AUTOMATED REASONING

SLIDES 4:

SOME USEFUL NOTIONS
Structures and Models
Herbrand Interpretations and Models
Soundness and Completeness Properties
Soundness of Resolution
Completeness of Resolution (Outline)

KB - AR - 09

Properties of Infrence Systems: 4ai

Slides 4 include some material on the properties of inference systems, including material on first order structures. The notion of a Herbrand interpretation, a first order structure with a very particular domain, is introduced and it is explained why Herbrand Interpretations are important for soundness and completeness of resolution. The "Useful Theorem" on Slide 4bii and the Skolemisation property on 4di capture this. These properties mean that when proving theorems about resolution it is sufficient to restrict considerations to Herbrand interpretations only, substantially simplifying the proofs. Also, when using refutation as a proof technique to show (un)satisfiability of data, it is sound to consider the clausal form representation of the data.

The proofs of the theorems in Slides 4 (if not given here) can be found either in Appendix1 or in Chapter Notes 1 at www.doc.ic.ac.uk/~kb. ***Details of proofs in Appendix1 are not examinable***.

If you are not too familiar with first order structures, try this example of a structure for the sentences on Slide 4aii: $\forall x[(P(x) \to P(f(x))]$, $P(a)$, $P(b)$, $Q(c, g(c))$ with the signature on the slide.
Take Domain = Lists over the (English) alphabet and the following mapping of terms to Domain
a is "a", b is "the", c is "hit"
f(x) is the word formed by appending 's" to x
P(x) is true if x is a correct English word

a) Which of $\forall x (P(x) \to P(f(x))$, P(a) or P(b) are true in this structure?
b) Choose interpretations for g and Q that make Q(c,g(c)) true in the structure.

Sometimes the notation ||S||[x/d] is used, which means the interpretation of the sentence S in the structure I in which free occurrences of x are replaced by Domain element d. For example, ||P(x)||[x/'a'] means "the interpretation of P('a')", which in the above structure is " 'a' is a correct English word". Note that $\forall$x S is true in a structure I if ||S||[x/d] is true in I for every d in the Domain, and $\exists$x S is true in a structure I if ||S||[x/d] is true in I for some d in the Domain.

Structures
4aii

- (First order) sentences are written in a language L, which uses predicates and terms constructed from names in the signature Sig(L) = <P, F, C>, where P = predicates, F = function symbols, C = constants.

- A structure for L (also referred to as an interpretation) consists of a non-empty domain D, and an interpretation (i.e. a meaning) for each symbol in Sig(L):

  c $\in$ C is interpreted by an element of D
  f (of arity n) $\in$ F is interpreted by a function of arity n from D^n to D
  p (of arity m) $\in$ P is interpreted by a relation of arity m on D^m.

Example: Sig(L) = <{P,Q},{f,g},{a,b,c}>
S = {$\forall x (P(x) \to P(f(x)))$, P(a), P(b), Q(c, g(c))}
a, b and c are constants, f and g have arity 1, P has arity 1 and Q has arity 2

Structure: Domain = {integers}
- a is 0, b is 2, c is ?
- f is interpreted as the function x -> x+2 (i.e. the "add 2" function)
- P(x) is true iff x is even
- Q(x,y) is true iff ?

- Choose an interpretation for g, c, Q so all sentences are true in the structure

P(a) is interpreted as "0 is even" and P(b) as "2 is even"; both are true.
$\forall x (P(x) \to P(f(x)))$ is interpreted as "$\forall$x (x is even $\to$ x+2 is even)"; it's true

Structures (more formally)
4aiii

- Given a structure I for L with domain D, s.t.
  c $\in$ C is interpreted by an element I(c) of D
  f (of arity n) $\in$ F is interpreted by a function I(f) of arity n from D^n to D
  p (of arity m) $\in$ P is interpreted by a relation I(P) of arity m on D^m

- The interpretation in I of a ground term or atom in language L is defined by:

||c|| = I(c) for a constant c
||f(t1, ..., tn)|| = I(f)(||t1||, ..., ||tn||) for a function f of arity n
||P(t1, ..., tn)|| = I(P)(||t1||, ..., ||tm||) for a predicate P of arity m

- ||x|| =x for a bound variable x

- The truth of a sentence S written in L under interpretation I is defined by:

S is an atom: S is true iff ||S|| is true
S =¬S: S is true iff ||S1|| is false
S = S1 op S2: S is true iff ||S1|| op ||S2|| is true
S = $\forall$x(S1): S is true iff ||S1||(x/d) is true for every d in D
S = $\exists$x(S1): S is true iff ||S1||(x/d) is true for some d in D
||S1||(x/d) means d replaces occurrences of x in ||S1||

A structure I for L is a *model* for a set of sentences S (written in L) if for every sentence s in S ||s|| is true under I
If S has a model it is *satisfiable*. If S has no models S is *unsatisfiable*.

The Special Interpretation called a Herbrand Structure

4aiv

Given: Sig(L) = <{P,Q},{f},{a,b,c}> S = {∀x (P(x) → P(f(x))), P(a) , P(b)}

A Herbrand Structure for Sig(L):

- Domain ={a,b,c, f(a), …, f(f(a)),…,f(f(f(a))),…} (i.e. the set of terms in L)
- a is a, b is b, c is c
- f (x) is f(x), g(x) is g(x) (for all x)

NOTE: mapping of constants and functors is fixed.

i.e. elements are, in effect, mapped to (interpreted as) themselves

- P(a) = P(f(a)) = P(f(f(a))) =… = True
- P(b) = P(f(b)) = … = P(c) = P(f(c)) =…= False
- Sentences P(a) and ∀x[P(x) → P(f(x))] are true, but P(b) is false.

A *Herbrand interpretation* can be represented by a subset of the set of atoms:
e.g. { P(a), P(f(a)), P(f(f(a))), ... } (the true atoms)
This Herbrand Interpretation is a Herbrand model.

NOTE1: For a Herbrand Structure, ||S|| is usually simply written as S; otherwise there would be much clutter such as " 'a' " to represent elements of the domain - ie the names of terms.

NOTE2: If there are any function symbols in Sig then the Domain is infinite.
There is assumed always one constant in Domain so that Domain ≠ ∅.
If clauses S have a H-model S is *H-satisfiable*. If not S is *H-unsatisfiable*.

Herbrand Interpretations

4av

Some Definitions: Let L be a language for a set of clauses S.

The *Herbrand Universe* HU of L is the set of terms using constants and function symbols in Sig(L).

The *Herbrand Base* HB of L is the set of ground atoms using terms from HU.

An *Herbrand Interpretation* HI of L is an assignment of T or F to the atoms in HB.

An *Herbrand model* of S is an Herbrand interpretation of L that makes each clause in S True.

Example: S=Px ∨ Ry ∨ ¬Qxy, ¬Sz ∨ ¬Rz, Sa, ¬Pf(a) ∨ ¬Pf(b)
Sig(L) = < {P,Q,R,S}, {f}, {a,b} >

- Herbrand Universe = {a,b,f(a),f(b),f(f(a)),f(f(b)), ...}
- Herbrand Base = {Pa,Pb, Pf(a), Pf(b), ... Sa, Sb, Sf(a), Sf(b), ..., Ra, Rb, Rf(a), ..., Qab, Qaa, Qbb,Qba, Qf(a)a, ...}
- One Herbrand interpretation = {Pa=Pb=Pf(a)=...=T, Qaa =F,Qbb=F, other Q atoms =T, all R and S atoms assigned F except Sa =T}
- This is **no**t a Herbrand *model* of S because {¬Pf(a),¬Pf(b)} is False.
- The HI that makes all S atoms =T and all P, Q, R atoms = F **is** a model of S.

General Soundness and Completeness Properties

4bi

A |− B - means B can be deduced from A using inference rules of some system
- eg resolution, or natural deduction.
- Generally A is a set of sentences and B is a single sentence

For resolution, the data – the givens and negated conclusion – is converted to clauses so that the converted clauses form A and the empty clause [] forms B

Given a language L and sets of sentences A and B written in L:
A |= B - (A logically implies B) means that
whenever a structure M (of L) is a model of A,
then M is a model of at least one sentence in B.
Usually B is a single sentence, so M must be a model of B in this case.

S |= B is equivalent to S, ¬B |= ⊥, (i.e. S,¬B have no models). ⊥ is always false.

The two relations |= and |− are equivalent,
as expressed in the *Soundness* and *Completeness* properties:

Soundness - if A |− B then A |= B
Completeness - if A |= B then A |− B

Soundness and Completeness of Resolution

4bii

Let C0 be a set of clauses. Let ⇒* denote "yields by ≥1 resolution or factor steps"

Soundness of Resolution: if C0 ⇒* [ ] then C0 |= ⊥ (or C0 has no models)

Completeness of Resolution - if C0 has no models then C0 ⇒* []

To show resolution is sound, we make use of two properties:

(1) **Useful Theorem (*)** which states that
S has a Herbrand model iff S has any model at all
≡ S has no Herbrand models iff S has no models

Hence to show C0 |= ⊥ it is sufficient to show C0 $\models_H$ ⊥

(2) a single resolution or factoring step is sound with respect to H-models:
if S ⇒ R then S $\models_H$ R (where R is a resolvent or factor from S)

where S $\models_H$ R holds iff
for every M, if M is a H-model of S then M is a H-model of R.
(Details and proofs are in Slides Appendix 1).

Proving Soundness of Resolution

4biii

Soundness of Resolution: if C0 $\Rightarrow *$ [ ] then C0 $\models \bot$ (or C0 has no models)

Using (1) and (2) from Slide 4bii we argue as follows

for a refutation C0⇒C0+C1⇒C0+C1+C2 ⇒ ... ⇒C0+...+[ ]:

by (2)

C0 has H-model ==> C0+C1 has H-model ==> C0+C1+C2 has H-model ==> ... ==> C0+C1+...+Cn has H- model ≡

C0+C1+...+Cn has no H-model ==> C0+C1+...+Cn-1 has no H-model ==> ... ==> C0+C1 has no H-model ==> C0 has no H-model

Now suppose that [] is a resolvent (Cn say);
since [] has no models, C0+C1+...+Cn has no H-models,
hence .... C0 has no H-models.

Hence by (1) C0 has no models at all

Completeness of Resolution

4ci

Theorem (Completeness) If a set of clauses S has no models then S ⇒ *****[]
(i.e. there is a resolution refutation of [] from S.)

Most methods used to show completeness rely on some very useful properties:

(a) A resolution refutation for a set of clauses S is similar in structure to a ground resolution refutation using ground instances of S (see slide 5aii for an example).

(b) If a set of clauses S is H-unsatisfiable (has no H-models) then there is a finite subset of ground instances of S also H-unsatisfiable (called *compactness*).

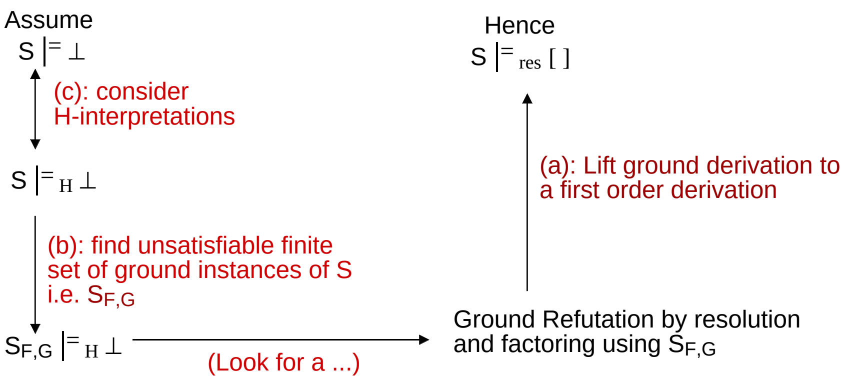
(c) A set of clauses S has no models iff S has no Hmodels (Useful Theorem (*))

(c) suggests it is sufficient to look at Herbrand Interpretations, then
(b) suggests to find a finite set of ground instances that are unsatisfiable, then
(a) suggests to find a ground proof first.

In fact, this is precisely what is usually done.

Structure of the Proof of Resolution Completeness

4cii



Assume that S $\models \bot$ and follow the arrows to show that S $\models_{res}$ [ ]

Details of steps from (b) to (a) are in Slides 5.

The General Case

4di

We want to show that resolution can be used to show unsatisfiability of any set of sentences. Recall that conversion to clauses used Skolemisation (Step 3):

3. *Skolemise* - remove existential-type quantifiers and replace bound variable occurrences of x in ∃xS by *Skolem constants* or *Skolem functions* with arguments that are the universal variables in whose scope they lie and which also occur in S.

All non-*Skolemisation* steps in the conversion to clausal form are equivalences.
Although Skolemised(S) $\models$ S, it is **not** true that S $\models$ Skolemised(S).
eg f(a) |= ∃x.f(x) – if f(a) is true then there is an x (namely a) s.t. f(x) is true.
But ∃x.f(x) does not imply f(a). Whatever x makes f(x) true need not be a.

However, it **is** true that Skolem(S) is unsatisfiable iff S is unsatisfiable. **(**)**
And this is what we need. (See Slides Appendix 1 for proof.)

In general:
To show Data |=Conclusion we convert {Data, ¬Conclusion} to clauses C.
Then
Data |=Conclusion iff {Data,¬Conclusion} is unsatisfiable (by definition)
iff C has no models (by (**))
iff C has no H-models (by (*) on 4bii)
iff C==>*[ ] (by Soundenss and Completeness of resolution)

Summary of Slides 4

1. Herbrand interpretations are first order structures which use a fixed mapping of terms in the Language to the structure. In particular, terms (constants or functional terms such as f(a)) map to the themselves.

2. Any set of clauses S has a model iff S has a Herbrand model.

3. Resolution is sound and complete: Derivation of [ ] from a set of clausesS by resolution and factoring implies that S|=⊥ and if S|=⊥ then there is a resolution (and factoring) derivation of [ ] from S.

4. Soundness of resolution depends on the soundness of a single resolution or factoring step: if S=>R then S|=R and hence S|=S+{R}.

5. Completeness of resolution is often proved by lifting a ground resolution derivation using ground instances of the given clauses S to give a resolution derivation from S.

6. Resolution can be used to show S |= C, for arbitrary sentences S and C by first converting S and ¬C to clauses (Clauses(S+¬C)) and then showing that Clauses(S+¬C) ==> [ ] by resolution. By the Soundness of Resolution this means that Clauses(S+¬C)|= ⊥ and by (**) on 4di that S+¬C are unsatisfiable and hence that S |= C.