AUTOMATED REASONING

SLIDES 9

SEMANTIC TABLEAUX

Standard Tableaux
Free Variable Tableaux
Soundness and Completeness
Different strategies
Lean Tap

KB - AR - 12

Non-Resolution Theorem Proving

9ai

Various different techniques have been considered as suitable alternatives to resolution and clausal form for automated reasoning. These include:

- Relax adherence to clausal form:
  Non-clausal resolution (Murray, Manna and Waldinger)
  Semantic tableau /Natural deduction but with unification (Hahlne, Manna and Waldinger, Reeves, Broda, Dawson, Letz, Baumgartner, Hahlne, Beckert and many others)
- Relax restriction to first order classical logic: Modal logics, resource logics (Constable, Bundy,Wallen , D'Agostino, McRobbie,)
  Add sorts (Walther, Cohn, Schmidt Schauss)
  Higher order logics (Miller, Paulson)
  Temporal logic with time parameters (Reichgeldt, Hahlne, Gore)
  Labelled deduction (Gabbay, D'Agostino, Russo, Broda}
- Heuristics and Metalevel reasoning:
  Use metalevel rules to guide theorem provers - includes rewriting, paramodulation (Bundy, Dershowitz, Hsiang, Rusinowitz, Bachmair)
  Abstractions (Plaistead)
  Procedural rules for natural deduction (Gabbay)
  Unification for assoc.+commut. operators (Stickel)
  Use models / analogy (Gerlenter, Bundy)
  Inductive proofs, proof plans (Boyer and Moore, Bundy et al)
  Tacticals / interactive proof / proof checkers, Isabelle (Paulson)

Considered in Part II (weeks 5-9 of the course) are Tableaux methods and rewriting for equality.

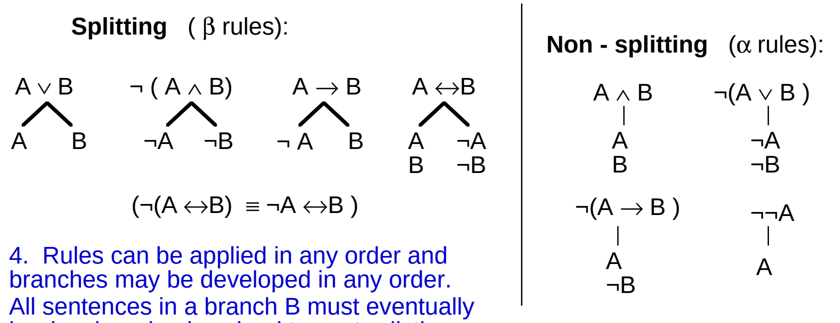
Theorem Proving with Semantic Tableaux

9bi

Proof Method: Refutation (but no translation into clausal form necessary)
Show by construction that no model can exist for given sentences
i.e. that all potential models are contradictory. Do this by following the consequences of the data - aim is to show that they all lead to a contradiction.

Development Rules:

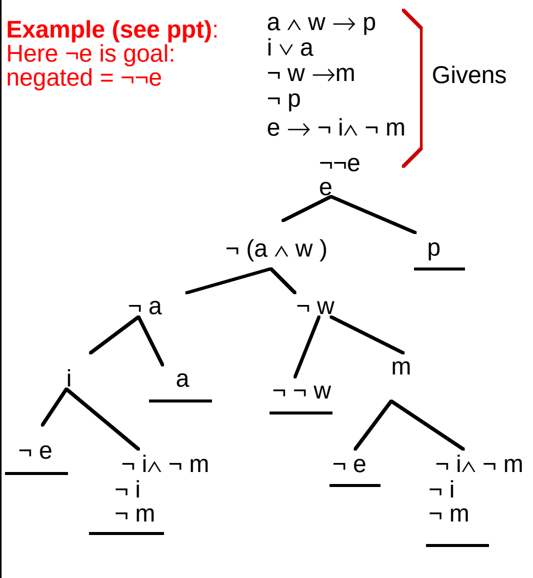
1. Conclusion is negated and added to givens (if conclusion is distinguished).
2. A tree is developed s.t. sentences in each branch give a partial model.
3. Two types of branch development using Non-splitting and Splitting rules:

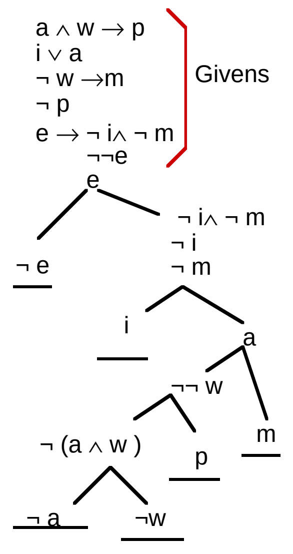


4. Rules can be applied in any order and branches may be developed in any order.
All sentences in a branch B must eventually be developed unless lead to contradiction.

9bii

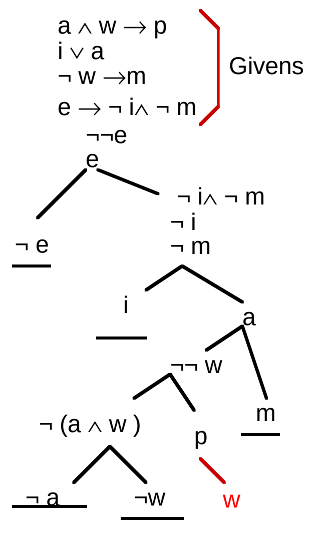
Example (see ppt):
Here ¬e is goal:
negated = ¬¬e





What if the Givens do not imply the conclusion?

9biii



Example: Givens do not |= ¬e
There is a model of Givens and ¬¬e

The branch ending in w does not close.
All data has been processed in the branch.
(It is called *saturated*.)

The assumption that a model of Givens+{¬¬e} exists is not contradicted; the branch is consistent and called open.

In fact, a model (valuation) can be read from the literals in an open branch. Each ***positive*** atom that occurs is assigned **true** and each atom that occurs ***negated is assigned* false**. Atoms that don't occur in the branch can be assigned arbitrarily, usually false.

Here: a, e, p, w =True and i, m=False.

Question: Why can no atom occur both positively and negatively in an open branch?

Semantic Tableaux

9biv

Semantic tableaux were introduced in 1954 by Beth. The standard method was automated in 1985, when the free variable method was also automated. The *Model Elimination* approach was introduced by both Kowalski and Loveland in 1970, but as a resolution refinement, not as a tableau method. This was later related to tableau methods 1990 onwards. The TABLEAUX Workshops (now part of IJCAR) are devoted to tableaux and their extended forms and related theorem proving methods.

The initial *branch* of a *semantic tableau* contains the given sentences that are to be refuted. Reasoning progresses by making assertions about the satisfiability of sub-formulas based on the satisfiability of the larger formulas of which they are a part. The α-rules (slide 9bi) can be read as "if α is in a branch and there is a model of the sentences in the branch, then there is a model of the sentences in the branch extended by α1 and α2" and the β-rules as "if β is in a branch and there is a model of the sentences in the branch, then there is a model of either the sentences in the branch extended by β1, or the sentences in the branch extended by β2", where α1, α2/β1, β2 are the two sub-formulae of the rules. If X and ¬X are in the branch then the sentences in the branch are clearly inconsistent and the branch is *closed*. If all branches in a tableau are closed (when the tableau is also said to be *closed*), then there are no possible consistent derivations of sub-formulas from the initially given sentences, and these sentences are unsatisfiable.

Two examples of closed propositional tableaux are on slide 9bii, illustrating that there can be differently sized tableaux for the same set of initial sentences. For propositional sentences the development of a tableau will always terminate as there is no need to develop a sentence in a branch more than once – to do so would duplicate one or more sub-formulas or atoms in that branch, which adds no information to the branch. However, every sentence in a branch must be developed in it. A fully developed (or completed) branch is called *open* if it is not closed, and similarly the tableau . A fully developed open branch will yield a model of the initial data.

The invariant property SATISFY:

9bv

- Each tableau extension rule maintains satisfiability:

if the sentences in a branch are satisfiable and a rule is applied, then
the new sentences in *at least one* descendant branch are satisfiable

e.g. If M is a model of a branch including i ∨ a, then M must assign true to at least one of i or a. Hence at least one of the two extended branches is still satisfied by M.

- A branch that contains both X and ¬ X is unsatisfiable and can be **closed** by the **closure rule**.

The property SATISFY is used to show Soundness of the tableau method (see Slides 9e).

Informally, if the givens are satisfiable, then at each step one branch remains satisfiable. If the tableau closes, all branches are unsatisfiable, so conclude that givens cannot be satisfiable.

Soundness and Completeness Statements for Tableaux:

The *soundness* theorem for tableaux states that "*if a set of sentences S is consistent, then the tableau developed from S will not fully close*". Equivalently, "*if the tableau developed from sentences S closes, then S is inconsistent*".

The *completenss* theorem for tableaux states that "*if a set of sentences S is unsatisfiable, then it is possible to find a fully closed tableau derived from them*".

The soundness theorem is a consequence of the property (SATISFY), which guarantees that the development rules maintain consistency in at least one descendant branch. Informally, therefore, if the original sentences are satisfiable, not all branches can close. Proofs of these properties are in Slides 9e.

First Order Tableaux (Standard Version)

There are two kinds of quantifier rules for tableaux; the *standard rules* for universal-type quantifiers (either ∀ or ¬∃) are similar to the usual ∀–elimination rule for natural deduction. That is, occurrences of the bound variable in the scope of the quantifier may be replaced by any term in the language, including terms involving other universally bound variables at the same outer level as (and hence in the scope of) the quantifier being eliminated. e.g.∀x∀y.P(x,y) could become ∀y.P(f(a),y) or even ∀y. P(y,y). The problem with this form of the rule is that the substitutions have to be guessed. (See example on 9ciii.) These rules are not often used in theorem provers (although they may be used in proofs **about** tableau provers). Instead, *free variable rules* are used instead. (Continued on 9dv.)

Rules for universal quantifiers

There are two versions of rules for dealing with ∀ (called γ rules)**:**

Standard version (not often used now except in proofs of tableau properties)
Free variable version (see slide 9civ)

Standard Universal Rules

```
∀x P[x]        ¬∃x P[x]
   |              |
  P[t1]         ¬ P[t1]
```

where t1 is a ground term from the language of the branch.

e.g. ∀x P(x,f(x)) is in a branch B
⇒ P(s,f(s)) and also
⇒ P(g(s),f(g(s)))

where s is a constant and g is a functor used in B

Example from 9ciii: from ¬∃y (pr(y)∧div(y,n)) obtain

¬(pr(n) ∧div(n,n)) Substitute constant n for variable y, or

¬(pr(f(g(n))) ∧div(f(g(n)),n)) Substitute term f(g(n)) for variable y

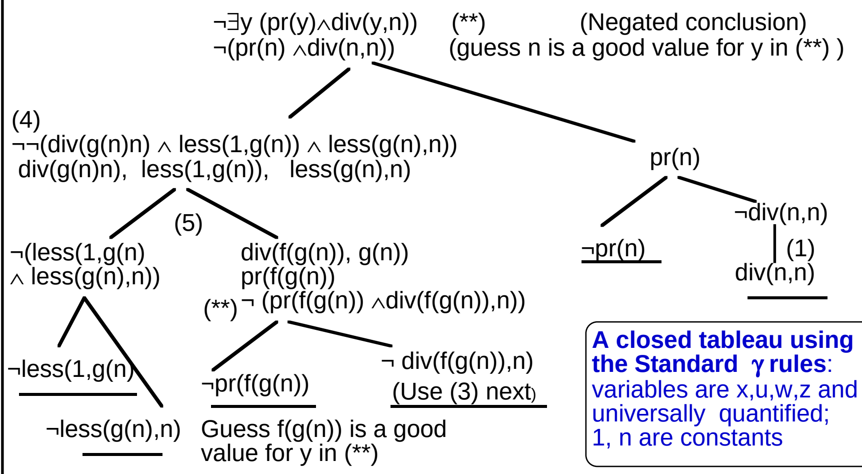
Two different instances are used, but it could be many more

In this **standard** version of the γ rules, each ∀ sentence in a branch B should eventually be used for every term that can be constructed from constants and functors appearing in the branch

Question: What problems do you forsee with these rules?

Example (see ppt)

(1) div(x,x), (2) less(1,n), (3) div(u,w) ∧ div(w,z) → div(u,z)
(4) ¬(div(g(x),x) ∧ less(1,g(x)) ∧ less(g(x),x)) → pr(x)
(5) less(1,x)∧less(x,n)→div(f(x),x)∧pr(f(x)) Show ∃y (pr(y)∧div(y,n))


¬∃y (pr(y)∧div(y,n)) (**) (Negated conclusion)
¬(pr(n) ∧div(n,n)) (guess n is a good value for y in (**))
(4)
¬¬(div(g(n)n) ∧ less(1,g(n)) ∧ less(g(n),n))
div(g(n)n), less(1,g(n)), less(g(n),n)
pr(n)
(5)
¬(less(1,g(n)
∧ less(g(n),n))
div(f(g(n)), g(n))
pr(f(g(n))
(**) ¬ (pr(f(g(n)) ∧div(f(g(n)),n))
¬pr(n)
¬div(n,n)
(1)
div(n,n)
¬less(1,g(n)
¬less(g(n),n)
¬pr(f(g(n))
¬ div(f(g(n)),n)
(Use (3) next)
Guess f(g(n)) is a good value for y in (**)

A closed tableau using the Standard γ rules:
variables are x,u,w,z and universally quantified;
1, n are constants

Free Variable Rules for Universal Quantifiers

9civ

Standard tableau rules for universal quantifiers have two problems:
i) guessing what to substitute for bound variables (of universal type), and
ii) non-termination due to infinite number of choices

In clausal reasoning resolution replaced guessing substitutions.
Free variable rules improve on standard rules by delaying γ rule substitutions.

Free variable γ rules

∀x P[x]	¬∃x P[x]
|	|
P[x1]	¬ P[x1]

where x1 is a new *free variable* in the tableau

e.g. ∀x P(x,f(x)) ⇒ P(x1,f(x1))
(x1 is new to tableau)

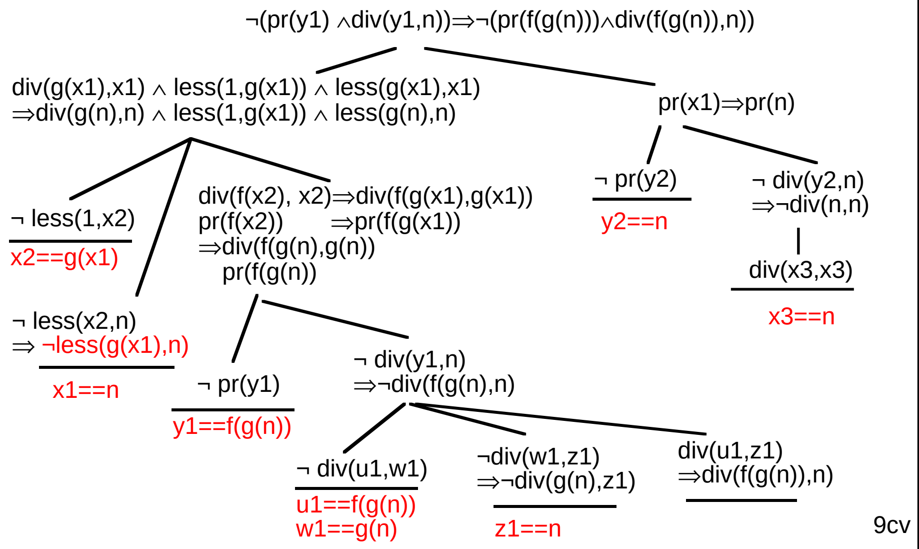
In a *free variable* tableau the CHOICE of substitution in a γ-rule application is delayed until closure.

The closure rule uses unification of terms in the matching literals to bind the free variables and find a substitution to achieve complementarity.

Wherever a free variable (x1 say) occurs in the tableau, it must be bound to the same term (if it is bound at all).

Example using Free Variable Universal quantifier rules (ppt)

(1) div(x,x), (2) less(1,n), (3) div(u,w) ∧ div(w,z) → div(u,z)
(4) ¬(div(g(x),x) ∧ less(1,g(x)) ∧ less(g(x),x)) → pr(x)
(5) less(1,x)∧less(x,n)→div(f(x),x)∧pr(f(x)) (6) ¬∃y (pr(y)∧div(y,n))



9cv

<u>Free Variable First Order Tableaux</u>

9cvi

When *free-variable* rules are used for dealing with ∀ sentences, the substituted term is a new variable, which acts like a place-holder until a suitable term can be decided. (For this reason, free variables are also sometimes called *Unknowns*.) A global binding environment for a tableau is maintained, which records eventual bindings to free variables. This is analogous to the procedure in Prolog execution (which can, in fact, be viewed as a particular free-variable tableaux development for Horn clauses).

Use of the free variable ∀ rule applied to ∀x∀y.P(x,y) yields P(x1, y1), where x1 and y1 are (fresh) free variables (note there is still the freedom for x1 to be bound to the same term as y1). The occurs check is used when unifying at closure to prevent, for example, x1 subsequently being bound to f(x1), as this would lead to infinite terms. An example would be a branch containing the pair of literals P(x1,f(x1)) and ¬P(f(x2),x2). Unification to close the branch would require to unify {x1=f(x2), f(x1)=x2}, but the unification algorithm will fail, as after setting the substitution x1==f(x2) the second equation becomes f(f(x2))=x2, which will fail the occurs check.

Existential-type quantifiers are treated to a Skolemisation process - either at run-time, or before run-time (similar to the Skolemisation step when converting to clausal form). Whereas for the standard rules the process at run-time always results in a new *constant* being introduced, for the free-variable rules it may result in a new (Skolem) *function* term whose arguments are the free variables in the sentence in the scope of the ∃.

Some observations about the Tableaux ∀ quantifier rules

1) When using the standard rules, for each ∀ sentence there should be one instance in each (open) branch in which it occurs for each substitution of terms constructed from functors and constants in the branch.

There is a potentially infinite number of instances if a functor occurs in the branch.

2) When using free variable rules, can often improve on this requirement - a completely unbound instance of a ∀ sentence captures many instances.

3) In order to maintain the property of a single binding to any free variable throughout the tableau, bindings are propagated as they are found by unification using the closure rule.

4) To develop tableaux systematically, branches are expanded to a given maximum depth, or to a maximum number of free variables.

If there is no success then the limit is increased.

9cvii

Rules for Existential Quantifiers

9di

Standard (∃) Quantifier rules (δ rules):

$$\begin{array}{ccc} \exists x P[x] & \quad & \neg\forall x P[x] \\ | & & | \\ P[a] & & \neg P[a] \end{array}$$

where a is a <u>new</u> constant symbol not occurring in the tableau (*) (also called a parameter).

e.g. ∀y ∃xP(x,y,y)
⇒ ∃xP(x,b,b) (by ∀ rule)
(say b occurs in the tableau)
⇒ P(c,b,b) (by ∃ rule)
(c is new to the tableau)

(*) – in fact, the parameter only needs to be new to the branch).

Free variable δ rules

$$\begin{array}{ccc} \exists x P[x] & \quad & \neg\forall x P[x] \\ | & & | \\ P[a] & & \neg P[a] \end{array}$$

where "a" is a term <u>new</u> to the tableau. It is dependent on the free variables occurring in ¬∀xP[x] or ∃xP[x], and is a functor of those variables, but if no free variables then "a" is a constant.

e.g. ∃xP(x,a) ⇒ P(d,a)

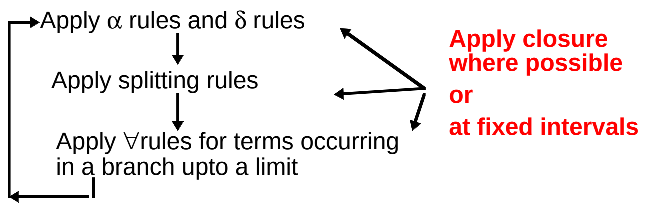
∀y,w ∃xP(x,y,w) ⇒ (by ∀ rule)
∃xP(x, y1,w1) ⇒ (by ∃ rule)
P(f(y1,w1), y1,w1)
(x is dependent on y1, w1)
(f is new to the tableau)

Strategies for Developing Tableaux

9dii

Systematic:

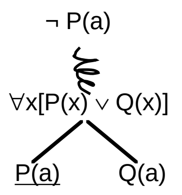
Why do we need a limit?


Apply α rules and δ rules
Apply splitting rules
Apply ∀rules for terms occurring in a branch upto a limit
Apply closure where possible
or
at fixed intervals

Heuristics:

Combine ∀ rule with a splitting rule and possibly closure

∀ + "∨" splitting + closure + choosing 'a' to substitute for the bound variable x.



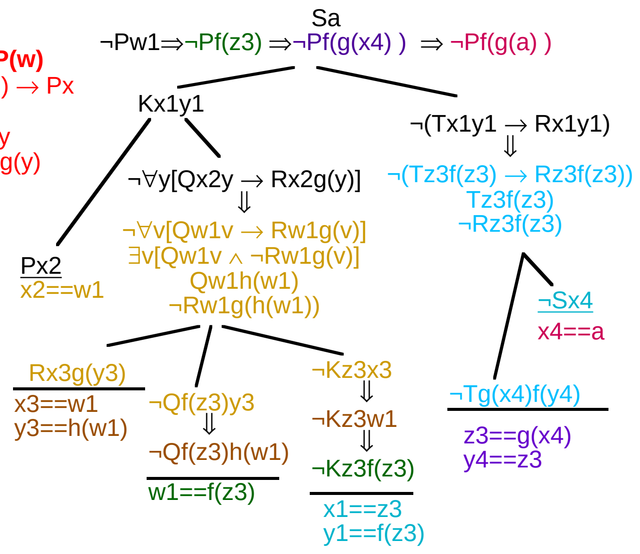
Extend the rules for A ∧/∨ B to deal with more than one operator of the same kind.

Use splitting rules that will close a branch (reduces branching)

9diii

Free variable rules (see ppt)

Show (1) - (5) |= ∃w. P(w)
(2) (∀v[Qxv → Rxg(v)]) → Px
(3) Sx → ¬Tg(x)f(y)
(4) (Txy → Rxy) → Kxy
(5) Qf(z)y ∧ Kzx → Rxg(y)
(1) Sa
(6) ∀w ¬Pw
(negated conclusion)



Variables x,y,z in (2)-(6) are universally quantified.
a is a constant.

Work from L to R.

Some observations about the Tableaux ∃ quantifier rules

1) It is often easier to initially Skolemise sentences in order to eliminate existential-type quantifiers from the data

The process is the same as we looked at earlier.
eg in clause (4) in 9cv "g" is a Skolem term
Original sentence:
$\forall x\ \exists y\ [\neg(div(y,x) \wedge less(1,y) \wedge less(y,x)\ ) \rightarrow pr(x)]$

becomes
$\forall x\ [\neg(div(g(x),x) \wedge less(1,g(x)) \wedge less(g(x),x)\ ) \rightarrow pr(x)]$

The free variable existential rules are sometimes called "run-time Skolemisation" as their effect is often similar to Skolemisating at the start.

2) Each ∃ sentence in a branch B is developed once in each branch below B.

9div

Constructing Free Variable Tableaux 9dv

When deciding on closures in a free variable tableau, you may do it in one of two extreme ways, which could be called "unify as you go", or "unify at the end". All free-variable tableaux on Slides 9 -11 are constructed using unify-as-you-go. In this kind of construction, whenever a closure is made that requires a binding to be made to one or more free variables, the substitution is applied to *all occurrences in the tableau* of those newly bound variables. This guarantees consistency of the bindings during tableau construction as it enforces the requirement that only one binding may be made to any free variable. The propagation is shown on the slides by an arrow (⇒ or ⇓). This approach is useful for most applications, especially those using data structures, when it may be necessary for a piece of data to be used many times. It has given rise to many different refinements for clausal data. See Slides 10/11. Branches of a tableau may be developed in any order, although usually we use left to right which allows some useful heuristics to be applied; different development orders result in different propagation orders and often a different tableau.

In this kind of construction it can be shown that it is unnecessary to put two unconstrained unbound variants of a universal sentence in a branch. However, as soon as such a variant is (even partially) bound, then there is scope for a second variant. eg after applying the ∀ rule to $\forall x[P(f(x)) \vee Q(x)]$, consider the branch including P(f(x1)); there is no need to use the sentence again as it would result in P(f(x2)), with both x1 and x2 unbound. If later P(f(x1)) happened to be used in closure, binding x1 to *a* (say), then sibling branches beneath P(f(x1))⇒P(f(a)) could use a second instance P(f(x2)), where the binding of x2 could be constrained to be different from *a*. This is analogous in not requiring development of a ground sentence in a branch more than once.

The alternative method of unify-at-the-end is shown and briefly discussed in Appendix 2; it is optional. In this construction, it is noted when a branch can close and what the corresponding binding is, but no propagation takes place. When every branch is potentially closed the possible substitutions are combined. If they do not successfully combine then alternative closures in one or more branches are sought. The approach is useful if it is known that one (or only a few) occurrences of a piece of data will be needed.

Benefits of the Tableau Approach

9dvi

- Uses the original structure of knowledge - no need to convert to clausal form and so no exponential expansion in presence of ↔ sentences;

- Extends to non-classical logics very easily - most non-classical automated techniques use tableaux as they can mimic the semantics closely;

- Lends itself to linear reasoning - at each extension step made to a leaf L use a sentence that closes one branch using L.
eg Logic Programming can be seen as tableau development;

- Can incorporate equality - we will see later how tableau incorporate equality quite naturally

- Free variable tableaux may terminate without closure for satisfiable sentences, when standard tableaux would be infinite;

- There are many refinements for free variable tableau using clausal data, which are often derived from resolution refinements.

In Slides 10 we'll look at Model Elimination, the basis for many of them.

The Standard Tableau Method is Sound

9ei

A closed tableau for S implies that S is unsatisfiable

• First show that each tableau extension rule maintains SATISFY:
if the sentences in a branch are satisfiable and a rule is applied,
then the new sentences in *at least one* descendant branch are satisfiable.

• *eg: the rule for* $\rightarrow$*:* If A $\rightarrow$ B is in a branch X and there is a model for the sentences in X, then this model at least makes A false or B true. Hence the same model will ensure satisfiability in one of the two extension branches.

• *eg: the rule for* $\forall$:
Suppose $\forall$x P[x] occurs in a branch B and M is a model for sentences in B.
Then if P[t] is added and t is aready interpreted in M then M is still a model; otherwise, M can be extended by interpreting t as some domain element and still remain a model. (If data is Skolemised at the start then Sig(S) is known and the first case always applies.)

• Other cases are similar (see 9eii).

• (SATISFY) implies that satisfiable initial sentences can never lead to a fully closed tableau, as there is always a branch with a model which must be open. (Formally use an induction argument on depth of the tableau.)

• Therefore a fully closed tableau indicates unsatisfiability of the initial sentences.

<u>Proving the Soundness of Tableau:</u>

9eii

The tableau soundness proof relies on the SATISFY property on Slide 9bv. The cases for boolean operators are all simple and similar to the case given on Slide 9ei.

For the standard $\forall$ case, the argument on 9ei is justified as follows. As in resolution, it is simplest to assume the domain is non-empty (else there are complications). Therefore, assume the domain of M is non-empty. The assignment for t will either already be made in M, or, if t is new and no assignment for t is yet made in M, then t can be any domain element. Either way M will remain a model since, in M, P[x] is true for every x in the domain.

In case x is captured by another universal quantifier, as in $\forall$x,u. P(x,u) becoming $\forall$u.P(u,u), then since $\forall$u. P(x,u) is true in M for every domain element substituted for x, in particular, for any domain element substituted for x it follows that P(x,x) is true in M, which is what $\forall$u P(u,u) is true means.

For the standard $\exists$ case, one can either include in the signature additional *parameters* to be used as needed (and which are similar to Skolem constants), or introduce new names as the need arises. In both cases it is assumed that M does not have an assignment for the name t introduced by the rule. Since $\exists$x.P[x] is true, P[a] must be true for some element a in the domain of M. The assignment for the new name t can be the witness *a*. The cases for the free variable $\forall$ and $\exists$ rules are considered on 9evii.

The proof uses the following notion: A *branch is saturated* if all possible development rules have been applied to the sentences in the branch. e.g. for a β-rule, if $\beta 1$ op $\beta 2$ belongs to the branch then either $\beta 1$ or $\beta 2$ belongs to the branch.

<u>Proof of Tableau Soundness</u>

9eiii

If a closed tableau is developed from S then S is unsatisfiable.
The proof uses induction on the completed depth of a tableau (defined next).

A tableau is *completed up to depth n* if each branch is either closed (after $\leq$ n steps), saturated and open (after $\leq$n steps), or neither and of length = n steps.

<u>Lemma:</u> **Let S be a satisfiable set of sentences. Forall n$\geq$0, P(n) holds, where P(n) states that if a tableau T is completed up to depth n using S then T has at least one satisfiable and open branch.**

<u>Proof (by induction on completed depth):</u> Assume that S is satisfiable. Then S does not contain a sentence A and its negation $\neg$A.
Base Case (n=0). S is satisfiable. Hence the initial branch after no steps is open and P(0) holds.

Induction Step (n>0). Assume as induction hypothesis (IH) that if a tableau is developed from a satisfiable set of sentences S, and every branch is completed up to depth k, 0≤k<n, then that tableau has at least one satisfiable and open branch. Let T be a tableau developed from S and completed up to depth n. Either: (i) all branches are closed (not possible by assumption), or saturated at depth n-1, whence the Lemma holds, or (ii) consider the reduced tableau T ', formed by undoing any step that lengthened a branch to depth n, which is therefore completed up to depth n-1. By (IH) T ' must have some open and satisfiable branch that has been developed to form T. Consider that branch. According to SATISFY at least one branch resulting from that step is open, leading to an open branch in T.

Hence by induction we conclude P(n). Finally, if S leads to a closed tableau, then we conclude S is not satisfiable, since a closed tableau has no open branch.

The standard tableau method is Complete

9eiv

For unsatisfiable S a closed tableau for S exists.

• Assume S is unsatisfiable and uses language L augmented with a set Δ of parameters for use in δ rule applications.
• Apply rules in a systematic and fair way (possibly by contemplating γ–rule applications for an "infinite" number of times), to obtain a maximally developed *saturated* tableau, in which all possible applications of the α, β, δ, γ rules are made in all branches. For any branch, all applications of the $\forall$ rule for terms using L and parameters used in the branch are made. e.g. can interleave the α, β, δ rules with a finite number of applications of the $\boldsymbol{\gamma}$-rule, as on slide 9dii
eg if p $\wedge$ q is in a branch B then both p and q are also in B

• If such a maximal tableau doesn't close, then at least one (possibly infinite) branch B is open (ie it does not contain X and $\neg$X) and a model of the sentences in B can be found, based on the literals occurring in B. (See slides 9ev and 9evi.)

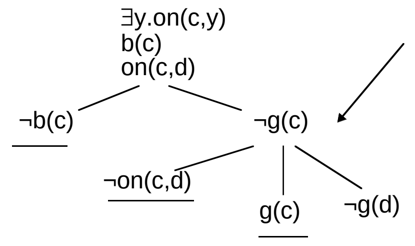
• A H-interpretation is constructed using as domain the set of terms built from symbols occurring in L+Δ', such that atoms in B are assigned true and all other atoms are assigned false. Δ' is the subset of Δ occurring in sentences in B

• Therefore, if the initial sentences are unsatisfiable and so do not have a model, they must lead to a closed tableau.

Example of a Saturated Tableau (ppt)

Given: b(c)   ∃y.on(c,y)   ¬b(x)∨¬g(x)   ¬on(x,z)∨g(x)∨ ¬g(z)

(Variables x,z are universally quantified and parameter is d)



• Domain of H-model in the open branch = {c,d}

• The atoms are on(c,d), b(c), so these are True.

• All other atoms are false: on(c,c), on(d,d), on(d,c), b(d), g(c), g(d).

Check:
• Clearly b(c) and ∃y.on(c,d) are true.
• For each x, g(x) is false so ¬b(x)∨¬g(x) is true.
• For z = c, ¬on(x,z)∨g(x)∨ ¬g(z)   is true as g(c) is false.
• Similarly for z=d.

Extensions by any other instances (4 possibilities in all) duplicate at least one literal in the remaining open branch. Such extensions are unnecessary.
Exercise: Check this is the case.

Constructing a Model from an open branch in a Saturated Tableau

• Suppose a standard tableau has been fully developed from initial sentences S, as described on Slide 9div, and that there is an open branch B.

• Let T be the set of sentences in B and M be a first order H-interpretation of T with domain terms built from symbols in T and constructed as follows:
Each atom in T is true in M; all other atoms are false in M.

• We show M is a model of T.
• Suppose not: then some sentence in T is false in M.
• Let X be the smallest sized sentence in T s.t. M makes X false.
• Whatever type of sentence X is, its being false leads to a contradiction:

• X cannot be an atom, by construction.
• *eg: case X is ¬ Y (Y atom)* : If ¬Y is false in M, then Y is true. But then Y occurs in T and closure would have occurred.
• *eg: case X is A ∨ B* : If A ∨ B is false in M then both A and B are false in M and smaller than A ∨ B; but at least one of A or B is in T, contradicting that X is the smallest false sentence in T.
• *eg: case X is ∀xP[x]* : If ∀xP[x] is false in M then P[t] is false for some domain element t. But by construction P[t] occurs in T and is smaller than ∀xP[x] (assume size is depth of parse tree of X).
• Other cases are similar.

Soundness and Completeness of free variable tableaux

Soundness:

When a free variable tableau closes, there may be free variables in it not yet bound. These can be bound (consistently) to *any* ground term yielding a ground tableau, which will still close. Then use Soundness of a standard tableau.

Completeness (outline):

A closed *standard* tableau may be *lifted* to a tableau using free variables:

Each use of the standard ∀-rule is made into a use of the γ-rule, with a fresh set of variables, and each closure then becomes one or more equations to be solved by unification.

Since the tableau is closed, a unifier satisfying the set of equations derived in this way exists and hence a most general unifier (mgu) exists also.

This mgu can be obtained by the unification algorithm in one attempt ("unify at the end") or in a distributed attempt, corresponding to the different branch closures ("unify as you go").

LeanTap: A Free Variable Tableau Theorem Prover

```
%prove(currentFormula,todo,branchLits,freevars,maxvars)
%Conjunction case (alpha rule)
prove((A,B),UE,Ls,FV,V) :- !, prove(A,[B|UE],Ls,FV,V).
%Disjunction case - split (beta rule)
prove((A;B),UE,Ls,FV,V) :- !, prove(A,UE,Ls,FV,V),
                              prove(B,UE,Ls,FV,V).
%Universal case - keep data all(X,Fm)
prove(all(X,Fm),UE,Ls,FV,V) :- !,
   \+ length(FV,V),copy_term((X,Fm,FV),(X1,Fm1,FV)),
   append(UE,[all(X,Fm)],UE1),
   prove(Fm1,UE1,Ls,[X1|FV],V).
%Closure case
prove(Lit,_,[L|Ls],_,_) :-
   (Lit = -Neg; -Lit = Neg) ->
   (unify_with_occurs_check(Neg,L); prove(Lit,[],Ls,_,_)).
%Literal not matching case
prove(Lit,[N|UE],Ls,FV,V) :-prove(N,UE,[Lit|Ls],FV,V).
```

Formulas are in Skolemised negated normal form (negations next to atoms).

```
nnf(-(-(p=>q)=>(q=>p)), ((p,-q),(q,-p)))
nnf(-(((p=>q)=>p)=>p), (((p,-q);p),-p))
nnf(ex(Y,all(X,(f(Y)=>f(X)))), all(X,(-f(s);f(X))))
(The code generates all(X,(f(Y)=>f(X))) as Skolem term s)
```

```
:- op(400,fy,-),op(500,xfy,&),op(600,xfy,v),
op(650,xfy,=>),  op(700,xfy,<=>).
nnf(Fml,NNF) :- nnf(Fml,[],NNF).
nnf(Fm,FV,NNF) :-
   (Fm = -(-A)      -> Fm1 = A;
    Fm = -all(X,F)  -> Fm1 = ex(X,-F);
    Fm = -ex(X,F)   -> Fm1 = all(X,-F);
    Fm = -(A v B)   -> Fm1 = -A & -B;
    Fm = -(A & B)   -> Fm1 = -A v -B;
    Fm = (A => B)   -> Fm1 = -A v B;
    Fm = -(A => B)  -> Fm1 = A & -B;
    Fm = (A <=> B)  -> Fm1 = (A & B) v (-A & -B);
    Fm = -(A <=> B) -> Fm1 = (A & -B) v (-A & B)),!,
   nnf(Fm1,FV,NNF).

nnf(all(X,F),FV,all(X,NNF)) :- !, nnf(F,[X|FV],NNF).
nnf(ex(X,Fm),FV,NNF) :- !,
   copy_term((X,Fm,FV),(Fm,Fm1,FV)), nnf(Fm1,FV,NNF).
nnf(A & B,FV,(NNF1,NNF2)) :- !,
   nnf(A,FV,NNF1), nnf(B,FV,NNF2).
nnf(A v B,FV,(NNF1;NNF2)) :- !,
   nnf(A,FV,NNF1),nnf(B,FV,NNF2).
nnf(Lit,_,Lit).
```

<u>LeanTap Prover</u>

The LeanTap theorem prover was developed by Bernard Beckert, Joachim Possega and Reiner Hahlne. It was the first ``Lean" theorem prover, meaning a ``very small Prolog program" that exploits Prolog unification in clever ways to implement a theorem prover for first order logic. A different, but equally good, prover is given on Slides 10 called LeanCop both part of the family of Lean provers. It is always impressive to see how compact a theorem prover in Prolog can be.

For simplicity, LeanTap uses formulas in Skolemised Negation Normal Form (NNF). This means that before trying to develop a tableau existential type quantifiers are replaced by Skolem functions and negations are pushed inwards so they are next to atoms; however, no distribution of $\wedge$ over $\vee$, or $\vee$ over $\wedge$, is applied. This sentence structure allows to simplify the top level of LeanTap, so only conjunctions and disjunctions, universal quantifers and literals need be considered. (LeanCop uses clausal form, which is a sub-case of NNF – ie distribution of $\vee$ over $\wedge$ is performed. Closure is checked for literals only. The Skolemisation step in `nnf' uses the name of the formula being Skolemised as the new Skolem constant.

There are several websites covering LeanTap - just type it into Google and see!

Example queries to run:

```
F= (-h(a), all(X,(f(X);h(X))),all(Z,(-g(Z);-f(b))),
     all(Y,(-f(Y);-h(b))),all(X,(g(X);-f(X)))),
      prove(F,[],[],[],4)

nnf(-(-e)&(a & w =>p)&(i v a)& -p&(e =>-i & -m)&(-w =>m), F),
      prove(F,[],[],0)
```

1. Add write instructions so that information about the structure of the tableau is printed out, including closure.

2. Explain details of universal and closure cases of prove and existential case of nnf.

3. Run other examples covered in slides and exercises.

What improvements could be made?
a) add a loop check. (Note that (for example) h(X) and h(Y) do not form a loop - so must be careful to match terms identically.
b) add a higher level predicate `prove1` that calls prove recursively, each time increasing the freevars limit by 1.
c) limit the depth of each branch instead of the number of freevars.

9fiv

More Notes about LeanTap 9fv

The interesting cases in the program are for universal quantifiers and literals.

i) *Case all(X,Fm)*: Note first that a limit is put on the total number of free variables used in the tableau (argument `maxvars`). If this is not reached then a copy of the formula `Fm` is made in which the current free variables are not copied, so they are preserved across the tableau. Bound variable *X* occurrences are copied throughout.

ii) *Case Lit in Fm*: Either the literal causes closure by unification (with the occurs check in place) with a complementary literal in `branchLits`, or it does not. In the latter case the Lit is added to `branchLits`.

Conversion to NNF is carried out recursively as on Slide 9fii. The existential case `ex(X,Fm)` introduces a Skolem term in a clever way. It instantiates X with the formula Fm. Thus each unique exists term introduces a different Skolem formula, which is dependent on the free variables in the formula `Fm`.

Summary of Slides 9

9gi

1. Semantic Tableau methods provide an alternative to resolution for theorem proving. They are also based on refutation and for a given set of sentences S attempt to demonstrate that S can have no models.

2. In the ``standard'' tableau method, rules for dealing with universal ($\forall$) sentences require substitution of ground terms for the bound variable. This is avoided in the ``free variable'' tableau method; instead fresh variables are substituted for bound variables, which are bound on branch closure using unification.

3. In the ``unify-as-you-go'' development strategy the bindings of free variables are immediately propagated to all occurrences of the variables in the tableau. An alternative is the ``unify-at-the-end'' development strategy, in which potential bindings for free variables are recorded and on (potential) closure of all branches in the tableau are combined. In effect, the difference between the two strategies is whether to combine unifiers as they are generated, or to wait until all have been generated. Only the first strategy is covered in the course.

4.The tableau method is sound and complete. The free variable soundness and completeness properties are derived from those of the standard tableau method.

5. The soundness property of tableau depends on the SATISFY property, 9gii which states that, for a consistent branch, the tableau rules maintain consistency in at least one descendant branch.

6. The completeness property depends on the notion of saturation, the development of a tableau to include all possible applications of each rule in every branch.

7. There are several implementations of the tableau method. The LeanTap approach uses Prolog and results in a very compact program. It exploits Prolog's use of variables to implement the unfication and propagation of free variables.

8. The tableau method has several benefits, including: it uses the original structure of the data, can be extended to many logics such as modal/temporal logic, can easily incorporate equality, and linear and many other refinements can be defined for the tableau method.

9. If a tableau terminates finitely without closing every branch, then a model can be found for any remaining open branches (possibly a different one for each open branch). The slides showed how to construct the model for standard tableaux. It is also possible to do so for free variable tableaux.