

Quasi-Ordered Gap Embedding*

—Extended Abstract—

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Kruskal’s Tree Theorem [3], stating that finite trees are well-quasi-ordered under homeomorphic embedding, and its extensions, have played an important rôle in both logic and computer science. In proof theory, it was shown to be independent of certain logical systems by exploiting its close relationship with ordinal notation systems (cf. [6]), while in computer science it provides a common tool for proving the termination of many rewrite-systems via the *recursive path* and related orderings [1]. For demonstrating termination of rewriting, it is beneficial to use a *partial* (or *quasi-*) ordering on labels, rather than a total one.

In [7], it was shown that many important order-theoretic properties of the well-partial-ordered precedence relations on function symbols carry over to the induced termination ordering. This is done by defining a general framework for precedence-based termination orderings via (so-called) *relativized ordinal notations*. Based on a few examples, it is further conjectured that every such application of a partial-order to an ordinal notation system carries the order-theoretic properties of the partial-order to the relativized notation system. An example of such a construction, using Takeuti’s ordinal diagrams, is introduced in [5] under the name *quasi-ordinal-diagrams*. The definition of these diagrams is the only result we know of that deals with gap embedding of trees and *quasi-ordered* labels.

Kříž [2] proved a conjectured extension by Harvey Friedman of the Tree Theorem, which states that finite trees labelled by ordinals are well-quasi-ordered under gap embedding. Our work extends this further to finite trees with well-quasi-ordered labels, with the following restriction (which is shown necessary): *Every node is comparable with all its ancestors*.

Let \mathcal{T} be a set of *ordered* (rooted, plane-planted) finite trees, with nodes well-quasi-ordered by \prec , and with the above restriction. Let t^\bullet denote the root of tree t . There is a (*gap*) *subtree* relation \triangleright on trees (which includes all immediate subtrees) with the following properties:

$$s \triangleright t \triangleright u \wedge t^\bullet \prec u^\bullet \Rightarrow s \triangleright u \quad (1)$$

$$s \triangleright t \triangleright u \wedge s^\bullet \prec t^\bullet \Rightarrow s \triangleright u \quad (2)$$

$$s \triangleleft t \Rightarrow s^\bullet \prec t^\bullet \vee t^\bullet \prec s^\bullet \quad (3)$$

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There is also an (*gap*) *embedding* relation \hookrightarrow on trees with the properties:

$$s \hookrightarrow t \trianglelefteq u \wedge t^\bullet \lesssim u^\bullet \Rightarrow s \hookrightarrow u \quad (4)$$

$$s \hookrightarrow t \trianglelefteq u \wedge s^\bullet \lesssim u^\bullet \Rightarrow s \hookrightarrow u \quad (5)$$

A *sequence* s is a partial function $s : \mathbf{N} \rightarrow \mathcal{T}$. If $s(i)$ is not defined we write $s(i) = \perp$. It is very convenient to extend the subtree relation and node ordering to empty positions of a sequence, so that: $t \triangleright \perp$ and $t^\bullet \lesssim \perp^\bullet$.

Let Seq be the set of ω -sequences of trees from \mathcal{T} . Define:

$$\begin{aligned} Ds &:= \{i \in \mathbf{N} \mid s(i) \neq \perp\} \\ \text{Bad} &:= \{s \in \text{Seq} \mid \forall i < j \in Ds. s(i) \not\hookrightarrow s(j)\} \\ \text{Sub } h &:= \{s \in \text{Seq} \mid \forall i \in Ds. h(i) \triangleright s(i)\} \\ \text{Inc } k &:= \{s \subseteq k \mid \forall i < j \in Ds. s^\bullet(i) \lesssim s^\bullet(j)\} \end{aligned}$$

where \triangleright is the *proper* (gap) subtree relation. We'll say that s is *infinite* when Ds is.

Since \lesssim is a well-quasi-ordering, $\text{Inc } k$ is nonempty, as long as k is infinite.

Theorem 1. $\text{Bad} = \emptyset$.

In other words, for every $s \in \text{Seq}$ there exist $i < j \in Ds$ such that $s(i) \hookrightarrow s(j)$. This extends the result of Kříž to quasi-ordered labels.

Assuming the contrary, the proof builds a minimal counterexample, minimal in the sense that no infinite sequence of proper (gap) subtrees of its elements is also bad (which leads to a contradiction—as in the original proof by Nash-Williams [4]). The construction of the minimal bad sequence proceeds by ordinal induction as follows:

$H(0)$	$h \in \text{Bad}$ if $\text{Bad} \cap \text{Sub } h = \emptyset$ then return h $h_0 \in \text{Inc } \text{lex}(h)$
$H(\alpha + 1)$	if $\text{Bad} \cap \text{Sub } h_\alpha = \emptyset$ then return h_α $k := \text{lex}(h_\alpha)$ $\forall i \in \mathbf{N}. f(i) := \begin{cases} k(i) & \text{if } h_\alpha^\bullet(i) \lesssim k^\bullet(i) \\ \perp & \text{otherwise} \end{cases}$ $g \in \text{Inc } f$ $\forall i \in \mathbf{N}. h_{\alpha+1}(i) := \begin{cases} h_\alpha(i) & \text{if } i < \min Dg \\ g(i) & \text{otherwise} \end{cases}$
$H(\lambda)$	$\forall i \in \mathbf{N}. \ell(i) := \lim_{\gamma \rightarrow \lambda} h_\gamma(i)$ if $\text{Bad} \cap \text{Sub } \ell = \emptyset$ then return ℓ $h_\lambda \in \text{Inc } \text{lex}(\ell)$

where the construct $s \in S$ chooses an arbitrary s from S ($s = \perp$ if $S = \emptyset$). The function $\text{lex} : \text{Bad} \rightarrow \text{Bad}$ chooses a bad sequence of subtrees with (lexicographically) minimal

labels:

$\text{lex}(h) :$	$K := \text{Bad} \cap \text{Sub } h$ for $i := 1$ to ∞ do $m := \min\{a^\bullet(i) \mid a \in K\}$ $k(i) := \{a(i) \mid a \in K, a^\bullet(i) = m\}$ $K := \{a \in K \mid a(i) = k(i)\}$ $k := K$ return k
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By construction, we have (for all α and i):

$$Dh_\alpha \supseteq Dh_{\alpha+1} \tag{6}$$

$$h_\alpha(i) \supseteq h_{\alpha+1}(i) \tag{7}$$

$$h_\alpha^\bullet(i) \lesssim h_{\alpha+1}^\bullet(i) \tag{8}$$

For each sequence h_α (for every countable ordinal α and for all $i < j \in Dh_\alpha$):

$$h_\alpha(i) \not\prec h_\alpha(j) \tag{9}$$

$$h_\alpha^\bullet(i) \lesssim h_\alpha^\bullet(j) \tag{10}$$

The successor step of (9,10) is proved by induction; the only interesting case is $i < \min Dg \leq j$, when

$$h_{\alpha+1}^\bullet(i) = h_\alpha^\bullet(i) \lesssim h_\alpha^\bullet(j) \lesssim f^\bullet(j) = k^\bullet(j) = h_{\alpha+1}^\bullet(j)$$

from which (9) follows using (5). By considering the limit case, it can be seen that for all $\alpha < \beta$:

$$Dh_\alpha \supseteq Dh_\beta \tag{11}$$

Finally, it can be shown that:

1. The constructed sequences h_α are all infinite.
2. The constructed sequences h_α are each distinct.
3. The construction eventually terminates with a minimal bad sequence.

By induction, this result may be extended also to the case where every path in the tree can be partitioned into some bounded number of subpaths with comparable labels.

Moreover, the absence of such a bound yields a bad sequence with respect to gap-embedding: Let a, b, c be three incomparable elements of the node ordering. The following is an antichain with respect to gap embedding:

$$c - a - c \quad c - b - a - c \quad c - a - b - a - c \quad c - b - a - b - a - c \dots$$

Consequently, the extension of Theorem 1 to set of trees with bounded number of comparable subpaths shows that the above counterexample is *canonical*: Every bad sequence with respect to gap embedding must contain paths of unbounded incomparability.

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