499 Modal and Temporal Logic

Classical systems

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Autumn 2007

Further reading: B.F. Chellas, *Modal logic: an introduction*. Cambridge University Press, 1980. Chapters 7–9.

Normal systems

Semantics Let $\mathcal{M} = \langle W, R, h \rangle$ be a standard, relational ('Kripke') model. The truth conditions for $\Box A$ and $\Diamond A$ are

$$\mathcal{M}, w \models \Box A \;\Leftrightarrow\; \forall t (w\,R\,t \Rightarrow \mathcal{M}, t \models A)$$
$$\mathcal{M}, w \models \Diamond A \;\Leftrightarrow\; \exists t (w\,R\,t \;\&\; \mathcal{M}, t \models A)$$

In terms of truth sets:

$$\mathcal{M}, w \models \Box A \;\Leftrightarrow\; R[w] \subseteq \|A\|^{\mathcal{M}}$$
$$\mathcal{M}, w \models \Diamond A \;\Leftrightarrow\; R[w] \cap \|A\|^{\mathcal{M}} \neq \emptyset$$

where $R[w] \stackrel{\text{def}}{=} \{t \text{ in } \mathcal{M} : w\,R\,t\}$.

Normal systems Normal systems of modal logic are defined in terms of the schemas

Df$\Diamond$. $\quad \Diamond A \leftrightarrow \neg\Box\neg A$

K. $\quad \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

and the rule of inference ('necessitation')

RN. $\quad \dfrac{A}{\Box A}$

or equivalently, instead of the schema K and the rule RN, by the rule RK:

RK. $\quad \dfrac{(A_1 \wedge \ldots \wedge A_n) \rightarrow A}{(\Box A_1 \wedge \ldots \wedge \Box A_n) \rightarrow \Box A} \quad (n \geq 0)$

The smallest normal modal logic is called K. To name normal systems we write

$$K\xi_1 \ldots \xi_n$$

for the normal modal logic that results when the schemas $\xi_1, \ldots, \xi_n$ are taken as theorems; i.e., $K\xi_1 \ldots \xi_n$ is the smallest normal system of modal logic containing (every instance of) the schemas $\xi_1, \ldots, \xi_n$.

Classical systems of modal logic

(See Chellas [1980], Ch. 7–9.)
Classical systems of modal logic are defined in terms of the schema

Df$\Diamond$. $\quad \Diamond A \leftrightarrow \neg\Box\neg A$

and the rule of inference

RE. $\quad \dfrac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$

Definition 1 (Classical system) *A system of modal logic is* classical *iff it contains* Df$\Diamond$ *and is closed under* RE.

And remember: every system of modal logic, by definition, also contains all tautologies *PL*, and is closed under modus ponens and uniform substitution. (Though not by the definition in Chellas: Chellas does not require closure under uniform substitution. It's a tiny point of detail.)

The smallest classical modal logic is called E. To name classical systems we write

$$E\xi_1 \ldots \xi_n$$

for the classical modal logic that results when the schemas $\xi_1, \ldots, \xi_n$ are taken as theorems; i.e., $E\xi_1 \ldots \xi_n$ is the smallest classical system of modal logic containing (every instance of) the schemas $\xi_1, \ldots, \xi_n$.

Monotonic and regular systems

Classical systems are sometimes classified further. (You don't need to remember the names!!)

RE. $\quad \dfrac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$

RM. $\quad \dfrac{A \rightarrow B}{\Box A \rightarrow \Box B}$

RR. $\quad \dfrac{(A \wedge B) \rightarrow C}{(\Box A \wedge \Box B) \rightarrow \Box C}$

RK. $\quad \dfrac{(A_1 \wedge \ldots \wedge A_n) \rightarrow A}{(\Box A_1 \wedge \ldots \wedge \Box A_n) \rightarrow \Box A} \quad (n \geq 0)$

Definition 2 (Monotonic system) *A system of modal logic is* monotonic *iff it contains* Df$\Diamond$ *and is closed under* RM.

Definition 3 (Regular system) *A system of modal logic is* regular *iff it contains* Df$\Diamond$ *and is closed under* RR.

Theorem 4

(1) *Every monotonic system of modal logic is classical.*
(2) *Every regular system of modal logic is monotonic.*
(3) *Every normal system of modal logic is regular.*

Proof The derivations for parts (1) and (3) are left as an exercise. For (2): suppose the system is regular, i.e. closed under RR:

1.	$A \rightarrow B$	ass.
2.	$A \wedge A \rightarrow B$	1, *PL*
3.	$(\Box A \wedge \Box A) \rightarrow \Box B$	2, RR
4.	$\Box A \rightarrow \Box B$	3, *PL*

Theorem 5

(1) *Every monotonic system of modal logic contains* M.
(2) *Every regular system of modal logic contains* M *and* C.
(3) *Every regular system of modal logic contains* M, C, R *and* K.

Proof Exercise.

Theorem 6 *Let* Σ *be a system of modal logic containing* Df$\Diamond$. *Then:*

(1) Σ *is monotonic iff it contains* M *and is closed under* RE.
(2) Σ *is regular iff it contains* C *and is closed under* RM.
(3) Σ *is regular iff it contains* C *and* M *and is closed under* RE.

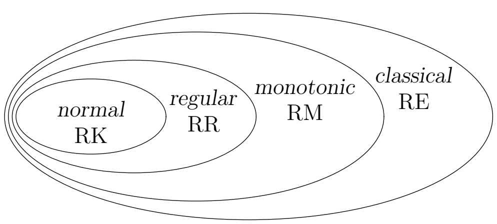
Proof It only remains to show the right-to-left halves.
For (1):

1.	$A \rightarrow B$	ass.
2.	$A \leftrightarrow (A \wedge B)$	1, *PL*
3.	$\Box A \leftrightarrow \Box(A \wedge B)$	2, RE
4.	$\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$	M
5.	$\Box A \rightarrow \Box B$	4, *PL*

For (2):

1.	$(A \wedge B) \rightarrow C$	ass.
2.	$\Box(A \wedge B) \rightarrow \Box C$	1, RM
3.	$(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$	C
4.	$(\Box A \wedge \Box B) \rightarrow \Box C$	2, 3, *PL*

Part (3) follows from parts (1) and (2).



RM	"="	RE + M
RR	"="	RE + MC
RK	"="	RE + MCN

M.	$\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$
C.	$(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$
N.	$\Box\top$

Other schemas

The schemas P, D, T, B, 4, 5 also come up frequently.

P.	$\neg\Box\bot$
D.	$\Box A \rightarrow \Diamond A$
T.	$\Box A \rightarrow A$
B.	$A \rightarrow \Box\Diamond A$
4.	$\Box A \rightarrow \Box\Box A$
5.	$\Diamond A \rightarrow \Box\Diamond A$

We will look at some of their properties later.

Note that for a *normal* system Σ, schema P is in Σ iff D is in Σ. That is *not the case* for non-normal systems in general.

Models for classical systems: 'neighbourhood semantics'

Here we look at 'neighbourhood semantics', also known as Montague-Scott semantics — the most general kind of possible-worlds semantics compatible with retaining the classical truth-table semantics for the truth-functional operators. In these notes, we shall call these models 'ν-models' for short.

Chellas [1980] calls these models 'minimal models'. This last name is best avoided, however, since 'minimal model' is now used very extensively for a completely different notion in the study of non-monotonic and defeasible logics. (See *Knowledge Representation* course in Spring term.)

The idea is that each world w of W has associated with it a set $\nu(w)$ of propositions — these are the propositions necessary at w. What is a proposition? In possible world semantics (any kind), a proposition is identified with a set of possible worlds, i.e., with a subset of W. So the set of propositions necessary at w, $\nu(w)$, is a set of propositions, i.e. a set of subsets of W. Note that $\nu(w)$ may be *any* set of propositions, including the empty set; there are no assumptions about ν except that it is a function from W to $\wp(\wp(W))$.

Definition 7 (ν-model) *A (neighbourhood, or Montague-Scott, or Chellas-minimal)* model *(or ν-model for short) is a structure*

$$\mathcal{M} = \langle W, \nu, h \rangle$$

where W is a set of worlds, $h : \mathcal{L} \to \wp(W)$ is a valuation of the propositional atoms (as usual), and ν is a mapping from W to sets of subsets of W, i.e., $\nu : W \to \wp(\wp(W))$. The component $\langle W, \nu \rangle$ is called a neighbourhood frame *(or here ν-frame for short).*

Definition 8 (ν-model: truth conditions) *Let w be a world in a model $\mathcal{M} = \langle W, \nu, h \rangle$.*

(1) $\mathcal{M}, w \models \Box A \;\Leftrightarrow\; \|A\|^{\mathcal{M}} \in \nu(w)$
(2) $\mathcal{M}, w \models \Diamond A \;\Leftrightarrow\; (W - \|A\|^{\mathcal{M}}) \notin \nu(w)$

The second part of this definition is obviously designed so that the notions of necessity and possibility again come out to be dual, i.e., so that $\Diamond$ has the meaning $\neg\Box\neg$.

Validity of a formula in a class of ν-models or in a class of ν-frames is defined as usual.

A relational ('Kripke') frame is a special case of a ν-frame — I'll explain exactly what kind of ν-frame it is later.

Example Here is a ν-frame:

$$\begin{aligned} W &= \{a, b, c\} \\ \nu(a) &= \{\,\{a,b,c\},\ \{b\},\ \{a,c\}\,\} \\ \nu(b) &= \{\,\{a\},\ \{b\},\ \{c\}\,\} \\ \nu(c) &= \{\,\emptyset,\ \{a\}\,\} \end{aligned}$$

(There is no picture to draw because this is not a relational frame.)
Here is one possible valuation on this frame:

$$\begin{aligned} h(p) &= \{\,a, b\,\} \\ h(q) &= \{\,b, c\,\} \end{aligned}$$

Call this combination the model $\mathcal{M}$. We see that:

$\mathcal{M}, a \models \Box\top$ (since $\|\top\|^{\mathcal{M}} = W = \{a,b,c\}$)
$\mathcal{M}, a \models \Box(p \wedge q)$ (since $\|p \wedge q\|^{\mathcal{M}} = \{b\}$)
$\mathcal{M}, a \models \Box\,((p \vee q) \wedge \neg(p \wedge q))$ (since $\|(p \vee q) \wedge \neg(p \wedge q)\|^{\mathcal{M}} = \{a,c\}$)
$\mathcal{M}, a \not\models \Box p$ (since $\|p\|^{\mathcal{M}} \notin \nu(a)$)
$\mathcal{M}, a \models \neg\Box p$ $\quad \mathcal{M}, a \models \Diamond\neg p$

$\mathcal{M}, b \models \Box(p \wedge \neg q)$ (since $\|p \wedge \neg q\|^{\mathcal{M}} = \{a\}$)
$\mathcal{M}, b \models \Box(p \wedge q)$ (since $\|p \wedge q\|^{\mathcal{M}} = \{b\}$)
$\mathcal{M}, b \models \Box(q \wedge \neg p)$ (since $\|q \wedge \neg p\|^{\mathcal{M}} = \{c\}$)
$\mathcal{M}, b \not\models \Box\top$ (since $W \notin \nu(b)$)
$\mathcal{M}, b \models \neg\Box\top$ $\quad \mathcal{M}, b \models \Diamond\bot$
$\mathcal{M}, b \models \Diamond\top$ (because $\mathcal{M}, b \not\models \Box\bot$)

$\mathcal{M}, c \models \Box\bot$ (since $\|\bot\|^{\mathcal{M}} = \emptyset$)
$\mathcal{M}, c \models \Box(p \wedge \neg q)$ (since $\|p \wedge \neg q\|^{\mathcal{M}} = \{a\}$)

Theorem 9 *Let* C *be a class of* ν*-models. Then:*

(1) $\models_{\mathsf{C}} \Diamond A \leftrightarrow \neg\Box\neg A$.

(2) *If* $\models_{\mathsf{C}} A \leftrightarrow B$ *then* $\models_{\mathsf{C}} \Box A \leftrightarrow \Box B$.

Proof For part (1): Let w be a world in any ν-model $\mathcal{M} = \langle W, \nu, h\rangle$ in class C.

$$\begin{aligned} \mathcal{M}, w \models \Diamond A &\Leftrightarrow (W - \|A\|^{\mathcal{M}}) \notin \nu(w) \\ &\Leftrightarrow \|\neg A\|^{\mathcal{M}} \notin \nu(w) \text{ (by definition of } \|.\|^{\mathcal{M}}) \\ &\Leftrightarrow \mathcal{M}, w \not\models \Box\neg A \\ &\Leftrightarrow \mathcal{M}, w \models \neg\Box\neg A \end{aligned}$$

For part (2): Suppose C is a class of ν-models such that $\models_{\mathsf{C}} A \leftrightarrow B$. Then $\|A\|^{\mathcal{M}} = \|B\|^{\mathcal{M}}$ for each $\mathcal{M}$ in C. It follows from this that for any world w in any model $\mathcal{M} = \langle W, \nu, h\rangle$ in C, $\|A\|^{\mathcal{M}} \in \nu(w)$ if and only if $\|B\|^{\mathcal{M}} \in \nu(w)$. So for any w in any $\mathcal{M}$ in C, $\mathcal{M}, w \models \Box A$ if and only if $\mathcal{M}, w \models \Box B$, which means that $\models_{\mathsf{C}} \Box A \leftrightarrow \Box B$.

So: the set of formulas valid in a class of ν-models (i) contains all instances of schema Df$\Diamond$, and (ii) is closed under the rule RE. In other words, it is a classical system of modal logic — in the *Chellas* sense (no requirement for closure under US).

This is the basis of soundness of classical systems with respect to ν-models. Completeness (via canonical models) comes later.

ν-models: alternative notation

The following alternative notation is sometimes easier to work with. (It is just a convenient notational variant, not some new kind of model.)

Given a function $\nu : W \to \wp(\wp(W))$ we can always define a function $f : \wp(W) \to \wp(W)$ such that $f(X) \stackrel{\text{def}}{=} \{w : X \in \nu(w)\}$. And every function ν can be defined in terms of such a function f. So we can define

$$w \in f(X) \Leftrightarrow X \in \nu(w)$$

In terms of f, the truth conditions for $\Box A$ are $\mathcal{M}, w \models \Box A \Leftrightarrow \|A\|^{\mathcal{M}} \in \nu(w) \Leftrightarrow w \in f(\|A\|^{\mathcal{M}})$, i.e.

$$\|\Box A\|^{\mathcal{M}} = f(\|A\|^{\mathcal{M}})$$

In similar fashion (check!):

$$\|\Diamond A\|^{\mathcal{M}} = W - f(W - \|A\|^{\mathcal{M}})$$

The schemas M, C, N

M.	$\Box(A \wedge B) \to (\Box A \wedge \Box B)$
C.	$(\Box A \wedge \Box B) \to \Box(A \wedge B)$
N.	$\Box\top$

Although each of these is valid in any class of standard (Kripke) models, each has a counterexample in a ν-model.

Theorem 10 *None of the schemas* M, C, *and* N *is valid in the class of all* ν*-models.*

Proof Construct counterexamples. (Exercise. The example of a ν-model on page 6 already contains suitable counterexamples.)

Theorem 11 *The schemas* M, C, *and* N *are valid in classes of* ν*-models satisfying the following conditions* (m), (c), *and* (n), *respectively:*

(m) $X \cap Y \in \nu(w) \Rightarrow X \in \nu(w)$ & $Y \in \nu(w)$

(c) $X \in \nu(w)$ & $Y \in \nu(w) \Rightarrow X \cap Y \in \nu(w)$

(n) $W \in \nu(w)$

Proof For (1). Let w be a world in any ν-model $\mathcal{M} = \langle W, \nu, h\rangle$ satisfying property (m).

$$\begin{aligned} \mathcal{M}, w \models \Box(A \wedge B) &\Rightarrow \|A \wedge B\|^{\mathcal{M}} \in \nu(w) \\ &\Rightarrow \|A\|^{\mathcal{M}} \cap \|B\|^{\mathcal{M}} \in \nu(w) \text{ (by definition of } \|.\|^{\mathcal{M}}) \\ &\Rightarrow \|A\|^{\mathcal{M}} \in \nu(w) \text{ and } \|B\|^{\mathcal{M}} \in \nu(w) \text{ (by condition (m))} \\ &\Rightarrow \mathcal{M}, w \models \Box A \text{ and } \mathcal{M}, w \models \Box B \end{aligned}$$

For (2). Let w be a world in any ν-model $\mathcal{M} = \langle W, \nu, h\rangle$ satisfying property (c).

$$\begin{aligned} \mathcal{M}, w \models \Box A \wedge \Box B &\Rightarrow \|A\|^{\mathcal{M}} \in \nu(w) \text{ and } \|B\|^{\mathcal{M}} \in \nu(w) \\ &\Rightarrow \|A\|^{\mathcal{M}} \cap \|B\|^{\mathcal{M}} \in \nu(w) \text{ (by condition (c))} \\ &\Rightarrow \|A \wedge B\|^{\mathcal{M}} \in \nu(w) \text{ (by definition of } \|.\|^{\mathcal{M}}) \\ &\Rightarrow \mathcal{M}, w \models \Box(A \wedge B) \end{aligned}$$

For (3). Let w be a world in any ν-model $\mathcal{M} = \langle W, \nu, h\rangle$ and suppose $W \in \nu(w)$. $W = \|\top\|^{\mathcal{M}}$, and so $\|\top\|^{\mathcal{M}} \in \nu(w)$, from which follows $\mathcal{M}, w \models \Box\top$.

Notice that condition (m) is equivalently expressed in terms of closure under supersets:

(rm) if $X \subseteq Y$ then $X \in \nu(w) \Rightarrow Y \in \nu(w)$.

(The proof is in the exercises.)

Notice: how conditions (m), (rm), (c) and (n) turn out if we use the alternative characterisation of ν-models employing the function $f : \wp(W) \to \wp(W)$ whereby $\|\Box A\|^{\mathcal{M}} = f(\|A\|^{\mathcal{M}})$. Recall then $w \in f(X) \Leftrightarrow X \in \nu(w)$:

(m_f) $f(X \cap Y) \subseteq f(X) \cap f(Y)$ equivalently, (rm_f) $X \subseteq Y \Rightarrow f(X) \subseteq f(Y)$

(c_f) $f(X) \cap f(Y) \subseteq f(X \cap Y)$

(n_f) $f(W) = W$

The schemas P, D, T, B, 4, 5

P.	$\neg\Box\bot$
D.	$\Box A \rightarrow \Diamond A$
T.	$\Box A \rightarrow A$
B.	$A \rightarrow \Box\Diamond A$
4.	$\Box A \rightarrow \Box\Box A$
5.	$\Diamond A \rightarrow \Box\Diamond A$

Theorem 12 *None of the schemas* P, D, T, B, 4, 5 *is valid in the class of all ν-models.*

Proof Construct counterexamples (Exercise).

Theorem 13 *The schemas* P, D, T, B, 4, 5 *are valid in classes of ν-models satisfying the following conditions* (p), (d), (t), (b), (iv), *and* (v), *respectively:*

(p) $\emptyset \notin \nu(w)$
(d) $X \in \nu(w) \;\Rightarrow\; (W - X) \notin \nu(w)$
(t) $X \in \nu(w) \;\Rightarrow\; w \in X$
(b) $w \in X \;\Rightarrow\; \{w' \in W : (W - X) \notin \nu(w')\} \in \nu(w)$
(iv) $X \in \nu(w) \;\Rightarrow\; \{w' \in W : X \in \nu(w')\} \in \nu(w)$
(v) $X \notin \nu(w) \;\Rightarrow\; \{w' \in W : X \notin \nu(w')\} \in \nu(w)$

Proof Let w be any world in a ν-model $\mathcal{M} = \langle W, \nu, h\rangle$.
For condition (p):

$$\begin{aligned} \mathcal{M}, w \models \neg\Box\bot \;&\Leftrightarrow\; \mathcal{M}, w \not\models \Box\bot \\ &\Leftrightarrow\; \|\bot\|^{\mathcal{M}} \notin \nu(w) \\ &\Leftrightarrow\; \emptyset \notin \nu(w) \end{aligned}$$

Suppose $\mathcal{M}$ satisfies condition (d):

$$\begin{aligned} \mathcal{M}, w \models \Box A \;&\Rightarrow\; \|A\|^{\mathcal{M}} \in \nu(w) \\ &\Rightarrow\; (W - \|A\|^{\mathcal{M}}) \notin \nu(w) \quad \text{by condition (d)} \\ &\Rightarrow\; \|\neg A\|^{\mathcal{M}} \notin \nu(w) \\ &\Rightarrow\; \mathcal{M}, w \models \Diamond A \end{aligned}$$

Suppose $\mathcal{M}$ satisfies condition (t):

$$\begin{aligned} \mathcal{M}, w \models \Box A \;&\Rightarrow\; \|A\|^{\mathcal{M}} \in \nu(w) \\ &\Rightarrow\; w \in \|A\|^{\mathcal{M}} \quad \text{by condition (t)} \\ &\Rightarrow\; \mathcal{M}, w \models A \end{aligned}$$

Suppose $\mathcal{M}$ satisfies condition (b):

$$\begin{aligned} \mathcal{M}, w \models A \;&\Rightarrow\; w \in \|A\|^{\mathcal{M}} \\ &\Rightarrow\; \{w' \in W : (W - \|A\|^{\mathcal{M}}) \notin \nu(w')\} \in \nu(w) \quad \text{by condition (b)} \\ &\Rightarrow\; \{w' \in W : \|\neg A\|^{\mathcal{M}} \notin \nu(w')\} \in \nu(w) \\ &\Rightarrow\; \{w' \in W : \mathcal{M}, w \not\models \Box\neg A\} \in \nu(w) \\ &\Rightarrow\; \{w' \in W : \mathcal{M}, w \models \neg\Box\neg A\} \in \nu(w) \\ &\Rightarrow\; \|\neg\Box\neg A\|^{\mathcal{M}} \in \nu(w) \\ &\Rightarrow\; \mathcal{M}, w \models \Box\neg\Box\neg A \\ &\Rightarrow\; \mathcal{M}, w \models \Box\Diamond A \end{aligned}$$

The proofs for conditions (iv) and (v) are on the exercise sheet.

Notice: how much more concisely these model conditions turn out when expressed using the function $w \in f(X) \Leftrightarrow X \in \nu(w)$:

(p_f) $f(\emptyset) = \emptyset$
(d_f) $f(X) \subseteq W - f(W - X)$
(t_f) $f(X) \subseteq X$
(b_f) $X \subseteq f(W - f(W - X))$
(iv_f) $f(X) \subseteq f(f(X))$
(v_f) $W - f(X) \subseteq f(W - f(X))$

(See exercise sheet.)
As should be clear from the above, the f function often allows us to read off direct from a schema the model condition that defines a class of ν-models that validate the schema. There is nothing mysterious about this: remember $\|\Box A\|^{\mathcal{M}} = f(\|A\|^{\mathcal{M}})$.

For example (1): notice that condition (v_f) as stated above would seem to correspond to a schema $(5')\neg\Box A \rightarrow \Box\neg\Box A$, not to the schema $(5)\Diamond A \rightarrow \Box\Diamond A$. The model condition corresponding to (5) would be

$$W - f(W - X) \subseteq f(W - f(W - X))$$

But $(5')$ and (5) are logically equivalent and (check!) the two model conditions are equivalent also (they hold for all subsets X of W).

For example (2): consider the schema

G.	$\Diamond\Box A \rightarrow \Box\Diamond A$

Schema G is valid in the class of ν-models satisfying the model condition that, for every $X \subseteq W$:

(g_f) $W - f(W - f(X)) \subseteq f(W - f(W - X))$

Check that this is true! (See exercise sheet.)

Soundness

We already know (Theorem 9) that $\Diamond A \leftrightarrow \neg\Box\neg A$ is valid in any class C of ν-models, and that $\models_{\mathsf{C}} A \leftrightarrow B$ implies $\models_{\mathsf{C}} \Box A \leftrightarrow \Box B$. Soundness of classical systems with respect to ν-models follows straightforwardly.
For the record:

Theorem 14 *Let $\xi_1, \ldots, \xi_n$ be schemas valid respectively in classes $\mathsf{C}_1, \ldots, \mathsf{C}_n$ of ν-models. Then the system of modal logic $E\xi_1 \ldots \xi_n$ is sound with respect to the class $\mathsf{C}_1 \cap \ldots \cap \mathsf{C}_n$.*

Proof By Theorem 9, Df$\Diamond$ is valid in any class of ν-models, and the rule RE, and all rules of propositional logic, preserve validity in any such class. Further: if $\xi_1, \ldots, \xi_n$ are valid respectively in $\mathsf{C}_1, \ldots, \mathsf{C}_n$ then they are valid in the intersection of these classes. So every theorem of $E\xi_1 \ldots \xi_n$ is valid in this intersection, which means that $E\xi_1 \ldots \xi_n$ is sound with respect to that class.

Completeness (via canonical models)

The basic idea is exactly the same as for canonical models for normal systems. We want to establish completeness of a (classical) system Σ with respect to some class C of ν-models, i.e. we want to prove that for all formulas A

$$\models_{\mathsf{C}} A \quad \Rightarrow \quad \vdash_\Sigma A$$

We try to find a canonical model $\mathcal{M}^\Sigma$ for system Σ, i.e. a model $\mathcal{M}^\Sigma$ such that

$$\mathcal{M}^\Sigma \models A \quad \Leftrightarrow \quad \vdash_\Sigma A$$

Now if we can show that this canonical model belongs to class C, i.e. that model $\mathcal{M}^\Sigma$ satisfies the model conditions that characterise the class C, then we have completeness. Because (as usual): suppose that $\mathcal{M}^\Sigma$ is a canonical model for system Σ; then if $\mathcal{M}^\Sigma$ belongs to the class of models C:

$$\models_{\mathsf{C}} A \quad \Rightarrow \quad \mathcal{M}^\Sigma \models A \qquad \Rightarrow \quad \vdash_\Sigma A$$

Sometimes, it is easier to go the other way: construct a model $\mathcal{M}$ that is clearly in class C. Then show that $\mathcal{M}$ is a canonical model for the system Σ.

Now it just remains to figure out how to construct a canonical ν-model for a classical system Σ.

Canonical ν-models: Sketch

I am NOT going to present all the details of canonical ν-models. They are not difficult but I don't want to spend too much time on it. Here is the basic idea. **Details are not examinable**.

Given a system Σ of modal logic, a canonical ν-model will be one in which, as usual, the set of worlds W will be the set of Σ-maxi-consistent sets of formulas, and the valuation h will be such that $w \in h(p) \Leftrightarrow p \in w$ for every world w and every atom p. We will also need a suitable constraint on the 'neighbourhood' function ν.

For classical systems, there are *many* canonical models, i.e., many choices of ν that will give us the property we want:

$$\mathcal{M}^\Sigma, w \models A \quad \Leftrightarrow \quad A \in w$$

For example, the *smallest canonical* ν-model for a classical system Σ is the model $\mathcal{M}^\Sigma = \langle W, \nu, h\rangle$ such that:

(1) W is the set of Σ-maxi-consistent sets.
(2) $\nu(w) = \{|A|_\Sigma : \Box A \in w\}$.
(3) For every w in $\mathcal{M}$ and every atom p, $w \in h(p) \Leftrightarrow p \in w$, i.e. $h(p) = |p|_\Sigma$.

Now it is very easy to check (try it) that, for the smallest canonical model, and every formula A:

- $\Box A \in w \;\Leftrightarrow\; |A|_\Sigma \in \nu(w)$, and hence:
- $\mathcal{M}^\Sigma, w \models A \;\Leftrightarrow\; A \in w$ (by induction on the structure of A), and hence:
- $\mathcal{M}^\Sigma \models A \;\Leftrightarrow\; \vdash_\Sigma A$

From the above it follows immediately that:

Theorem 15 *E is complete with respect to the class of ν-models.*

For classical systems, there are other canonical models besides the 'smallest' one defined above. I won't go through the details here.

Example (just one) The classical system ET is complete with respect to the class of ν-models satisfying the following condition, for every w in $\mathcal{M}$, $X \subseteq W$:

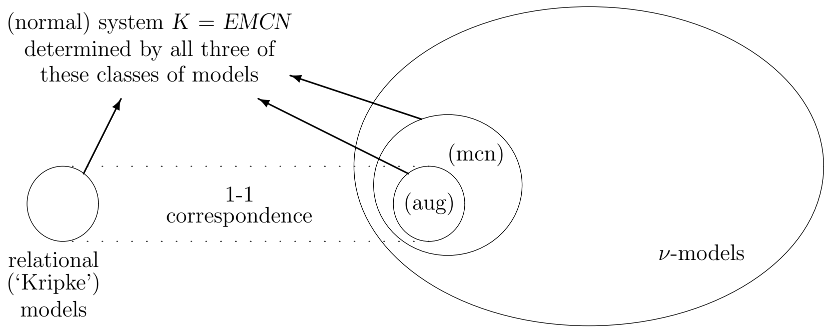
$$\text{(t)} \quad X \in \nu(w) \Rightarrow w \in X$$

Proof Let $\mathcal{M} = \langle W, \nu, h\rangle$ be the smallest canonical ν-model for the system ET. Then $\nu(w) = \{|A|_{ET} : \Box A \in w\}$.
We show that this canonical model satisfies (t): $X \in \nu(w) \Rightarrow w \in X$.
Suppose $X \in \nu(w)$. Then $X = |A|_{ET}$ for some $\Box A \in w$. But $\vdash_{ET} \Box A \rightarrow A$ means that also $A \in w$, and this means (by definition of proof set) that $w \in |A|_{ET}$, i.e. $w \in X$ as required.

Relational (Kripke) models and ν-models

ν-models are more general than relational (Kripke) models. (Obviously, since classical systems subsume normal systems as a special case.) Relational (Kripke) models correspond to a particular class of ν-models in which the 'neighbourhood function' has certain properties.

(You do not have to memorise any of the details here. This is just for your information. **Details are not examinable**.)



System K — the smallest normal system — is determined by (is sound and complete with respect to) the following classes of models:

- relational ('Kripke') models
- the class of ν-models satisfying conditions (m), (c), and (n). I will call these (mcn) models for short. (Sometimes they are called 'filters'.)
- a sub-class of (mcn) models — those satisfying the additional property

$$\text{(aug)} \qquad X \in \nu(w) \;\Leftrightarrow\; \bigcap \nu(w) \subseteq X \subseteq W$$

There is a 1-1 correspondence between the (aug) ν-models and relational ('Kripke') models.

To check this and to obtain the diagram on the previous page, it nows remains to show the following. (The proofs are all easy exercises. I omit them. This is probably already more than you want to know.)

1. For every relational (Kripke) model $\mathcal{M} = \langle W, R, h \rangle$ there is a ν-model $\mathcal{M}^\nu = \langle W, \nu, h \rangle$ satisfying condition (aug) such that, for every formula A,

   $$\mathcal{M}, w \models A \;\Leftrightarrow\; \mathcal{M}^\nu, w \models A \qquad \textit{`pointwise equivalent'}$$

   and vice-versa (for every ν-model satisfying (aug) there is a relational (Kripke) model that is pointwise equivalent).
2. The class of ν-models satisfying (aug) is a sub-class of those satisfying (mcn). (Easy.)
3. (aug) $\neq$ (mcn): there are (mcn) models which do not satisfy (aug). (Not so easy. Here is one. Take W to be the set of real numbers. Take $\nu(w) = \{(w, w+\delta) : \delta > 0\}$.)
4. K is determined by the class of (mcn) models. (Easy to prove. Compare the method for *EMCT* shown earlier.)
5. K is also determined by the class of (aug) models. (Given the smallest canonical model it is easy to construct one that satisfies (aug). Then one shows that this model is canonical for $K = EMCN$.)

I repeat: the details are given here for your interest only. There is no need to memorise any of this, not even the picture on the previous page.