4. Prove the following statements (also given in the notes under ‘Example’)

(i) The inconsistent logic is a normal logic.
(ii) PL is not a normal logic.
(iii) If \( \{ \Sigma_i | i \in I \} \) is a collection of normal logics, then \( \cap_{i \in I} \Sigma_i \) is a normal logic.
(iv) If \( F \) is any class of relational (‘Kripke’) frames then \( \Sigma_F \), the set of formulas valid in \( F \), is a normal logic.

5. Prove that every normal logic has the following rules of inference and theorems:

<table>
<thead>
<tr>
<th>Rule</th>
<th>Premises</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>RM0.</td>
<td>( A \rightarrow B )</td>
<td>( \Box A \rightarrow \Box B )</td>
</tr>
<tr>
<td>RM1.</td>
<td>( \Box A \rightarrow \Box (A \wedge B) )</td>
<td>( \Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) )</td>
</tr>
<tr>
<td>RM2.</td>
<td>( \Box (A \rightarrow B) )</td>
<td>( (\Box A \wedge \Box B) \rightarrow (\Box A \rightarrow \Box B) )</td>
</tr>
</tbody>
</table>

6. Prove that the normal logic KT5 is the same as the normal logic KT45. (Both are the logic called S5.)

What you need to do is to show that KT5 contains all instances of the schema 4:

4. \( \Box A \rightarrow \Box \Box A \)

Schemas T and 5 are as follows:

<table>
<thead>
<tr>
<th>Schema</th>
<th>Premises</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>T.</td>
<td>( \Box A \rightarrow A )</td>
<td>( \Box A \rightarrow \Box \Box A )</td>
</tr>
</tbody>
</table>

Hint: From T and 5 you can derive \( A \rightarrow \Box \Box A \).

From 5 you can derive \( \Box \Box A \rightarrow \Box A \).

7. (from 2003 exam)

Consider a language with modal operators O and \( \Box \). Models are Kripke structures \( (W; R_O, R_\Box, h) \) where \( W \) is a set of worlds, \( h \) is a valuation function for the atoms, and \( R_O \) and \( R_\Box \) are the accessibility relations for the operators O and \( \Box \) respectively.

Suppose further that O is a normal modality of type KD (\( R_O \) is serial) and \( \Box \) is a normal modality of type KT (\( R_\Box \) is reflexive).

Suppose now that the language is extended with another modal operator Oblig defined as follows:

\[ \text{Oblig } A \overset{\text{df}}{=} O A \wedge \neg A \]

Show that Oblig has the following properties

| NoN. | \( \neg \Box T \) |
| D. | Oblig \( A \rightarrow \neg \Box A \) |
| C. | \( \Box (Oblig \ A \wedge Oblig \ B) \rightarrow Oblig (A \wedge B) \) |

You may use any standard properties of normal logics, such as \( \Box (A \wedge B) \rightarrow \Box A \), as long as you identify them clearly.

/More Overleaf …
8. Prove that the following are theorems of every classical ET5 system:

- P. \( \Diamond \top \)
- N. \( \Box \top \)

The next two questions are more demanding. You might want to consult the solutions as you go along.

9. Show that the following ‘reduction laws’ are all theorems of the normal system S4 (=KT4):

- \( \Box A \leftrightarrow \Box \Box A \)
- \( \Diamond A \leftrightarrow \Diamond \Diamond A \)
- \( \Diamond \Box A \leftrightarrow \Diamond \Diamond \Diamond A \)
- \( \Box \Diamond A \leftrightarrow \Box \Box \Box A \)

Now show that every modality (i.e., every sequence of \( \Box \), \( \Diamond \) and \( \neg \), in any order) is equivalent in S4 to one of 14 distinct modalities.

10. By comparison with the previous question, try to identify the reduction laws for the system S5 (=KT45 = KT5), and hence determine how many distinct modalities there are in S5.