499 Modal and Temporal Logic

Canonical models for normal logics

(Completeness via canonicity)

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Further reading:
B.F. Chellas, *Modal logic: an introduction*. Cambridge University Press, 1980.
P. Blackburn, M. de Rijke, Y. Venema, Chapter 4, *Modal Logic*. Cambridge University Press, 2002.

Notation

$\mathcal{M} \models A$ — A is valid in model $\mathcal{M}$ (A is true at all worlds in $\mathcal{M}$)
$\mathcal{F} \models A$ — A is valid in the frame $\mathcal{F}$ (valid in all models with frame $\mathcal{F}$)
$\models_{\mathsf{C}} A$ — A is valid in the class of models C (valid in all models in C)
$\models_{\mathsf{F}} A$ — A is valid in the class of frames F (valid in all frames in F)

The truth set, $\|A\|^{\mathcal{M}}$, of the formula A in the model $\mathcal{M}$ is the set of worlds in $\mathcal{M}$ at which A is true. $\|A\|^{\mathcal{M}} =_{def} \{w \text{ in } \mathcal{M} : \mathcal{M}, w \models A\}$

Reminder

- $\vdash_\Sigma A$ means that A is a theorem of Σ. $\vdash_\Sigma A$ iff $A \in \Sigma$.
- $\Gamma \vdash_\Sigma A$ iff $\vdash_\Sigma (A_1 \wedge \cdots \wedge A_n) \to A$ for some set of formulas $\{A_1, \ldots, A_n\} \subseteq \Gamma$ $(n \geq 0)$.
- Γ is Σ-inconsistent iff $\Gamma \vdash_\Sigma \bot$, i.e., iff $\vdash_\Sigma (A_1 \wedge \cdots \wedge A_n) \to \bot$ for some set of formulas $\{A_1, \ldots, A_n\} \subseteq \Gamma$ $(n \geq 0)$. Γ is Σ-consistent iff Γ is not Σ-inconsistent.
  A useful property: Γ is Σ-consistent iff there is no A such that both $\Gamma \vdash_\Sigma A$ and $\Gamma \vdash_\Sigma \neg A$.
- Γ is a *Σ-maxi-consistent set* iff Γ is Σ-consistent, and for every formula A, if $\Gamma \cup \{A\}$ is Σ-consistent, then $A \in \Gamma$.
- The proof set $|A|_\Sigma$ is the set of Σ-maxi-consistent sets that contain A.
- *Lindenbaum's lemma:* If Γ is Σ-consistent then there exists a Σ-maxi-consistent set Δ such that $\Gamma \subseteq \Delta$.
- Three useful properties of any Σ-maxi-consistent set Γ and formula A:
  - for any formula A, either $A \in \Gamma$ or $\neg A \in \Gamma$;
  - if $\Gamma \vdash_\Sigma A$ then $A \in \Gamma$ (actually, $A \in \Gamma \Leftrightarrow \Gamma \vdash_\Sigma A$)
  - Γ is closed under MP (modus ponens)

Reminder — Normal system

The set of formulas Σ is a *system of modal logic* iff it contains all propositional tautologies (*PL*) and is closed under modus ponens (MP) and uniform substitution (US).

A system of modal logic is *normal* iff it contains the schemas $\Diamond A \leftrightarrow \neg\Box\neg A$ (Df$\Diamond$) and K and is closed under RN.

$$\Box(A \to B) \to (\Box A \to \Box B) \qquad \text{(K.)}$$

$$\frac{A}{\Box A} \qquad \text{(RN.)}$$

Or equivalently: a system of modal logic is *normal* iff it contains the schema Df$\Diamond$ and is closed under RK.

$$\frac{(A_1 \wedge \ldots \wedge A_n) \to A}{(\Box A_1 \wedge \ldots \wedge \Box A_n) \to \Box A} \quad (n \geq 0) \qquad \text{(RK.)}$$

Soundness

Definition 1 (Soundness) *Let C be a class of models (or frames). A logic Σ is* sound *with respect to C if, for any formula A, $\vdash_\Sigma A$ implies $\models_{\mathsf{C}} A$.*

$$\boxed{\vdash_\Sigma A \Rightarrow \models_{\mathsf{C}} A}$$

If we define Σ_{C} to be the set of all formulas valid in the class C: $\Sigma_{\mathsf{C}} =_{def} \{A \mid \models_{\mathsf{C}} A\}$, then Σ is sound with respect to C if $\Sigma \subseteq \Sigma_{\mathsf{C}}$.
It follows that if Σ is sound with respect to C, then every logic $\Sigma' \subseteq \Sigma$ is also sound with respect to C.

Small point of detail For those looking at the book by Chellas: recall (first set of notes) that Chellas's definition (2.11, p46) of a modal logic does not require closure under uniform substitution. So according to Chellas, Σ_{C} is a (normal) modal logic for *any* class of (Kripke) models C; according to the definition in Blackburn et al (which requires closure under US, as above), Σ_{C} is only a (normal) modal logic when the class of models C is actually a class of *frames*.

Theorem 2 *[Chellas Thm 5.1, p162] Let $\xi_1, \ldots, \xi_n$ be schemas valid respectively in classes of relational models/frames $\mathsf{C}_1, \ldots, \mathsf{C}_n$. Then the system of modal logic $K\xi_1 \ldots \xi_n$ is sound with respect to the class $\mathsf{C}_1 \cap \cdots \cap \mathsf{C}_n$.*

Proof Very easy.

Corollary The system K is sound with respect to every class of relational ('Kripke') models/frames.

Completeness

Definition 3 (Completeness) *Let* C *be a class of models (or frames). A logic* Σ *is* complete *with respect to* C *if for any formula* A, $\models_{\mathsf{C}} A$ *implies* $\vdash_{\Sigma} A$.

$$\boxed{\models_{\mathsf{C}} A \ \Rightarrow \vdash_{\Sigma} A}$$

Notice: Σ is complete with respect to C if $\Sigma_{\mathsf{C}} \subseteq \Sigma$.
It follows that if Σ is complete with respect to C then every logic $\Sigma' \supseteq \Sigma$ is also complete with respect to C.

Determination: soundness and completeness

Thus, if we prove that a logic Σ is both sound and complete with respect to some class of models/frames C, we have established a perfect match between the syntactical and semantical perspectives: $\Sigma = \Sigma_{\mathsf{C}}$.

When logic Σ is sound *and* complete with respect to the class of models (or frames) C, Σ is *determined* by C.

Note: a logic may be determined by more than one class of models. For example, the logic S5 ($= KT5 = KT45 = KTB5 = KTB45$) is determined both by the class of *equivalence* frames and also by the class of *universal* frames.

Given a semantically specified logic Σ_{C} (that is, the logic of some class C of interest) we often want to find a simple collection of formulas Γ such that Σ_{C} is the logic generated by Γ. In such a case, we sometimes say that Γ *axiomatizes* C.

Note For those looking at the book by Blackburn et al: Blackburn et al (p194) call this *weak completeness*. They also define a *strong completeness*: a logic Σ is *strongly complete* with respect to a class of models (or frames) C if for any set of formulas $\Gamma \cup \{A\}$, if $\Gamma \models_{\mathsf{C}} A$ then $\Gamma \vdash_{\Sigma} A$.
Here, $\Gamma \models_{\mathsf{C}} A$ means that for every model $\mathcal{M}$ in class C, and for every world w in $\mathcal{M}$, if $\mathcal{M}, w \models \Gamma$ then $\mathcal{M}, w \models A$.
Weak completeness is the special case of strong completeness in which Γ is empty. Thus strong completeness with respect to some class of structures (models, frames) implies weak completeness with respect to that same class. The converse does *not* hold. Example (Blackburn et al, p194): the system $KL = K \cup \{\Box(\Box A \to A) \to \Box A)\}$ is weakly complete with respect to the class of finite transitive trees, but is not strongly complete with respect to this class, or indeed with respect to any class of frames whatsoever.
We won't bother with strong completeness in these notes.

Why are we interested?

Many reasons (besides the purely technical):

To compare different logic systems We want to know whether two (syntactically presented) logics Σ_1 and Σ_2 are the same. This is often a non-trivial matter. If $\Sigma_1 = \Sigma_2$ it is usually not so bad: we show that the defining schemas and rules of Σ_1 can be derived in Σ_2, and vice-versa. If $\Sigma_1 \subset \Sigma_2$, $\Sigma_1 \subseteq \Sigma_2$ is usually not so bad (as above), but $\Sigma_2 \not\subseteq \Sigma_1$ is not so easy: we can't just say we tried to derive Σ_2 from Σ_1 but couldn't manage it. (We might not be very good at it.)

Soundness and completeness results allow us to reason about the corresponding semantical structures which can often be easier.

To validate computing systems If we have a specification given semantically (say as a transition system/Kripke structure) soundness and completeness results allow us to reason about it using proof-theoretic tools, such as automated theorem provers. Conversely, if we have a syntactical specification of a computing system (a set of formulas describing its intended behaviour, say) soundness and completeness guarantees that we can reason about its properties using model theoretic tools, such as model checkers.

Note: the inconsistent logic

The inconsistent logic (the set of all formulas) is a normal modal logic. (Trivial – exercise in earlier set of notes).

Trivially, the inconsistent logic is complete for any class of frames/models.

But the inconsistent logic is not sound for any class of frames/models.

Completeness (via canonical models)

Here is one way of establishing completeness. (It does not always work!)

The basic idea is this. We want to establish completeness of a system Σ with respect to some class C of models, i.e. we want to prove that for all formulas A

$$\models_{\mathsf{C}} A \quad \Rightarrow \quad \vdash_\Sigma A$$

We try to find a model $\mathcal{M}^\Sigma$ for system Σ with the special property that

$$\mathcal{M}^\Sigma \models A \quad \Rightarrow \quad \vdash_\Sigma A$$

Actually we usually go for the stronger property $\mathcal{M}^\Sigma \models A \quad \Leftrightarrow \quad \vdash_\Sigma A$.

Such a model is called a *canonical model* for the system Σ.

Now if we can show that this canonical model belongs to class C, i.e. that model $\mathcal{M}^\Sigma$ satisfies the model conditions that characterise the class C, then we have completeness. Because: suppose $\models_{\mathsf{C}} A$. Then since $\mathcal{M}^\Sigma$ is in class C, $\mathcal{M}^\Sigma \models A$. And since $\mathcal{M}^\Sigma \models A$ implies $\vdash_\Sigma A$ when $\mathcal{M}^\Sigma$ is a canonical model, we have the completeness result $\models_{\mathsf{C}} A \Rightarrow \vdash_\Sigma A$ as required.

$$\models_{\mathsf{C}} A \;\Rightarrow\; \mathcal{M}^\Sigma \models A \;\Rightarrow \vdash_\Sigma A$$

Sometimes, it is easier to go the other way: construct a model $\mathcal{M}$ that is clearly in class C. Then show that $\mathcal{M}$ is a canonical model for the system Σ.

Now it just remains to figure out how to construct a canonical model for a system Σ. The key construct is *maxi-consistent sets* for the system Σ.

Canonical models for normal systems

The basic idea, whether we are dealing with normal systems or non-normal ones (not covered in this course), is this. We want completeness of the system Σ

— with respect to some class C of models, $\models_{\mathsf{C}} A \Rightarrow \vdash_\Sigma A$;
— with respect to some class F of frames, $\models_{\mathsf{F}} A \Rightarrow \vdash_\Sigma A$.

We can do this (sometimes!) by finding a *canonical model* $\mathcal{M}^\Sigma = \langle \mathcal{F}^\Sigma, h^\Sigma \rangle$ for system Σ, which is a model such that

$$\mathcal{M}^\Sigma \models A \Leftrightarrow \vdash_\Sigma A.$$

Now if we can show $\mathcal{M}^\Sigma \in \mathsf{C}$ (resp. $\mathcal{F}^\Sigma \in \mathsf{F}$) then we have completeness, because then

$$\models_{\mathsf{C}} A \;\Rightarrow\; \mathcal{M}^\Sigma \models A \quad (\mathcal{M}^\Sigma \in \mathsf{C}), \quad \text{and} \quad \mathcal{M}^\Sigma \models A \Rightarrow \vdash_\Sigma A \quad \text{(canonical model)}$$

Or in terms of frames: $\models_{\mathsf{F}} A \;\Rightarrow\; \mathcal{F}^\Sigma \models A$, and $\mathcal{F}^\Sigma \models A \;\Rightarrow\; \mathcal{M}^\Sigma \models A \Rightarrow \vdash_\Sigma A$.

Proofs of completeness via canonical models do not always work. See e.g. Blackburn *et al*, Chapter 4, for some other methods for normal modal logics. (Moreover, not every normal logic is the logic of some class of frames. Many temporal logics are like this. See Blackburn *et al* for examples.)

Definition 4 (Canonical model for normal system Σ) *Let Σ be a normal system. The* canonical model *for Σ is $\mathcal{M}^\Sigma = \langle W^\Sigma, R^\Sigma, h^\Sigma \rangle$ such that:*

(1) *W^Σ is the set of Σ-maxi-consistent sets.*
(2) *For every w, w' in $\mathcal{M}^\Sigma$: $w\, R^\Sigma\, w' \Leftrightarrow \forall A\, (\Box A \in w \Rightarrow A \in w')$.*
(3) *For every atom p, $h^\Sigma(p) = \{w \mid p \in w\}$, i.e. $h^\Sigma(p) = |p|_\Sigma$.*

$\mathcal{F}^\Sigma = \langle W^\Sigma, R^\Sigma \rangle$ is the canonical frame *for Σ.*
h^Σ is called the canonical valuation *(or sometimes the 'natural valuation').*

Note: Chellas calls this the 'proper canonical model' for Σ. This is to leave open the possibility that there are other models of the form $\langle W^\Sigma, R, h^\Sigma \rangle$ with a different relation R that can also be used as canonical models for Σ. We will follow the more common usage and simply say 'the canonical model' for the model $\mathcal{M}^\Sigma = \langle W^\Sigma, R^\Sigma, h^\Sigma \rangle$ defined above.

We'll record the main results in a moment. The key thing to remember is the definition of R^Σ (the other components are easy to remember):

$$\boxed{w\, R^\Sigma\, w' \;\Leftrightarrow\; \forall A\, (\Box A \in w \Rightarrow A \in w')}$$

Notice that this can be expressed equivalently as follows:

$$w\, R^\Sigma\, w' \;\Leftrightarrow\; \{\, A \mid \Box A \in w \,\} \subseteq w'$$

Sometimes this form of the definition is easier to manipulate.

Also, the following definition of R^Σ is equivalent. You might find it easier to see what it is saying.

$$\boxed{w\, R^\Sigma\, w' \;\Leftrightarrow\; \forall A\,[\, A \in w' \Rightarrow \Diamond A \in w\,]}$$

This version can be expressed as follows

$$w\, R^\Sigma\, w' \;\Leftrightarrow\; \{\, \Diamond A \mid A \in w' \,\} \subseteq w$$

It is worth remembering both $\Box$ and $\Diamond$ versions. In completeness proofs it is often useful to use one or the other or both.

Why are these definitions of R^Σ equivalent? Consider:

$$\begin{aligned}
w\, R^\Sigma\, w' \;&\Leftrightarrow\; \forall A\,[\,\Box A \in w \Rightarrow A \in w'\,] \\
&\Leftrightarrow\; \forall A\,[\,A \notin w' \Rightarrow \Box A \notin w\,] \\
&\Leftrightarrow\; \forall A\,[\,\neg A \in w' \Rightarrow \neg\Box A \in w\,] \qquad (w,\ w' \text{ are } \Sigma\text{-maxi-consistent sets}) \\
&\Leftrightarrow\; \forall A\,[\,\neg A \in w' \Rightarrow \Diamond\neg A \in w\,] \\
&\Leftrightarrow\; \forall A'\,[\,A' \in w' \Rightarrow \Diamond A' \in w\,] \qquad (\text{this last step is not entirely obvious !!})
\end{aligned}$$

Here is the result that justifies the last step above, the one that isn't obvious. (It was one of the exercises on the tutorial sheet for maxi-consistent sets):

Theorem 5 *[Chellas Thm 4.29, p158] Let Γ and Γ' be Σ-maxi-consistent sets in a normal system Σ. Then:*

$$\{\, A \mid \Box A \in \Gamma \,\} \subseteq \Gamma' \quad\Leftrightarrow\quad \{\, \Diamond A \mid A \in \Gamma' \,\} \subseteq \Gamma$$

In other words: $\forall A\,[\,\Box A \in \Gamma \Rightarrow A \in \Gamma'\,] \quad\Leftrightarrow\quad \forall A\,[\,A \in \Gamma' \Rightarrow \Diamond A \in \Gamma\,]$.

Proof Left-to-right. Assume LHS. Suppose $A \in \Gamma'$. We need to show $\Diamond A \in \Gamma$.

$$\begin{aligned}
A \in \Gamma' \Rightarrow\;& \neg A \notin \Gamma' && (\Gamma' \text{ consistent}) \\
\Rightarrow\;& \Box\neg A \notin \Gamma && (\text{assumed LHS}) \\
\Rightarrow\;& \neg\Box\neg A \in \Gamma && (\Gamma \text{ maxi}) \\
\Rightarrow\;& \Diamond A \in \Gamma
\end{aligned}$$

The other direction is similar: Assume RHS. Suppose $\Box A \in \Gamma$. We need to show $A \in \Gamma'$.

$$\begin{aligned}
\Box A \in \Gamma \Rightarrow\;& \neg\Diamond\neg A \in \Gamma \\
\Rightarrow\;& \Diamond\neg A \notin \Gamma && (\Gamma \text{ consistent}) \\
\Rightarrow\;& \neg A \notin \Gamma' && (\text{assumed RHS}) \\
\Rightarrow\;& A \in \Gamma' && (\Gamma' \text{ maxi})
\end{aligned}$$

Let's record the main results.

Theorem 6 (Truth lemma) *Let $\mathcal{M}^\Sigma = \langle W^\Sigma, R^\Sigma, h^\Sigma \rangle$ be the canonical model for a normal system Σ. Then for every w in $\mathcal{M}^\Sigma$ and every formula A:*

$$\mathcal{M}^\Sigma, w \models A \;\Leftrightarrow\; A \in w$$

In other words, $\|A\|^{\mathcal{M}^\Sigma} = |A|_\Sigma$.

Proof The proof is by induction on the structure of A. The key step is the case where A is of the form $\Box B$. The other cases, where A is of the form $\neg A'$, $A' \wedge A''$, $A' \vee A''$, $A' \rightarrow A''$, are very straightforward. In case you can't imagine how it goes, here are the details.

Base case. Suppose A is an atom p. $\mathcal{M}^\Sigma, w \models p \;\Leftrightarrow\; w \in h^\Sigma(p) \;\Leftrightarrow\; p \in w$.

Inductive step. Suppose the result holds for formulas A and B. It remains to show that it holds also for $\neg A$, $A \wedge B$, $A \vee B$, $A \rightarrow B$, $\Box A$. (In fact, we don't need to do all of these: any two of the truth-functional connectives will do, since the others can be defined in terms of them.)

Case $\neg A$: $\mathcal{M}^\Sigma, w \models \neg A \;\Leftrightarrow\; \mathcal{M}^\Sigma, w \not\models A \;\Leftrightarrow\;$ (by the inductive hypothesis) $A \notin w$. w is a Σ-maxi-consistent set, so $A \notin w \Leftrightarrow \neg A \in w$, as required.
Or more succinctly, using the notation of truth sets and proof sets: $\|\neg A\|^{\mathcal{M}^\Sigma} = W - \|A\|^{\mathcal{M}^\Sigma} =$ (by the inductive hypothesis) $W - |A|_\Sigma = |\neg A|_\Sigma$.

Case $A \wedge B$: $\mathcal{M}^\Sigma, w \models A \wedge B \;\Leftrightarrow\; \mathcal{M}^\Sigma, w \models A$ and $\mathcal{M}^\Sigma, w \models B$
$\Leftrightarrow$ (by the inductive hypothesis)$A \in w$ and $B \in w \;\Leftrightarrow\; A \wedge B \in w$.
Or: $\|A \wedge B\|^{\mathcal{M}^\Sigma} = \|A\|^{\mathcal{M}^\Sigma} \cap \|B\|^{\mathcal{M}^\Sigma} =$ (by the inductive hypothesis) $|A|_\Sigma \cap |B|_\Sigma = |A \wedge B|_\Sigma$.

Case $A \vee B$: Similar to the proof for $A \wedge B$. Details omitted.

Case $A \rightarrow B$: Similar to the proofs for $A \wedge B$ and $A \vee B$.

Case $\Box A$: This is the key bit. We need to show that $\mathcal{M}^\Sigma, w \models \Box A \;\Leftrightarrow\; \Box A \in w$ (assuming the inductive hypothesis).

$$\begin{aligned}
\mathcal{M}^\Sigma, w \models \Box A \;&\Leftrightarrow\; \forall w'\,[\,w\, R^\Sigma\, w' \Rightarrow \mathcal{M}^\Sigma, w' \models A\,] \\
&\Leftrightarrow\; \forall w'\,[\,w\, R^\Sigma\, w' \Rightarrow A \in w'\,] \qquad \text{by the inductive hypothesis}
\end{aligned}$$

So we need to show that $\Box A \in w \;\Leftrightarrow\; \forall w'\,[\,w\, R^\Sigma\, w' \Rightarrow A \in w'\,]$.

Lemma $\Box A \in w \;\Leftrightarrow\; \forall w'\,[\,w\, R^\Sigma\, w' \Rightarrow A \in w'\,]$.
Left-to-right: suppose $\Box A \in w$ and $w\, R^\Sigma\, w'$. Then $A \in w'$ follows immediately from the definition of R^Σ.
Right-to-left: suppose $\Box A \notin w$. We need to show that $\exists w'\,[w\, R^\Sigma\, w'$ and $A \notin w']$.

$$\begin{aligned}
\exists w'\,[w\, R^\Sigma\, w' \text{ and } A \notin w'] \;&\Leftrightarrow\; \exists w'\,[w\, R^\Sigma\, w' \text{ and } \neg A \in w'] \qquad (w' \text{ is } \Sigma\text{-maxi-consistent}) \\
&\Leftrightarrow\; \exists w'\,[\{B \mid \Box B \in w\} \subseteq w' \text{ and } \neg A \in w'] \qquad (\text{definition of } R^\Sigma) \\
&\Leftrightarrow\; \exists w'\,[\{B \mid \Box B \in w\} \cup \{\neg A\} \subseteq w']
\end{aligned}$$

By Lindenbaum's lemma, it is enough to show that $\{B \mid \Box B \in w\} \cup \{\neg A\}$ is Σ-consistent. Suppose not: suppose $\{B \mid \Box B \in w\} \cup \{\neg A\}$ is Σ-inconsistent. Then $\vdash_\Sigma (B_1 \wedge \cdots \wedge B_n) \rightarrow A$ for some $\{\Box B_1, \ldots, \Box B_n\} \subseteq w$. But Σ is normal and w is a Σ-maxi-consistent set, so w must contain also $(\Box B_1 \wedge \cdots \wedge \Box B_n) \rightarrow \Box A$. Since all of $\Box B_1, \ldots, \Box B_n$ belong to w, then $\Box A \in w$. This contradicts the hypothesis that $\Box A \notin w$.

Theorem 7 *Let $\mathcal{M}^\Sigma$ be the canonical model for a normal system Σ. Then:*

$$\mathcal{M}^\Sigma \models A \Leftrightarrow \vdash_\Sigma A$$

Proof This follows immediately from previous theorem.
We know that $\vdash_\Sigma A$ iff A is a member of every Σ-maxi-consistent set, i.e., $\vdash_\Sigma A$ iff $A \in w$ for every w in $\mathcal{M}^\Sigma$. But by the previous theorem, $A \in w$ iff $\mathcal{M}^\Sigma, w \models A$, and $A \in w$ for every w in $\mathcal{M}^\Sigma$ is therefore $\mathcal{M}^\Sigma \models A$.

Notice that Theorem 6 (truth lemma) provides a stronger condition than we actually need. It says that for all formulas A:

$$\forall w\, [\, \mathcal{M}^\Sigma, w \models A \;\Leftrightarrow\; A \in w\,]$$

For Theorem 7 we need only

$$\forall w\, [\, \mathcal{M}^\Sigma, w \models A\,] \;\Leftrightarrow\; \forall w \in \mathcal{M}^\Sigma\, [\, A \in w\,]$$

which is obviously a weaker condition.

This stronger condition means that Theorem 6 (truth lemma) can be used for what Blackburn *et al* call 'strong completeness' results (which we are ignoring).

Since we have shown above that there exists a canonical model for any normal modal logic Σ, and since this model is obviously a relational ('Kripke') model, we immediately have the following:

Theorem 8 *Every normal modal logic is complete with respect to the class of relational ('Kripke') models/frames.*

Of course, not all normal logics will be *sound* with respect to all relational ('Kripke') models. But the smallest normal logic, system K, is sound with respect to all relational ('Kripke) models. And so:

Theorem 9 *The smallest normal modal logic, system K, is sound and complete with respect to the class of relational ('Kripke') models/frames.*

Note again The inconsistent logic (the set of all formulas) is a normal modal logic. (Trivial – exercise in earlier set of notes).

What is its canonical model? Answer: it doesn't have one. The worlds of the canonical model are the maxi-consistent sets, and there aren't any maxiconsistent sets for the inconsistent logic. A model must have at least one world.

Trivially, the inconsistent logic is complete for any class of frames/models.

But the inconsistent logic is not sound for any class of frames/models.

(Deliberately left blank)

Examples

Example The normal modal logic S4 (= *KT4*) is sound and complete with respect to the class of reflexive, transitive frames.

Proof *Soundness:* As usual, this is easy. We just need to check that schemas T ($\Box A \to A$) and 4 ($\Box A \to \Box\Box A$) are valid in the class of reflexive, transitive frames. Exercise.

Completeness: We show that the relation R^{S4} of the canonical model for S4 belongs to the class in question, i.e. that R^{S4} defined as

$$w\, R^{S4}\, w' \;\Leftrightarrow\; \forall A\, [\, \Box A \in w \;\Rightarrow\; A \in w'\,]$$

is both reflexive and transitive.

Reflexive: We need to show (for all formulas A and worlds/S4-maxi-consistent sets w) that $\forall A\, [\, \Box A \in w \;\Rightarrow\; A \in w\,]$.
Suppose $\Box A \in w$. Then since S4 contains the schema T ($\Box A \to A$) and w is S4-maxi-consistent, it follows that $A \in w$. Done.

Here is the first step in full, in case it is not obvious:

$$\begin{array}{rl} \Box A \to A \in w & \text{(4 is in S4, and } w \text{ is S4-maxi)} \\ \text{Suppose } \Box A \in w & \\ \text{Then } A \in w & (w \text{ is S4-maxi, and hence closed under MP)} \end{array}$$

Transitive: We need to show $w\, R^{S4}\, w'$, $w'\, R^{S4}\, w'' \;\Rightarrow\; w\, R^{S4}\, w''$ for all w, w', w'' in the canonical model.
Suppose (1) $w\, R^{S4}\, w'$, i.e., $\{\, A \mid \Box A \in w\,\} \subseteq w'$ and (2) $w'\, R^{S4}\, w''$, i.e., $\{\, A \mid \Box A \in w'\,\} \subseteq w''$. We need to show $w\, R^{S4}\, w''$, i.e., $\forall A\, [\, \Box A \in w \;\Rightarrow\; A \in w''\,]$.

So: suppose $\Box A \in w$. We need to show $A \in w''$.

$$\begin{array}{rcll} \Box A \in w & \Rightarrow & \Box\Box A \in w & \text{(4 is in S4, and } w \text{ is maxi)} \\ \Box\Box A \in w & \Rightarrow & \Box A \in w' & (w\, R^{S4}\, w') \\ \Box A \in w' & \Rightarrow & A \in w'' & (w'\, R^{S4}\, w'') \end{array}$$

Done.

Again, here is the first step in full, in case it is not obvious:

$$\begin{array}{rl} \Box A \to \Box\Box A \in w & \text{(4 is in S4, and } w \text{ is S4-maxi)} \\ \text{Suppose } \Box A \in w & \\ \text{Then } \Box\Box A \in w & (w \text{ is S4-maxi, hence closed under MP)} \\ \text{Hence } \Box A \in w \;\Rightarrow\; \Box\Box A \in w & \end{array}$$

Here is an example to show that the alternative, equivalent definition of R^Σ (Theorem 5) is sometimes very convenient.

Example The normal modal logic B (= *KB*) is sound and complete with respect to the class of symmetric frames.

Proof *Soundness:* Check that schema B ($A \to \Box\Diamond A$) is valid in the class of symmetric frames. Easy exercise.

Completeness: We show that the relation R^B of the canonical model for B= *KB* is symmetric.

We need to show (for all B-maxi-consistent sets w, w') that
$\{\, A \mid \Box A \in w\,\} \subseteq w' \Rightarrow \{\, A \mid \Box A \in w'\,\} \subseteq w$.
Equivalently (Theorem 5) we show $\{\, A \mid \Box A \in w\,\} \subseteq w' \Rightarrow \{\, \Diamond A \mid A \in w\,\} \subseteq w'$.
Or equivalently again, that $\forall A\, [\, \Box A \in w \;\Rightarrow\; A \in w'\,]$ ($w\, R^B\, w'$) implies $\forall A\, [\, A \in w \;\Rightarrow\; \Diamond A \in w'\,]$ ($w'\, R^B\, w$).

Suppose (1) $w\, R^B\, w'$, and (2) $A \in w$. Need to show $\Diamond A \in w'$.

$$\begin{array}{rcll} A \in w & \Rightarrow & \Box\Diamond A \in w & \text{(B, and } w \text{ is maxi)} \\ \Box\Diamond A \in w & \Rightarrow & \Diamond A \in w' & (w\, R^B\, w') \end{array}$$

Done.

Again, first step in full:

$$\begin{array}{rl} A \to \Box\Diamond A \in w & \text{(B, and } w \text{ is } KB\text{-maxi)} \\ \text{Suppose } A \in w & \\ \text{Then } \Box\Diamond A \in w & (w \text{ is } KB\text{-maxi, and hence closed under MP)} \end{array}$$

Example A different kind of proof ...
Show KD is complete with respect to serial frames (for all w, there exists w' such that $w\, R\, w'$).
For the canonical frame $\langle W^{KD}, R^{KD} \rangle$

$$\begin{aligned} w\, R^{KD}\, w' &\Leftrightarrow \forall A\, [\, \Box A \in w \Rightarrow A \in w'\,] \\ &\Leftrightarrow \{A \mid \Box A \in w\} \subseteq w' \end{aligned}$$

So we want to show that for every w in W^{KD}

$$\exists w'\, \{A \mid \Box A \in w\} \subseteq w'$$

By Lindenbaum's lemma it is sufficient to show that

$$\{A \mid \Box A \in \Gamma\}$$

is KD-consistent for any KD-maxi-consistent set Γ.
(Easy exercise.)

Multi-modal normal logics

You can easily check that the definitions and theorems above can all be generalised straightforwardly to the multi-modal case. (The structural induction is hardly affected. Try it.)

Example Suppose we have a logic Σ with two 'box' operators K_a and K_b, interpreted on frames of the form $\langle W, R_a, R_b \rangle$ where R_a and R_b are the accessibility relations corresponding to operators K_a and K_b, respectively. The logic of K_a and K_b individually is normal. (You can read $\mathsf{K}_a A$ and $\mathsf{K}_b A$ as 'a knows that A' and 'b knows that A', respectively.)
If

$$\vdash_\Sigma \mathsf{K}_b A \to \mathsf{K}_a A$$

then the canonical frame $\langle W^\Sigma, R_a^\Sigma, R_b^\Sigma \rangle$ has the property

$$R_a^\Sigma \subseteq R_b^\Sigma$$

Suppose $w\, R_a^\Sigma\, w'$. We need to show $w\, R_b^\Sigma\, w'$,i.e.,

$$\forall A\, [\, \mathsf{K}_b A \in w \;\Rightarrow\; A \in w' \,]$$

Suppose $\mathsf{K}_b A \in w$. We need to show $A \in w'$.

$$\begin{array}{lll} \mathsf{K}_b A \in w \;\Rightarrow\; \mathsf{K}_a A \in w & & \text{(axiom, and } w \text{ is maxi)} \\ \mathsf{K}_a A \in w \;\Rightarrow\; A \in w' & & (w\, R_a^\Sigma\, w') \end{array}$$

Done.

Again, just to clear, here is the first step in full:

$$\begin{array}{rl} \mathsf{K}_b A \to \mathsf{K}_a A \in w & \quad \text{(axiom, and } w \text{ is maxi)} \\ \text{Suppose } \mathsf{K}_b A \in w & \\ \text{Then } \mathsf{K}_a A \in w & \quad (w \text{ is maxi, and so closed under MP)} \end{array}$$

Example (the minimal normal temporal logic) (Blackburn *et al*, p204–206)
The basic temporal language has two 'diamonds' F and P, whose respective duals are G and H. F and G look forwards along the flow of time, and P and H look backwards.
Usually, the language is interpreted on a frame $\langle W, R \rangle$ with the truth conditions for P and H modified to make sure they look backwards along R.
Suppose we interpret such a language on frames of the form $\langle W, R_\mathrm{F}, R_\mathrm{P} \rangle$. For temporal logics, we are only interested in frames where the relations R_F and R_P are mutually converse: $w\, R_\mathrm{F}\, w'$ iff $w'\, R_\mathrm{P}\, w$.
It is easy to check that the following schema is valid in all such frames:

$$(A \to \mathsf{HF}A) \wedge (B \to \mathsf{GP}B) \qquad (*)$$

Now we show that if Σ with G and H both normal contains schema (*) then the canonical frame $\langle W^\Sigma, R_\mathrm{F}^\Sigma, R_\mathrm{P}^\Sigma \rangle$ is such that:

$$w\, R_\mathrm{F}^\Sigma\, w' \;\Leftrightarrow\; w'\, R_\mathrm{P}^\Sigma\, w$$

for all w, w'.

For left-to-right: Suppose $w\, R_\mathrm{F}^\Sigma\, w'$, i.e.,

$$\forall A\, [\, \mathsf{G}A \in w \;\Rightarrow\; A \in w' \,] \quad \text{or equivalently} \quad \forall A\, [\, A \in w' \;\Rightarrow\; \mathsf{F}A \in w \,]$$

We show $w'\, R_\mathrm{P}^\Sigma\, w$, i.e.,

$$\forall A\, [\, \mathsf{H}A \in w' \;\Rightarrow\; A \in w \,] \quad \text{or equivalently} \quad \forall A\, [\, A \in w \;\Rightarrow\; \mathsf{P}A \in w' \,]$$

The second version is easier. Suppose $A \in w$. We show $\mathsf{P}A \in w'$.

$$\begin{array}{rll} A \in w \;\Rightarrow\; \mathsf{GP}A \in w & & \text{(schema (*) and } w \text{ maxi)} \\ \mathsf{GP}A \in w \;\Rightarrow\; \mathsf{P}A \in w' & & (w\, R_\mathrm{F}^\Sigma\, w') \end{array}$$

Done. (The other direction is similar.)

Sahlqvist theorems

From Ian Hodkinson's notes ...

Theorem (Sahlqvist Correspondence Theorem) Let A be a Sahlqvist formula. There is a corresponding first-order frame property that holds of a frame iff A is valid in the frame. (This property can be obtained from A by a simple algorithm.)

Here is its completeness twin ...

Definition (Canonical for a Property) Let A be a formula, and P be a property. The formula A is *canonical for* P if

- the canonical frame for any normal logic Σ containing A has property P; and
- A is valid in any class of frames with property P.

(Blackburn *et al*, p204.)

Example all instances of 4 are canonical for transitivity, because the presence of 4 forces canonical frames to be transitive, and 4 is valid in all transitive frames.

Theorem (Sahlqvist Completeness Theorem) Every Sahlqvist formula is canonical for the property it defines. That is: if A is a Sahlqvist formula defining property P, then A is valid in any class of frames with property P, and the canonical frame for any normal logic Σ containing A has property P.
(Proof omitted.)

So: given a set of Sahlvist formulas ξ, the normal modal logic $K\xi$ is (strongly) complete with respect to the first-order class of frames defined by ξ.

Other definitions

The first one is quite often encountered ...

Definition (Canonical logic) A normal logic Σ is *canonical* if, for all $A \in \Sigma$, A is valid in the canonical frame for Σ.
(A normal logic is canonical if all its formulas are valid in its canonical frame.)

Not all normal logics are canonical.
Example: $KL = K \cup \{\Box(\Box A \rightarrow A) \rightarrow \Box A)\}$ is not canonical. (Blackburn *et al*, p211.)

And a definition I can never remember (I wouldn't bother with it, personally) ...

Definition (Canonical formula) A formula A is *canonical* if, for any normal logic Σ, $A \in \Sigma$ implies that A is valid in the canonical frame for Σ.