AUTOMATED REASONING

SLIDES 10:

CLAUSAL TABLEAUX

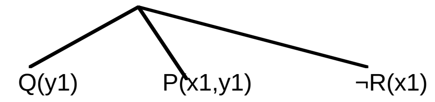
Model Elimination
Short-cuts: Lemmas and Merging
LeanCop Theorem Prover

KB - AR - 09

Clausal Tableaux and Linear Strategies

- In **Clausal Tableaux** all sentences are clauses.
- **Clause Extension rule** is derived from free variable γ-rule and ∨-splitting.

eg using Q(y) ∨ P(x,y) ∨ ¬R(x)


Q(y1)
P(x1,y1)
¬R(x1)

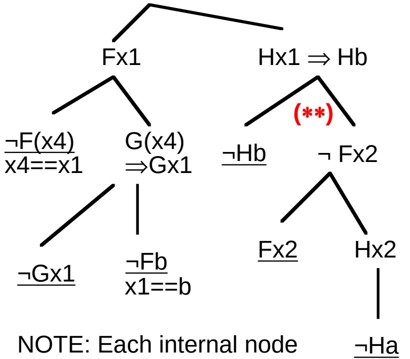
Closure rule is the free variable closure rule

- Development follows a **Linear strategy :**
- Select an initial clause called **top** in set of support (i.e top is necessary for closure to occur).
- Select a branch B (usually work from left to right) and a clause C with a literal that is complementary to current leaf L of B. (Re)order literals in C to close L in selected branch with leftmost literal of C.
- May also be able to close other branches below L with other literals in C.
- Either: propagate bindings as they are made (usual method), or record potential closures for later solution.

- Called a **connection** tableau, or **Model Elimination** (ME) tableau.
- Do not need to use a clause that results in a literal being duplicated in a branch. Then called a **regular** tableau.
- **Note**: P(x1) and P(x2) are not duplicates! x1 and x2 could end up being bound to different values.

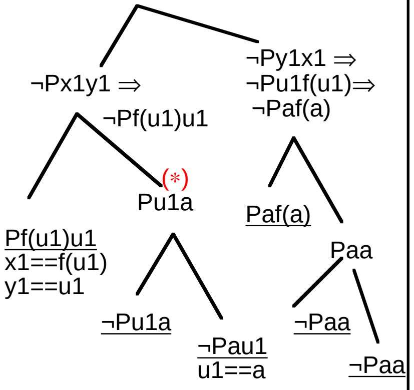
Example Model Elimination Tableaux

¬Ha , ¬Gx ∨ ¬Fb, ¬Fx ∨ ¬Hb,
Gx ∨ ¬Fx, Fx ∨ Hx


Fx1
Hx1 ⇒ Hb
¬F(x4)
x4==x1
G(x4)
⇒Gx1
¬Hb
(**)
¬ Fx2
¬Gx1
¬Fb
x1==b
Fx2
Hx2
¬Ha
x2==a

NOTE: Each internal node matches leftmost leaf literal immediately below. Reorder used clause if needed. eg at (**)

¬Pxy ∨ ¬Pyx, Pf(u)u ∨ Pua,
Pvf(v) ∨ Pva


¬Px1y1 ⇒
¬Pf(u1)u1
¬Py1x1 ⇒
¬Pu1f(u1)⇒
¬Paf(a)
Pf(u1)u1
x1==f(u1)
y1==u1
(*)
Pu1a
Paf(a)
Paa
¬Pu1a
¬Pau1
u1==a
¬Paa
¬Paa

Note there is no possible closure at (*) between ¬Pf(u1)u1 and Pu1a due to occurs check.

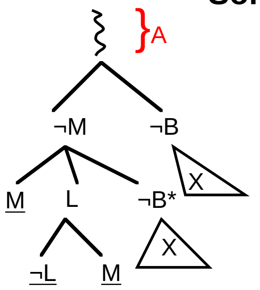
Model Elimination Tableaux:

The examples of Model Elimination tableaux shown on 10bi illustrate several features of connection tableaux. In the left-hand example notice that in the extension below Fx1 an explicit introduction of x4 for x in the use of the clause ¬Fx ∨ Gx is made. The resulting literal ¬Fx4 is matched with Fx1 to give closure with binding (x4==x1}. This binding is then propagated through the tableau (indicated by ⇒). These steps can be combined, and in subsequent steps are, to save unnecesary introduction of new free variables. Thus in the next step below Gx1, a copy of ¬Gx ∨ Fb is taken, implicitly using new free variable x3, to enable closure between ¬Gx3 and Gx1; x3 is immediately bound to x1 and only the value after closure is shown. This saves some clutter in depicting the tableau. Note also the reordering of ¬Fx ∨ Hb so the leftmost branch closes below Hb.

In the example on the right the introduction of fresh variable u1 in the first step is made explicit, so the copy (of Pf(u)u ∨ Pua) uses free variable u1. This is reasonable here, as it is the older variable x1 that is bound (x1==f(u1)), not the new one, u1. The bindings must be propagated in the tableau, so ¬Px1y1 becomes ¬Pf(u1)u1 and ¬Py1x1 becomes ¬Pu1f(u1).
(In fact, since y1 is also bound to u1, it isn't necessary to introduce u1 here either, since an implicit u1 could be bound to y1 leading to x1==f(y1). However, it is clearer to introduce u1, I think.) Notice that the possible closure between ¬P(x1,y1) = ¬Pf(u1)u1 and Pu1a fails. When u1 is later bound to a this is propagated to ¬Pu1f(u1), which becomes ¬Paf(a). Closing a branch by unifying the leaf with a literal higher in the same branch (eg beneath ¬Pau1) is sometimes called *ancestor resolution, or ancestor matching*.

It is also possible to delay propagation of unifications on closure until the end. In the second tableau a possible closure at depth 2 could be derived if the following unifiers could be combined: {x1==f(u1), y1==u1}, {x1==u1, y1==a}, {v1==y1, x1=f(v1)}, {y1==v1, x1==a}. (v1 is introduced in the right-hand branch using the free variable instance Pv1f(v1) ∨ Pv1a of clause 3.) These unifiers cannot be combined, so some other closed tableau must be found.

Some Short cuts: Merging



The refutation X (found beneath the rightmost occurrence of ¬B) could also be used below the occurrence at ¬B*. **Why?**

This step is valid only because the tableau is developed left to right; all ancestors of ¬B (indicated by (A)) are available also to ¬B*.

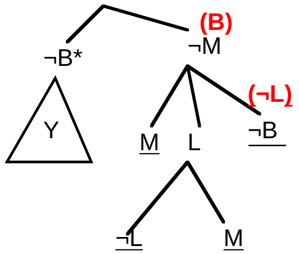
On encountering ¬B* and noticing that ¬B occurs also to the right in the ME tableau, can close ¬B* by *merging*.

Merging is the tableau version of factoring. In the first order case, analogous to safe factoring, merging is usually restricted so that variables in ¬B and any other unclosed branches on the right of ¬B* are not bound by the merge step unifier. Those in ¬B* may be.

eg1: if ¬B* is ¬G(a) and ¬B is ¬G(x1) then merging binds x1==a; it may be that ¬G(a) can be closed at ¬B* but not at ¬B, whereas ¬G(x1) does close at B but for x1==c (say).

eg2: ¬B* is ¬G(x1) and ¬B is ¬G(a) and a second sibling of ¬B is H(x1). If x1==a is no good for H(x1) it is better not to make the merge. Since one doesn't know this when at ¬B* merge is not the best option necessarily.

Some Short cuts: Re-use



In this tableau the second occurrence of ¬B occurs in the right hand branch below the sibling of ¬B* (i.e. ¬M) so merge is not available on encountering the first occurrence at ¬B*.

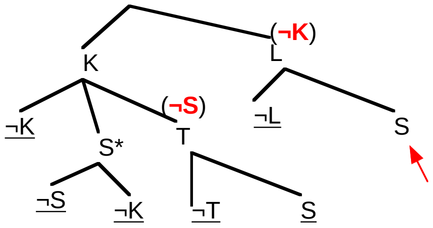
Instead, can use ***Re-use***: once a closure below a literal has been found, any other occurrences can use the same closure (as long as the necessary ancestors are available).

Can use closure Y below ¬B. Simulate this by placing (B) in the branch to represent closure below ¬B*, so when ¬B is encountered can use closure rule.

Similarly, can use (¬L) to represent closure beneath L in the 3rd branch. This is ok since the ancestors of L used in the closure beneath it are ¬M, and ¬M is in the 4th branch.

In general, re-use is usually used in two cases only: (i) when no ancestors were required in closure beneath a literal, or (ii) when the second closure is beneath a sibling branch of the first closure (both cases in example).

Example showing when re-use is inapplicable



After closing occurrence of S at S*, notice that ancestor K was necessary. Since K is not an ancestor of S in the right-most branch, cannot re-use here the closure made under S*.

Cannot apply re-use to S here

First order case:
Suppose K was the literal K(x1) and closure beneath it does not bind free variable x1. **What would this imply about K(x)?**

Can simulate this by adding ∀x ¬K(x) to right branch, representing that K(x) can be closed for any x. Some quite sophisticated short cuts can take place when variables remain unbound by closure - will return to this on slides 11.

Refinements of Model Elimination:

There are two simple refinements for ME-tableaux shown on 10ci/10cii, which are here called *merging* and *re-use*. (Note: in the Chapter Notes re-use was also called "Use of Lemmas".) Consider the case for **propositional tableau** first.

Important Note 1: merging and re-use *cannot both be used in a single tableau*; otherwise soundness is not in general maintained.

Important Note 2: *merging and re-use are only available for ME-tableau*; this is due to the left to right development of such tableaux.

Merging is the simplest. If a leaf literal L can be unified with another leaf literal L'in an open branch **to its right** (necessarily a sibling of L or a sibling of an ancestor of L), then the branch ending at L can be closed by *merge* without further steps. This is sound because when the (necessary) closure beneath L' is made, it can be repeated (retrospectively) beneath L. Any ancestors needed for the closure beneath L' will also be available beneath L, due to the tableau structure. *Merging is the tableau version of factoring*.

The other extension is called **re-use**. If a sub-tableau beneath a literal L at node n closes, then any other occurrences of L at nodes n' that may occur in open branches of the tableau can be closed also, as long as the ancestors needed to close L at n are also available at n'. If the subsequent occurrences of L appear at siblings of n or at descendants of siblings of n, then this will be so. Otherwise, it needs to be checked. In the simplest case, when no ancestors are needed, then any occurrence of L can be closed in the same manner as the occurrence of L at n is closed. The (re-use) rule can be implemented in a simple way by including ¬L in all branches that are known to share the necessary ancestors. Then closure will be made by the normal (ancestor matching) closure rule. Usually, implementations check only the 2 cases of sibling branches and no ancestors used, to receive the literal ¬L.

Merging in First Order Tableaux:

Assume the first occurrence (the one to be closed by merge) is L and that it is to be merged with a second occurrence L'. There are 2 basic cases to consider.

Case 1 is when bindings are required to be made to L but not to L'. This case is safe as long as the variables in L that are bound do not occur in other leaf literals in branches to the right of L or in ancestors of L. The reason is that the bindings would be propagated to those literals and they may not be appropriate to completing the tableau beneath them. This restricted case is sound because when the (necessary) closure beneath L' is made, it can be repeated beneath L, for after unification they are identical. Any ancestors needed for the closure beneath L' will also be available beneath L, due to the tableau structure.

(In fact, if the bindings affect only L and ancestors of L the merge is also safe, but see Slides 11 for a discussion of this case).

Case 2 is when bindings are required to be made to L'. This case is not usually implemented (see Slide 10ci for an example).

Re-Use in First Order Tableaux:

Assume the first occurrence occurs at leafnode *n* and the second occurrence occurs at *n*'. Either *n*' should be a descendant of a sibling of *n*, or, if closure beneath *n* involved no ancestors, then *n*' can also be a descendant of an ancestor of *n*. There are then 2 basic cases.

No ancestor involved in closure beneath *n*: if the literal at *n* has the form P[x] and there is a completed sub-tableau beneath it, which does not bind x, then this means that $\forall x P[x]$ can be proven. i.e. for any instance of P[x] a closed sub-tableau beneath it can be constructed. Thus $\forall x \neg P[x]$ can be added to the tableau. Note that, even if x occurs in other leaf literals and is later bound, this property still holds. If the literal at *n*' becomes bound by the application of Re-use, this does not affect soundness, but it may not lead to a closed tableau.

Some ancestor is involved in closure beneath *n*: This is a more complex property, as even if variables in the literal at *n* are not bound by the step, those variables could appear in an ancestor of *n*. If such variables do not also appear in any sibling of *n* or of *n*' then this case is also sound and worth considering (see Slides 11 for a brief discussion of this case). Otherwise, while still sound, the result may not lead to a closed tableau and is is not usually implemented.

Completeness of Model Elimination Tableau:

Let S be a set of minimally unsatisfiable ground clauses (ie removing any clause from S yields a satisfable set). Then a closed ME tableau exists for S starting from any top clause (from S). The proof is by induction on the number of non-unit clauses k in S, where $k \geq 0$. Therefore, let S be a minimally unsatisfiable set of ground clauses with k non-unit clauses. Assume as induction hypothesis (IH), that, for any minimally unsatisfiable set of ground clauses with $n<k$ non-unit clauses a ME tableau can be found. In order to show that a ME tableau exists for S there are 2 cases.

Case 1: k=0. In this case all clauses are unit clauses. If S is unsatisfiable then it must consist of two complementary unit clauses. One of these can be selected as the top clause and the tableau will close by extension using the other one.

Case 2: k>0. Choose as top clause a non-unit clause C, say $L1 \vee L2 \vee ... \vee Ln$. Then for each Li there must exist a clause that has a literal complementary to Li (ie containing $\neg Li$).
(**Exercise**: Show this. The proof requires to show that if for some Li such a clause did not exist then S could not be minimally unsatisfiable - eg consider pure literals.)
Consider the set of clauses S1' = S - {C} +{L1}. ie remove the clause C and add the unit clause L1. Then S1' is also unsatisfiable and L1 belongs to some minimally unsatisfiable subset of S1'. (**Exercise:** Show this.) S1' has <k non-unit clauses and the IH is applicable, using L1 as the top clause and the clause complementary to L1 as the second clause. (If this clause is a unit clause, that is not a problem.) Repeat the argument exemplified for L1 for each literal Li, i>1, in C.

It is easy to lift a ground ME tableau to the first order case, as described in Slide 9dvii.

You are encouraged to follow the proof construction to find a closed ME tableau for the ground instances $\neg Ha$, $\neg Fa \vee \neg Hb$, $Fa \vee Ha$, $Fb \vee Hb$, $Ga \vee \neg Fb$, $Ga \vee Fa$ with top clause $Fb \vee Hb$.
Exercise: Show how to adapt Case 2 for regular ME tableaux.

Constructing Model Elimination Tableaux:

Slide 10dii shows an outline program for constructing model elimination tableaux.

The predicate `show` implements the basic part of the construction (note that its clauses include only the 3 basic steps. Initial data is a list of clauses, given as the 3rd argument (arg3) and the list of leaf literals, given as arg1. The ancestor literals available to these leaf literals are in arg2, which is initially empty.

To avoid following an infinite branch, `show` has a fourth argument, the maximum depth of a tableau constructed by `show`. Each time `show` recurses, the maximum depth is reduced by 1. If it reaches 0 then only closure is allowed, not extension. The predicate `showd` controls the use of D, the Depth argument. Initially, D is a small value; it is increased if no closed tableau can be found at depth ≤D.

Various extensions of this basic structure are easy to implement, such as merging or re-use. (Remember, only <u>**one**</u> of these is possible in a given tableau.)

Later, you'll see LeanCop, which is a cleverly implemented version of the basic model elimination tableaux.

Implementing Model Elimination tableaux:

- Start with a top clause;
- Each literal at an internal node matches directly below with leftmost literal.
- A literal at a leaf node may match any literal in the branch above.
- Only one instantiation of any literal in the tableau may be made.

```
show([],A,C,D).
show([G|Rest],A,C,D) :- D≥0,complement(G,A),
show(Rest,A,C,D).
show([G|Rest],A,C,D) :- D>0,match(G,New, C),D1 is D-1,
                        show(New,[G|A],C,D1),show(Rest,A,C,D).
```

`match(G,New, C)` finds a clause in `C` with a literal `L` that unifies with, and is complementary to, `G` and has other literals `New`.

```
showd(Goals,C,D):- show(Goals,[ ],C,D), !.
showd(Goals,C,D) :- D2 is D+1,showd(Goals,C,D2).
```

`showd` controls attempts to show the Goals at ever increasing depth.

- The tableau is usually constructed in a depth first way, as in the program.
- **Initial call** is `showd(Top,C,D)` for some small initial `D` (eg `D=3`). `[Top]` is the top clause represented as a list of literals.

Exercise. Add a clause to `show` that will enforce regular tableaux.

LeanCop: A ME Theorem Prover

```
prove([],_,_,_).

prove([Lit|Cla],Mat,Path,PathLim) :-
  (-NegLit=Lit;-Lit=NegLit) ->
     (member(NegL,Path), %branch closure case
      unify_with_occurs_check(NegL,NegLit);
      append(MatA,[Cla1|MatB],Mat),
      copy_term(Cla1,Cla2), %find matching clause
      append(ClaA,[NegL|ClaB],Cla2),
      unify_with_occurs_check(NegL,NegLit),
      append(ClaA,ClaB,Cla3),
        (Cla1==Cla2 ->    %ground clause matched
          append(MatA,MatB,Mat1);
          length(Path,K), K<PathLim,%vars in clause matched
          append(MatA,[Cla1|MatB],Mat1)
        ),                %continue with same branch
      prove(Cla3,Mat1,[Lit|Path],PathLim)
      ),                  %continue with next branch
   prove(Cla,Mat,Path,PathLim).
```

Data: `Mat` is a list of clauses, each clause a list of Literals

---

```
prove(Mat,PathLim) :-
     append(MatA,[Cla|MatB],Mat),
     \+member(-_,Cla),     %top clause all positive
     append(MatA,MatB,Mat1),
     prove([!],[[-!|Cla]|Mat1],[],PathLim).
prove(Mat,PathLim) :-
     \+ground(Mat),    %if not propositional increase PathLim
     PathLim1 is PathLim+1,
     prove(Mat,PathLim1).

%Operator precedences (put at top of program)
:- op(400,fy,-),op(500,xfy,&),op(600,xfy,v),
    op(650,xfy,=>),  op(700,xfy,<=>).
```

Examples:
prove([[-h(a)], [f(X),h(X)], [-g(Z),-f(b)], [-f(Y),-h(b)[,[g(U),-f(U)]], 4)

prove([[-a,-w,p],[e],[i,a], [w,m], [-p], [-e,-i], [-e,-m]],0)

Exercises:
(1) Explain why PathLim doesn't need to increase for propositional case.
(Hint: look at test Cla1==Cla2).

(2) Add a test to enforce only regular tableau to be generated and searched.

The LeanCop Prolog Prover:

LeanCop is similar to LeanTap in that it is written in Prolog and is very compact. However, it is designed by different people: Jens Otten and Wolfgang Bibel – see the website (more up-to-date than LeanTap's) at http://www.leancop.de/

LeanCop is a Model Elimination prover, so takes clauses as input. The four arguments of `prove` are: ``current list of leaf literals, list of all clauses, current branch, current max depth of branch for search".

In one sense using clauses makes it simpler than LeanTap. In another, it makes it more complicated, as there are more possibilities for clever tricks. In particular, consider the line

```
        (Cla1==Cla2 ->    %ground clause matched
```

In case the test is true, this means that the result of the earlier call to `copy-term` did not introduce fresh variables because there were no free variables in `Cla1` to be copied. Therefore the clause `Cla1` is ground and there is no need to re-use it in the current branch in the future, so it can be discarded. Moreover, there is no need to increase `PathLim` – it is only increased when extension is by a non-ground clause instance, which potentially may have to be re-used.

As in LeanTap, if no closure is found at an initial depth, the depth is increased.

Summary of Slides 10

1. The tableau method can be applied to sets of clauses, when special development rules can be used to good effect. Since clausal form has already eliminated ∃ quantifiers, only one extension rule is required, derived from the free variable γ rule and $\vee$ rule. The closure rule uses unification.

2. The most usual development rules result in the Model Elimination method, or Connection tableaux. The first step selects a top clause. Thereafter, every extension must use a clause that has a literal which unifies with the leaf literal at the left-most open branch. This literal is placed left-most in its clause. The tableau is developed from Left to Right and depth-first.

3. If the development rules summarised in 2) are in force, then some short cuts can be incorporated, of which we considered Merging and Re-use. Merging is the tableau variant of factoring and Re-use allows whole derivations to be re-used.

4. At ground level, there are simple restrictions on merging and re-use to ensure soundness. In the general case the restrictions are tighter, and it is harder to show soundness.

5. The LeanCop theorem prover uses model elimination and uses Prolog in an elegant implementation.

6. Soundness of Model elimination follows from the soundness of ordinary free variable tableau.

7. Completeness must be proved separately, since the development imposes restrictions, which could compromise completeness.

One proof of completeness for the simple ground case uses induction on the number of non-unit clauses available in a branch is given. The ground tableau can be lifted as described on Slides 9 for general free variable tableaux.

Other proofs are possible, that construct any ground tableau using instances of the given clauses and then transform the constructed tableau into one that follows the refinement.