

**Fixpoint semantics for definite programs****SOLUTIONS****Question 1**

(a) Let  $IDB = \{p \leftarrow q, r; q \leftarrow s\}$  and  $EDB = \{r, s\}$ .

Clearly  $Cl_{IDB}(EDB) = \{p, q, r, s\}$ .

Compute  $Cl_{IDB}(EDB) = T'_{IDB} \uparrow^\omega(EDB)$ :

$$\begin{aligned} T'_{IDB} \uparrow^0(EDB) &= EDB = \{r, s\} \\ T'_{IDB} \uparrow^1(EDB) &= T'_{IDB}(\{r, s\}) = \{q\} \cup \{r, s\} \\ T'_{IDB} \uparrow^2(EDB) &= T'_{IDB}(\{q, r, s\}) = \{p, q\} \cup \{q, r, s\} \\ T'_{IDB} \uparrow^3(EDB) &= T'_{IDB}(\{p, q, r, s\}) = \{p, q\} \cup \{p, q, r, s\} \\ &\vdots \\ T'_{IDB} \uparrow^\omega(EDB) &= \{r, s\} \cup \{q, r, s\} \cup \{p, q, r, s\} \end{aligned}$$

Compare  $T_{IDB} \uparrow^\omega(EDB)$ :

$$\begin{aligned} T_{IDB} \uparrow^0(EDB) &= EDB = \{r, s\} \\ T_{IDB} \uparrow^1(EDB) &= T_{IDB}(\{r, s\}) = \{q\} \\ T_{IDB} \uparrow^2(EDB) &= T_{IDB}(\{q\}) = \emptyset \\ T_{IDB} \uparrow^3(EDB) &= T_{IDB}(\emptyset) = \emptyset \\ &\vdots \\ T_{IDB} \uparrow^\omega(EDB) &= \{r, s\} \cup \{q\} = \{q, r, s\} \end{aligned}$$

(b) Let  $IDB = \{p \leftarrow q, r; q \leftarrow s; r\}$  and  $EDB = \{s\}$ .

Clearly  $Cl_{IDB}(EDB) = \{p, q, r, s\}$ .

Compute  $Cl_{IDB}(EDB) = T'_{IDB} \uparrow^\omega(EDB)$ :

$$\begin{aligned} T'_{IDB} \uparrow^0(EDB) &= EDB = \{s\} \\ T'_{IDB} \uparrow^1(EDB) &= T'_{IDB}(\{s\}) = \{q, r\} \cup \{s\} \\ T'_{IDB} \uparrow^2(EDB) &= T'_{IDB}(\{q, r, s\}) = \{p, q, r\} \cup \{q, r, s\} \\ T'_{IDB} \uparrow^3(EDB) &= T'_{IDB}(\{p, q, r, s\}) = \{p, q, r\} \cup \{p, q, r, s\} \\ &\vdots \\ T'_{IDB} \uparrow^\omega(EDB) &= \{s\} \cup \{q, r, s\} \cup \{p, q, r, s\} \end{aligned}$$

Compare  $T_{IDB} \uparrow^\omega(EDB)$ :

$$\begin{aligned} T_{IDB} \uparrow^0(EDB) &= EDB = \{s\} \\ T_{IDB} \uparrow^1(EDB) &= T_{IDB}(\{s\}) = \{q, r\} \\ T_{IDB} \uparrow^2(EDB) &= T_{IDB}(\{q, r\}) = \{p, r\} \\ T_{IDB} \uparrow^3(EDB) &= T_{IDB}(\{p, r\}) = \{r\} \\ &\vdots \\ T_{IDB} \uparrow^\omega(EDB) &= \{s\} \cup \{q, r\} \cup \{p, r\} \cup \{r\} = \{p, q, r, s\} \end{aligned}$$

So here  $T_{IDB} \uparrow^\omega(EDB)$  happens to come out the same as  $Cl_{IDB}(EDB)$ . Part (a) was intended to show that this is not guaranteed.

**Question 2**

Part(a): Base case ( $n = 0$ ):  $T_P \uparrow^0(X) = X$  and  $T'_P \uparrow^0(X) = X$ .

Inductive hypothesis: suppose  $T_P \uparrow^k(X) \subseteq T'_P \uparrow^k(X)$ .

By monotony of  $T_P$ :

$$T_P(T_P \uparrow^k(X)) \subseteq T_P(T'_P \uparrow^k(X))$$

Since  $T_P(I) \subseteq T_P(I) \cup I$ :

$$T_P(T_P \uparrow^k(X)) \subseteq T'_P(T'_P \uparrow^k(X))$$

And so:

$$T_P \uparrow^{k+1}(X) \subseteq T'_P \uparrow^{k+1}(X)$$

That completes the proof.

Further:  $T_P \uparrow^\omega(X) = \bigcup_{n \geq 0} T_P \uparrow^n(X) \subseteq \bigcup_{n \geq 0} T'_P \uparrow^n(X) = T'_P \uparrow^\omega(X)$ .

Part (b): to show  $T_P \uparrow^n(\emptyset) \subseteq T'_P \uparrow^{n+1}(\emptyset)$ .

Base case ( $n = 0$ ):  $T_P \uparrow^0(\emptyset) = \emptyset$ , and  $\emptyset \subseteq T'_P \uparrow^1(\emptyset)$ .

Inductive hypothesis: suppose  $T_P \uparrow^k(\emptyset) \subseteq T'_P \uparrow^{k+1}(\emptyset)$ .

By monotony of  $T_P$ :

$$T_P(T_P \uparrow^k(\emptyset)) \subseteq T_P(T'_P \uparrow^{k+1}(\emptyset))$$

$$T_P \uparrow^{k+1}(\emptyset) \subseteq T'_P \uparrow^{(k+1)+1}(\emptyset)$$

That completes the proof.

And hence, to show:  $T'_P \uparrow^n(\emptyset) = T_P \uparrow^n(\emptyset)$ .

Base case ( $n = 0$ ):  $T'_P \uparrow^0(\emptyset) = \emptyset = T_P \uparrow^0(\emptyset)$ .

Inductive hypothesis: suppose  $T'_P \uparrow^k(\emptyset) = T_P \uparrow^k(\emptyset)$ .

$$\begin{aligned} T'_P \uparrow^{k+1}(\emptyset) &= T'_P(T'_P \uparrow^k(\emptyset)) \\ &= T'_P(T_P \uparrow^k(\emptyset)) \quad (\text{inductive hypothesis}) \\ &= T_P(T_P \uparrow^k(\emptyset)) \cup T_P \uparrow^k(\emptyset) \\ &= T_P \uparrow^{k+1}(\emptyset) \cup T_P \uparrow^k(\emptyset) \\ &= T_P \uparrow^{k+1}(\emptyset) \quad (\text{because } T_P \uparrow^k(\emptyset) \subseteq T_P \uparrow^{k+1}(\emptyset)) \end{aligned}$$

That completes the proof.

Further:  $T_P \uparrow^\omega(\emptyset) = \bigcup_{n \geq 0} T_P \uparrow^n(\emptyset) = \bigcup_{n \geq 0} T'_P \uparrow^n(\emptyset) = T'_P \uparrow^\omega(\emptyset)$ .

**Question 3** A set  $X$  of atoms is a Herbrand model of  $IDB \cup EDB$

$$\text{iff } T_{IDB \cup EDB}(X) \subseteq X \quad (1)$$

$$\text{iff } T_{IDB}(X) \cup EDB \subseteq X \quad (2)$$

$$\text{iff } T_{IDB}(X) \subseteq X \text{ and } EDB \subseteq X \quad (3)$$

So the least set of atoms satisfying (1) is also the least set of atoms satisfying (3), which is  $Cl_{IDB}(EDB)$  by definition.

You can also do this by checking that  $T'_{IDB \cup EDB} \uparrow^n(\emptyset) = T'_{IDB} \uparrow^n(EDB)$ .

**Question 4**  $Cl_{IDB}$  is a classical consequence operator:

•  $X \subseteq Cl_{IDB}(X)$  by definition of  $Cl_{IDB}$ .

• (i)  $Cl_{IDB}(X) \subseteq Cl_{IDB}(X)$  (obviously)

(ii)  $Cl_{IDB}(X)$  is closed under the rules  $IDB$  (by definition).

But the smallest set of atoms satisfying (i) and (ii) is by definition  $Cl_{IDB}(Cl_{IDB}(X))$ .

So  $Cl_{IDB}(Cl_{IDB}(X)) \subseteq Cl_{IDB}(X)$ .

• Suppose  $X_1 \subseteq X_2$ . Then  $X_1 \subseteq X_2 \subseteq Cl_{IDB}(X_2)$ . So:

(i)  $X_1 \subseteq Cl_{IDB}(X_2)$

(ii)  $Cl_{IDB}(X_2)$  is closed under the rules  $IDB$  (by definition).

But the smallest set of atoms satisfying (i) and (ii) is by definition  $Cl_{IDB}(X_1)$ . So  $Cl_{IDB}(X_1) \subseteq Cl_{IDB}(X_2)$ .

Notice that the third property (monotony of  $Cl_{IDB}$ ) does not depend on monotony of  $T_{IDB}$ . But if  $IDB$  are not definite, then  $T_{IDB}$  is not necessarily monotonic, and then there is no guarantee that  $Cl_{IDB}$  exists.

**Question 5** Since  $\{r\} \subseteq Cl_{IDB}(\{r\})$  there are four candidates for  $Cl_{IDB}(\{r\})$ , if it exists:

•  $Cl_{IDB}(\{r\}) = \{r\}$ ?

No:  $\{r\}$  is not closed under rules  $IDB$ :  $T_{IDB}(\{r\}) = \{p, q\} \not\subseteq \{r\}$ .

•  $Cl_{IDB}(\{r\}) = \{p, r\}$  or  $Cl_{IDB}(\{r\}) = \{q, r\}$ ?

No: not unique.  $\{p, r\}$  is closed under rules  $IDB$ :  $T_{IDB}(\{p, r\}) = \{p\} \subseteq \{p, r\}$ . Consider the subsets of  $\{p, r\}$ :  $\{r\}$  is not closed under rules  $IDB$  (see above), and  $\{p\}$  and  $\emptyset$  do not contain  $\{r\}$ .

However: the same holds for  $\{q, r\}$ , by the same argument.

So  $\{p, r\}$  is not the *unique* smallest set  $X$  such that  $\dots$ , and nor is  $\{q, r\}$ .

•  $Cl_{IDB}(\{r\}) = \{p, q, r\}$ ?

No: not minimal.  $\{p, q, r\}$  is closed under rules  $IDB$ :  $T_{IDB}(\{p, q, r\}) = \emptyset \subseteq \{p, q, r\}$ .

But  $\{p, q, r\}$  is not the smallest such set containing  $\{r\}$ : the subset  $\{p, r\}$  is also closed under  $IDB$ , for example (above).