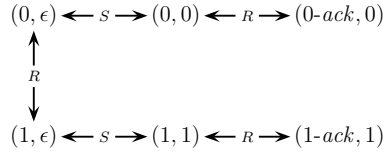


Tutorial Exercises 4 (mjs)

SOLUTIONS

Question 1



1. $IS_b \models \mathbf{recbit} \rightarrow (K_R(\mathbf{bit}=0) \vee K_R(\mathbf{bit}=1))$

recbit is true at global states $(0, 0)$, $(1, 1)$, $(0\text{-ack}, 0)$, $(1\text{-ack}, 1)$. So we need to check that $(K_R(\mathbf{bit}=0) \vee K_R(\mathbf{bit}=1))$ is true at each of these global states.

To show $IS_b, (0, 0) \models (K_R(\mathbf{bit}=0) \vee K_R(\mathbf{bit}=1))$: the global states R -accessible from $(0, 0)$ are $(0, 0)$ and $(0\text{-ack}, 0)$. (Or if you prefer, the global states that are indistinguishable for R from $(0, 0)$ are $(0, 0)$ and $(0\text{-ack}, 0)$.) In both of these global states, **recbit** is true, and so $IS_b, (0, 0) \models K_R(\mathbf{bit}=0)$ and $IS_b, (0, 0) \models (K_R(\mathbf{bit}=0) \vee K_R(\mathbf{bit}=1))$.

Now check the calculation for each of the remaining three global states $(1, 1)$, $(0\text{-ack}, 0)$, $(1\text{-ack}, 1)$.

2. $IS_b \models \mathbf{recack} \rightarrow \mathbf{recbit}$

recack is true at global states $(0\text{-ack}, 0)$, $(1\text{-ack}, 1)$. So we need to check that **recbit** is true at each of these global states, which it clearly is.

3. $IS_b \models (\mathbf{bit}=0) \rightarrow K_S(\mathbf{bit}=0)$

By direct evaluation in the model again. $(\mathbf{bit}=0)$ is true at global states $(0, \epsilon)$, $(0, 0)$, $(0\text{-ack}, 0)$. We need to check that $K_S(\mathbf{bit}=0)$ is true at each of these global states.

For $(0, \epsilon)$, the S -accessible (or S -indistinguishable) global states are $(0, \epsilon)$ and $(0, 0)$. Clearly $(\mathbf{bit} = 0)$ is true at both of these global states.

For $(0, 0)$, the S -accessible (or S -indistinguishable) global states are again $(0, \epsilon)$ and $(0, 0)$. Clearly $(\mathbf{bit} = 0)$ is true at both of these global states (we just checked this).

For $(0\text{-ack}, 0)$, the only S -accessible (or S -indistinguishable) global state is $(0\text{-ack}, 0)$ itself. Clearly $(\mathbf{bit}=0)$ is true at this global state.

4. $IS_b \models \mathbf{recack} \rightarrow K_S \mathbf{recack}$

As in the previous part, evaluate the formula at every global state in the model. **recack** is true at global states $(0\text{-ack}, 0)$ and $(1\text{-ack}, 1)$. The only global state S -accessible from $(0\text{-ack}, 0)$ is $(0\text{-ack}, 0)$ itself, and here **recack** is true. Similarly for $(1\text{-ack}, 1)$.

5. $IS_b \not\models \mathbf{recack} \rightarrow K_R \mathbf{recack}$

We just need to find one global state at which **recack** is true but $K_R \mathbf{recack}$ is false. Consider global state $(0\text{-ack}, 0)$: we show $K_R \mathbf{recack}$ is false at this global state. The global states R -accessible from $(0\text{-ack}, 0)$ are $(0\text{-ack}, 0)$ and $(0, 0)$. But **recack** is false at $(0, 0)$ and so $K_R \mathbf{recack}$ is false at $(0\text{-ack}, 0)$.

6. $IS_b \models \mathbf{recbit} \wedge (\mathbf{bit}=0) \rightarrow K_R(\mathbf{bit}=0)$

recbit \wedge $(\mathbf{bit}=0)$ is true at global states $(0, 0)$ and $(0\text{-ack}, 0)$. The global states R -accessible from $(0, 0)$ are $(0, 0)$ and $(0\text{-ack}, 0)$; the global states R -accessible from $(0, 0)$ are also $(0, 0)$ and $(0\text{-ack}, 0)$. And clearly $(\mathbf{bit}=0)$ is true at both of those global states.

7. $IS_b \models \mathbf{recack} \rightarrow (K_R(\mathbf{bit}=0) \vee K_R(\mathbf{bit}=1))$

One can evaluate this directly at each global state in the model. But it is much quicker to notice that it follows immediately by propositional logic from parts (1) and (2).

8. $IS_b \models \mathbf{recack} \rightarrow K_S(K_R(\mathbf{bit}=0) \vee K_R(\mathbf{bit}=1))$

Look at the previous part: $IS_b \models K_S \mathbf{recack} \rightarrow K_S(K_R(\mathbf{bit}=0) \vee K_R(\mathbf{bit}=1))$ follows from it by rule RM. (If $A \rightarrow B$ is valid in model IS_b then $K_S A \rightarrow K_S B$ is also valid in model IS_b , because that holds for all relational ('Kripke') models, of which IS_b is one.)

Now combine this with part (4) and propositional logic.

9. $IS_b \models \mathbf{recack} \wedge (\mathbf{bit}=0) \rightarrow K_S K_R(\mathbf{bit}=0)$

Parts (2) and (6) by propositional logic give $IS_b \models \mathbf{recack} \wedge (\mathbf{bit}=0) \rightarrow K_R(\mathbf{bit}=0)$. By rule RK ($n=2$) we get $IS_b \models K_S \mathbf{recack} \wedge K_S(\mathbf{bit}=0) \rightarrow K_S K_R(\mathbf{bit}=0)$.

Now parts (4) and (3) by propositional logic give $IS_b \models \mathbf{recack} \wedge (\mathbf{bit}=0) \rightarrow K_S \mathbf{recack} \wedge K_S(\mathbf{bit}=0)$.

Propositional logic again gives $IS_b \models \mathbf{recack} \wedge (\mathbf{bit}=0) \rightarrow K_S K_R(\mathbf{bit}=0)$.

10. $IS_b \not\models \mathbf{recack} \wedge (\mathbf{bit}=0) \rightarrow K_R K_S K_R(\mathbf{bit} = 0)$

recack \wedge $(\mathbf{bit}=0)$ is true at global state $(0\text{-ack}, 0)$ but $IS_b, (0\text{-ack}, 0) \not\models K_R K_S K_R(\mathbf{bit} = 0)$. Because: the global states R -accessible from $(0\text{-ack}, 0)$ are $(0, 0)$ and $(0\text{-ack}, 0)$, and $K_S K_R(\mathbf{bit}=0)$ is false at $(0, 0)$. This is because the global states S -accessible from $(0, 0)$ are $(0, 0)$ and $(0, \epsilon)$, and $K_R(\mathbf{bit}=0)$ is false at $(0, \epsilon)$. And this is because the global states R -accessible from $(0, \epsilon)$ are $(0, \epsilon)$ and $(1, \epsilon)$, and clearly $(\mathbf{bit}=0)$ is false at $(1, \epsilon)$.

Question 2

Validity of T and 4 in reflexive and transitive frames, respectively, has been shown in lecture notes and previous tutorial exercises. I'll repeat a proof for validity of 4 ($\Box A \rightarrow \Box\Box A$) in transitive frames for your convenience.

Let $\mathcal{M} = \langle W, R, h \rangle$ be any model in which R is transitive, and let w be any world in W .

Suppose $\mathcal{M}, w \models \Box A$. We have to show $\mathcal{M}, w \models \Box\Box A$.

So consider any world w' such that $w R w'$. We need to show $\mathcal{M}, w' \models \Box A$.

So consider any world w'' such that $w' R w''$. We need to show $\mathcal{M}, w'' \models A$.

We have $w R w'$ and $w' R w''$, and so also $w R w''$ because R is transitive. Since $\mathcal{M}, w \models \Box A$, and $w R w''$, we have $\mathcal{M}, w'' \models A$, as required.

Now validity of 5 ($\Diamond A \rightarrow \Box\Diamond A$) in symmetric transitive frames.

Let $\mathcal{M} = \langle W, R, h \rangle$ be any model in which R is both symmetric and transitive, and let w be any world in W .

Suppose $\mathcal{M}, w \models \Diamond A$. We have to show $\mathcal{M}, w \models \Box\Diamond A$.

So consider any world w' such that $w R w'$. We need to show $\mathcal{M}, w' \models \Diamond A$.

From $\mathcal{M}, w \models \Diamond A$, we know $\mathcal{M}, w'' \models A$ for some w'' such that $w R w''$.

We have $w R w'$ and so also $w' R w$ because R is symmetric.

But $w' R w$ and $w R w''$ implies $w' R w''$ (R is transitive).

So now we have $w' R w''$ and $\mathcal{M}, w'' \models A$, and so $\mathcal{M}, w' \models \Diamond A$ as required.

(It is easier to see the argument if you draw a picture.)

Question 3

Solution omitted from this version as Question 3 might be part of the assessed coursework.