499 Modal and Temporal Logic *Autumn 2008*

Tutorial Exercises 2 (mjs) (Maxi-consistent sets)

1. *From 2002 exam* S4 is the normal modal logic $KT4$. Prove that if $\{\Box A_1, \ldots, \Box A_n, \neg\Box B\}$ is S4-consistent then so is $\{\Box A_1, \ldots, \Box A_n, \neg B\}$.

   (This is not a question about maxi-consistent sets.)

2. This is not so much an 'exercise' as a few small observations about the properties of maxi-consistent sets.

   Let Γ be a S4-maxi-consistent set. (S4 is the normal modal logic $KT4$.) p and q are atoms.

   (i) Suppose $p \in \Gamma$. Does that imply $q \in \Gamma$?
   (Obviously not. This is an example that shows therefore that maxi-consistent sets are not closed under uniform substitution US.)
   Informally you can think of Γ as a 'possible world'. If p holds in a possible world, does that imply q also holds in that possible world? Obviously not—but can you give a more careful formal explanation?

   (ii) Suppose $p \in \Gamma$. Does that imply $\Box p \in \Gamma$?
   (Obviously not. This is an example that shows therefore that maxi-consistent sets, even of normal logics, are not closed under the rule of necessitation RN.)

   (iii) Suppose $p \in \Gamma$. Does that imply $\Diamond p \in \Gamma$?
   (Γ is a maxi-consistent set of S4=$KT4$ remember.)

   (iv) Suppose $A_1 \wedge \ldots \wedge A_n \rightarrow A \in \Gamma$. Does that imply $\Box A_1 \wedge \ldots \wedge \Box A_n \rightarrow \Box A \in \Gamma$?

   (v) Suppose $A_1 \wedge \ldots \wedge A_n \rightarrow A \in$ S4. Does that imply $\Box A_1 \wedge \ldots \wedge \Box A_n \rightarrow \Box A \in \Gamma$?

3. (This is one of the unproved theorems in the notes.) Prove that:

   (a) $\Gamma \vdash_\Sigma A$ iff $A \in \Delta$ for every Σ-maxi-consistent Δ such that $\Gamma \subseteq \Delta$.

   (b) $\vdash_\Sigma A$ iff $A \in \Delta$ for every Σ-maxi-consistent Δ.

   Hint: for the first one, one half is easy, the other half requires Lindenbaum's lemma. The second follows more or less immediately as a special case of the first.

4. (The following result is useful when we define canonical models for normal systems.)

   Prove that for any Σ-maxi-consistent sets Γ and Γ'

$$\{A \mid \Box A \in \Gamma\} \subseteq \Gamma' \quad \Leftrightarrow \quad \{\Diamond A \mid A \in \Gamma'\} \subseteq \Gamma$$

   or equivalently

$$\forall A\,[\,\Box A \in \Gamma \Rightarrow A \in \Gamma'\,] \quad \Leftrightarrow \quad \forall A\,[\,A \in \Gamma' \Rightarrow \Diamond A \in \Gamma\,]$$