499 Modal and Temporal Logic *Autumn 2008*

Tutorial Exercises 2 (mjs)

SOLUTIONS

1. We prove the contrapositive. Suppose $\{\Box A_1, \ldots, \Box A_n, \neg B\}$ is S4-inconsistent.

   Then either

   (i) $\vdash_{S4} (\Box A_i \wedge \cdots \wedge \Box A_k) \rightarrow \bot$

   or (ii) $\vdash_{S4} (\Box A_i \wedge \cdots \wedge \Box A_k \wedge \neg B) \rightarrow \bot$

   for some $\{\Box A_i, \ldots, \Box A_k\} \subseteq \{\Box A_1, \ldots, \Box A_n\}$.

   If case (i) then $\{\Box A_1, \ldots, \Box A_n, \neg\Box B\}$ is also S4-inconsistent.

   If case (ii) then $\vdash_{S4} (\Box A_i \wedge \cdots \wedge \Box A_k) \rightarrow B$.

   And so (S4 is normal, and rule RK) $\vdash_{S4} (\Box\Box A_i \wedge \cdots \wedge \Box\Box A_k) \rightarrow \Box B$.

   But (schema 4, and RPL) $\vdash_{S4} (\Box A_i \wedge \cdots \wedge \Box A_k) \rightarrow (\Box\Box A_i \wedge \cdots \wedge \Box\Box A_k)$

   and so $\vdash_{S4} (\Box A_i \wedge \cdots \wedge \Box A_k) \rightarrow \Box B$.

   Hence $\vdash_{S4} (\Box A_i \wedge \cdots \wedge \Box A_k \wedge \neg\Box B) \rightarrow \bot$ and so $\{\Box A_1, \ldots, \Box A_n, \neg\Box B\}$ is S4-inconsistent.

   (Note that this doesn't use schema T.)

   The following (slightly quicker) is also fine.

   If $\{\Box A_1, \ldots, \Box A_n, \neg B\}$ is S4-inconsistent then

   $$\vdash_{S4} (\Box A_i \wedge \cdots \wedge \Box A_k) \rightarrow B$$

   Now (same argument as above, details omitted)

   $$\vdash_{S4} (\Box A_i \wedge \cdots \wedge \Box A_k) \rightarrow \Box B$$

   So $\{\Box A_1, \ldots, \Box A_n, \neg\Box B\}$ is S4-inconsistent.

2. (i) You could say $\{p, q\}$ and $\{p, \neg q\}$ are both S4-consistent, so (by Lindenbaum's lemma) there are at least two distinct S4-maxi-consistent sets containing p — one has q and the other has $\neg q$.

   Or: if $p \in \Gamma$ implied $q \in \Gamma$ that would mean $p \rightarrow q \in \Gamma$ (by Theorem 6(8) of the lecture notes). Since Γ is arbitrary, we would have shown $p \rightarrow q \in \Gamma$ for every S4-maxi-consistent set Γ, and hence (by Theorem 8(2) of the notes—see next question of this sheet) that $\vdash_{S4} p \rightarrow q$, which is clearly not true.

   (ii) Same argument as above. $\{p, \Box p\}$ and $\{p, \neg\Box p\}$ are both S4-consistent.

   Or: by the same argument as above, we would have $\vdash_{S4} p \rightarrow \Box p$, which is clearly not true.

   (iii) Yes, $p \in \Gamma$ does imply $\Diamond p \in \Gamma$. Because ...

   $\vdash_{S4} p \rightarrow \Diamond p$. This is because S4 contains all instances of the schema T ($\Box A \rightarrow A$), of which one instance is $\Box\neg p \rightarrow \neg p$, which is propositionally equivalent to $p \rightarrow \neg\Box\neg p$. (Or: the 'dual schema' of T is $A \rightarrow \Diamond A$.)

   Since $\vdash_{S4} p \rightarrow \Diamond p$ and Γ is S4-maxi-consistent, $p \rightarrow \Diamond p \in \Gamma$. But $p \in \Gamma$ and Γ is closed under MP, so $\Diamond p \in \Gamma$.

   (iv) No. (Part (ii) is already a counter-example for the case $n = 0$.)

   (v) Yes. If $A_1 \wedge \cdots \wedge A_n \rightarrow A \in$ S4, then $\Box A_1 \wedge \cdots \wedge \Box A_n \rightarrow \Box A \in$ S4 (by the rule RK, and the fact that S4 is normal).

   If $\Box A_1 \wedge \cdots \wedge \Box A_n \rightarrow \Box A \in$ S4 then $\Box A_1 \wedge \cdots \wedge \Box A_n \rightarrow \Box A \in \Gamma$ because any S4-maxi-consistent set Γ contains all theorems of S4.

3. This is a theorem in the notes relating deducibility ($\vdash_\Sigma$) with maxiconsistent sets. We need to prove that:

   (a) $\Gamma \vdash_\Sigma A$ iff $A \in \Delta$ for every Σ-maxi-consistent Δ such that $\Gamma \subseteq \Delta$.

   (b) $\vdash_\Sigma A$ iff $A \in \Delta$ for every Σ-maxi-consistent Δ.

   Proof. Left to right: suppose $\Gamma \vdash_\Sigma A$. Suppose $\Gamma \subseteq \Delta$. Then $\Delta \vdash_\Sigma A$ (monotonicity of $\vdash_\Sigma$). For the other half: suppose $\Gamma \nvdash_\Sigma A$. We have to show there is a Σ-maxi-consistent Δ such that $\Gamma \subseteq \Delta$ and $A \notin \Delta$. ¿From $\Gamma \nvdash_\Sigma A$, it follows that $\Gamma \cup \{\neg A\}$ is Σ-consistent. By Lindenbaum's lemma there is therefore a Σ-maxi-consistent Δ such that $\Gamma \cup \{\neg A\} \subseteq \Delta$. Because $\{\neg A\} \subseteq \Delta$, i.e., $\neg A \in \Delta$, $A \notin \Delta$ as required.

   Part (b) is just the special case of part (a) where $\Gamma = \emptyset$, and so follows immediately remembering that $\emptyset \vdash_\Sigma A \Leftrightarrow \vdash_\Sigma A$.

4. We want to prove that for any Σ-maxi-consistent sets Γ and Γ'

$$\{A \mid \Box A \in \Gamma\} \subseteq \Gamma' \quad \Leftrightarrow \quad \{\Diamond A \mid A \in \Gamma'\} \subseteq \Gamma$$

or equivalently

$$\forall A\,[\,\Box A \in \Gamma \Rightarrow A \in \Gamma'\,] \quad \Leftrightarrow \quad \forall A\,[\,A \in \Gamma' \Rightarrow \Diamond A \in \Gamma\,]$$

Assume LHS. Now suppose $A \in \Gamma'$. We need to show $\Diamond A \in \Gamma$.
Suppose not. Suppose $\Diamond A \notin \Gamma$.

$$\begin{array}{rcll}
\Diamond A \notin \Gamma & \Rightarrow & \neg\Diamond A \in \Gamma & \quad (\Gamma \text{ is maxi}) \\
\neg\Diamond A \in \Gamma & \Rightarrow & \Box\neg A \in \Gamma & \\
\Box\neg A \in \Gamma & \Rightarrow & \neg A \in \Gamma' & \quad (\text{assumed LHS}) \\
\neg A \in \Gamma' & \Rightarrow & A \notin \Gamma' & \quad (\Gamma' \text{ is } \Sigma\text{-consistent}) \\
A \notin \Gamma' & & \multicolumn{2}{l}{\text{Contradiction (we assumed } A \in \Gamma')}
\end{array}$$

The other direction is similar. Here it is ...

Assume RHS. Now suppose $\Box A \in \Gamma$. We need to show $A \in \Gamma'$.
Suppose not. Suppose $A \notin \Gamma'$.

$$\begin{array}{rcll}
A \notin \Gamma' & \Rightarrow & \neg A \in \Gamma' & \quad (\Gamma' \text{ is maxi}) \\
\neg A \in \Gamma' & \Rightarrow & \Diamond\neg A \in \Gamma & \quad (\text{assumed RHS}) \\
\Diamond\neg A \in \Gamma & \Rightarrow & \neg\Diamond\neg A \notin \Gamma & \quad (\Gamma \text{ is } \Sigma\text{-consistent}) \\
\neg\Diamond\neg A \notin \Gamma & \Rightarrow & \Box A \notin \Gamma & \\
\Box A \notin \Gamma & & \multicolumn{2}{l}{\text{Contradiction (we assumed } \Box A \in \Gamma)}
\end{array}$$