Tutorial Exercises 2 (mjs)

SOLUTIONS

1. We prove the contrapositive. Suppose \{\Box A_1, \ldots, \Box A_n, \neg B\} is S4-inconsistent.

   Then either
   
   (i) \Gamma \vdash_{\text{S4}} (\Box A_1 \land \cdots \land \Box A_k) \rightarrow \bot
   
   or
   
   (ii) \Gamma \vdash_{\text{S4}} (\Box A_1 \land \cdots \land \Box A_k \land \neg B) \rightarrow \bot
   
   for some \{\Box A_1, \ldots, \Box A_n\} \subseteq \{\Box A_1, \ldots, \Box A_n\}.

   If case (i) then \{\Box A_1, \ldots, \Box A_n, \neg B\} is also S4-inconsistent.

   If case (ii) then \Gamma \vdash_{\text{S4}} (\Box A_1 \land \cdots \land \Box A_k) \rightarrow B.

   And so (S4 is normal, and rule RK) \Gamma \vdash_{\text{S4}} (\Box A_i \land \cdots \land \Box A_k \land \neg B) \rightarrow \Box B.

   But (schema 4, and RPL) \Gamma \vdash_{\text{S4}} (\Box A_i \land \cdots \land \Box A_k) \rightarrow (\Box A_i \land \cdots \land \Box A_k)

   and so \Gamma \vdash_{\text{S4}} (\Box A_i \land \cdots \land \Box A_k) \rightarrow \Box B.

   Hence \Gamma \vdash_{\text{S4}} (\Box A_i \land \cdots \land \Box A_k \land \neg B) \rightarrow \bot

   and so \{\Box A_1, \ldots, \Box A_n, \neg B\} is S4-inconsistent.

   (Note that this doesn’t use schema T.)

   The following (slightly quicker) is also fine.

   If \{\Box A_1, \ldots, \Box A_n, \neg B\} is S4-inconsistent then

   \Gamma \vdash_{\text{S4}} (\Box A_1 \land \cdots \land \Box A_k) \rightarrow B

   Now (same argument as above, details omitted)

   \Gamma \vdash_{\text{S4}} (\Box A_1 \land \cdots \land \Box A_k) \rightarrow B

   So \{\Box A_1, \ldots, \Box A_n, \neg B\} is S4-inconsistent.

3. This is a theorem in the notes relating deducibility (\vdash_{\Sigma}) with maxiconsistent sets. We need to prove that:

   (a) \Gamma \vdash_{\Sigma} A \iff A \in \Delta \text{ for every } \Sigma\text{-maxi-consistent } \Delta \text{ such that } \Gamma \subseteq \Delta.

   (b) \vdash_{\Sigma} A \iff A \in \Delta \text{ for every } \Sigma\text{-maxi-consistent } \Delta.

   Proof. Left to right: suppose \Gamma \vdash_{\Sigma} A. Suppose \Gamma \subseteq \Delta. Then \Delta \vdash_{\Sigma} A \text{ (monotonicity of } \vdash_{\Sigma}). \text{ For the other half: suppose } \Gamma \not\vdash_{\Sigma} A. \text{ We have to show there is a } \Sigma\text{-maxi-consistent } \Delta \text{ such that } \Gamma \subseteq \Delta \text{ and } A \not\in \Delta. \text{ From } \Gamma \not\vdash_{\Sigma} A \text{ it follows that } \Gamma \cup \{\neg A\} \text{ is } \Sigma\text{-consistent. By Lindenbaum’s lemma there is therefore a } \Sigma\text{-maxi-consistent } \Delta \text{ such that } \Gamma \cup \{\neg A\} \subseteq \Delta. \text{ Because } \{\neg A\} \subseteq \Delta, \text{ i.e., } \neg A \in \Delta, \text{ } A \not\in \Delta \text{ as required.}

   Part (b) is just the special case of part (a) where } \Gamma = \emptyset, \text{ and so follows immediately remembering that } \emptyset \vdash_{\Sigma} A \iff \vdash_{\Sigma} A.
4. We want to prove that for any Σ-maxi-consistent sets Γ and Γ’

\[
\{A \mid \Box A \in \Gamma\} \subseteq \Gamma' \iff \{\Diamond A \mid A \in \Gamma'\} \subseteq \Gamma
\]

or equivalently

\[
\forall A [\Box A \in \Gamma \Rightarrow A \in \Gamma'] \iff \forall A [A \in \Gamma' \Rightarrow \Diamond A \in \Gamma]
\]

Assume LHS. Now suppose \(A \in \Gamma'\). We need to show \(\Diamond A \in \Gamma\).

Suppose not. Suppose \(\Diamond A \notin \Gamma\).

\[
\Diamond A \notin \Gamma \Rightarrow \neg \Diamond A \in \Gamma \quad (\Gamma \text{ is maxi})
\]

\[
\neg \Diamond A \in \Gamma \Rightarrow \Box \neg A \in \Gamma
\]

\[
\Box \neg A \in \Gamma \Rightarrow \neg A \in \Gamma' \quad (\text{assumed LHS})
\]

\[

\neg A \in \Gamma' \Rightarrow A \notin \Gamma' \quad (\Gamma' \text{ is } \Sigma\text{-consistent})
\]

\[
A \notin \Gamma' \quad \text{Contradiction (we assumed } A \in \Gamma')
\]

The other direction is similar. Here it is . . .

Assume RHS. Now suppose \(\Box A \notin \Gamma\). We need to show \(A \in \Gamma'\).

Suppose not. Suppose \(A \notin \Gamma'\).

\[

A \notin \Gamma' \Rightarrow \neg A \in \Gamma' \quad (\Gamma' \text{ is maxi})
\]

\[

\neg A \in \Gamma' \Rightarrow \Diamond A \notin \Gamma \quad (\text{assumed RHS})
\]

\[

\Diamond A \notin \Gamma \Rightarrow \neg \Diamond A \notin \Gamma \quad (\Gamma \text{ is } \Sigma\text{-consistent})
\]

\[

\neg \Diamond A \notin \Gamma \Rightarrow \Box A \notin \Gamma
\]

\[

\Box A \notin \Gamma \quad \text{Contradiction (we assumed } \Box A \notin \Gamma\)
\]