_1\text{-CA})+\text{BI}$. *Ann. Pure Appl. Logic*, 33(2):131–155, 1987.

[Der82] Nachum Dershowitz. Orderings for term-rewriting systems. *Theoret. Comput. Sci.*, 17(3):279–301, 1982.

[DJ90] Nachum Dershowitz and Jean-Pierre Jouannaud. Rewrite systems. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B: Formal Methods and Semantics, pages 243–320. North-Holland, Amsterdam, 1990.

[FRS87] Harvey Friedman, Neil Robertson, and Paul Seymour. The metamathematics of the graph minor theorem. In *Logic and combinatorics (Arcata, Calif., 1985)*, volume 65 of *Contemp. Math.*, pages 229–261. Amer. Math. Soc., Providence, RI, 1987.

[Gor89] L. Gordeev. Generalizations of the one-dimensional version of the Kruskal-Friedman theorems. *Journal of Symbolic Logic*, 54(1):101–121, 1989.

[Gor90] L. Gordeev. Generalizations of the Kruskal-Friedman theorems. *Journal of Symbolic Logic*, 55(1):157–181, 1990.

[Hig52] G. Higman. Ordering by divisibility in abstract algebras. *Proc. London Mathematical Society*, 2:326–336, 1952.

[JR98] Jean-Pierre Jouannaud and Albert Rubio. Rewrite orderings for higher-order terms in η-long β-normal form and the recursive path ordering. *Theoret. Comput. Sci.*, 208(1-2):33–58, 1998. Rewriting techniques and applications (New Brunswick, NJ, 1996).

[Kru60] Joseph B. Kruskal. Well-quasiordering, the tree theorem, and Vazsonyi's conjecture. *Trans. American Mathematical Society*, 95:210–223, 1960.

[Kří89] I. Kříž. Well-quasiordering finite trees with gap-condition. proof of Harvey Friedman's conjecture. *Ann. of Math.*, 130:215–226, 1989.

[Kří95] I. Kříž. The structure of infinite Friedman trees. *Advances in Mathematics*, 115:141–199, 1995.

[NW63] C.ST.J.A. Nash-Williams. On well-quasi-ordering finite trees. *Proc. Cambridge Phil. Soc.*, 59:833–835, 1963.

[Oga95] M. Ogawa. Simple gap termination on term graph rewriting systems. In *Theory of Rewriting Systems and Its Application*, pages 99–108, 1995. Research report 918, RIMS, Kyoto Univ, Kyoto., also available at http://www.brl.ntt.co.jp/people/mizuhito.

[Oka88] Mitsuhiro Okada. Note on a proof of the extended Kirby-Paris theorem on labeled finite trees. *European J. Combin.*, 9(3):249–253, 1988.

[OT87] Mitsuhiro Okada and Gaisi Takeuti. On the theory of quasi-ordinal diagrams. In *Logic and combinatorics (Arcata, Calif., 1985)*, volume 65 of *Contemp. Math.*, pages 295–308. Amer. Math. Soc., Providence, RI, 1987.

[Rat94] Michael Rathjen. Admissible proof theory and beyond. In *Logic, methodology and philosophy of science, IX (Uppsala, 1991)*, volume 134 of *Stud. Logic Found. Math.*, pages 123–147. North-Holland, Amsterdam, 1994.

[RS84] Neil Robertson and P. D. Seymour. Graph width and well-quasi-ordering: a survey. In *Progress in graph theory (Waterloo, Ont., 1982)*, pages 399–406. Academic Press, Toronto, ON, 1984.

[Sim85] S.G. Simpson. Nonprovability of certain combinatorial properties of finite trees. In L. A. Harrington, editor, *Harvey Friedman's research on the foundation of mathematics*, pages 87–117. Elsevier, 1985.

[Sim99] S. G. Simpson. Open problems in reverse mathematics. url: http://www.math.psu.edu/simpson/cta/problems/problems.html, December 1999.

[Tak77] Gaisi Takeuti. *Proof Theory*. North-Holland, 1 edition, 1977.

[Wei92] Andreas Weiermann. Proving termination for term rewriting systems. In *Computer science logic (Berne, 1991)*, volume 626 of *Lecture Notes in Comput. Sci.*, pages 419–428. Springer, Berlin, 1992.