499 Modal and Temporal Logic

Normal modal logics
(Syntactic characterisations)

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Further reading:
B.F. Chellas, *Modal logic: an introduction*. Cambridge University Press, 1980.
P. Blackburn, M. de Rijke, Y. Venema, Ch4, *Modal Logic*. Cambridge Univ. Press, 2002.

These notes follow Chellas quite closely.

Notation

$\mathcal{M} \models A$	—	A is valid in model $\mathcal{M}$ (A is true at all worlds in $\mathcal{M}$)
$\mathcal{F} \models A$	—	A is valid on the frame $\mathcal{F}$ (valid in all models with frame $\mathcal{F}$)
$\models_{\mathsf{C}} A$	—	A is valid in the class of models C (valid in all models in C)
$\models_{\mathsf{F}} A$	—	A is valid in the class of frames F (valid on all frames in F)

Truth sets

The truth set, $\|A\|^{\mathcal{M}}$, of the formula A in the model $\mathcal{M}$ is the set of worlds in $\mathcal{M}$ at which A is true.

Definition 1 (Truth set)

$$\|A\|^{\mathcal{M}} =_{def} \{w \textit{ in } \mathcal{M} : \mathcal{M}, w \models A\}$$

Theorem 2 *[Chellas Thm 2.10, p38] Let $\mathcal{M} = \langle W, \ldots, h \rangle$ be a model. Then:*

(1) $\|p\|^{\mathcal{M}} = h(p)$, for any atom p.
(2) $\|\top\|^{\mathcal{M}} = W$.
(3) $\|\bot\|^{\mathcal{M}} = \emptyset$.
(4) $\|\neg A\|^{\mathcal{M}} = W - \|A\|^{\mathcal{M}}$.
(5) $\|A \wedge B\|^{\mathcal{M}} = \|A\|^{\mathcal{M}} \cap \|B\|^{\mathcal{M}}$.
(6) $\|A \vee B\|^{\mathcal{M}} = \|A\|^{\mathcal{M}} \cup \|B\|^{\mathcal{M}}$.
(7) $\|A \to B\|^{\mathcal{M}} = (W - \|A\|^{\mathcal{M}}) \cup \|B\|^{\mathcal{M}}$.

Proof Exercise (very easy).

$\|A\|^{\mathcal{M}}$ can be regarded as the *proposition* expressed by the formula A in the model $\mathcal{M}$.

Relational ('Kripke') semantics

Let $\mathcal{M} = \langle W, R, h \rangle$ be a standard, relational ('Kripke') model.
The truth conditions for $\Box A$ and $\Diamond A$ are

$$\mathcal{M}, w \models \Box A \;\Leftrightarrow\; \forall t\,(w\,R\,t \Rightarrow \mathcal{M}, t \models A)$$

$$\mathcal{M}, w \models \Diamond A \;\Leftrightarrow\; \exists t\,(w\,R\,t \;\&\; \mathcal{M}, t \models A)$$

In terms of truth sets:

$$\mathcal{M}, w \models \Box A \;\Leftrightarrow\; R[w] \subseteq \|A\|^{\mathcal{M}}$$

$$\mathcal{M}, w \models \Diamond A \;\Leftrightarrow\; R[w] \cap \|A\|^{\mathcal{M}} \neq \emptyset$$

where $R[w] =_{def} \{t \text{ in } \mathcal{M} : w\,R\,t\}$. $R[w]$ is the set of worlds accessible from w.

And you know that validities of certain formulas correspond to various frame properties, e.g.

$\Box A \to A$	reflexive frames
$\Box A \to \Box\Box A$	transitive frames
$\Box A \to \Diamond A$	serial frames
etc.	

Multi-modal systems

All this can be generalised easily to the multi-modal case.
Example: consider a language with 'box' operators K_a, K_b, K_c, …interpreted on models with structure $\langle W, R_a, R_b, R_c, \ldots, h \rangle$.
You can read $\mathsf{K}_a A$ as 'a knows that A' for example.

Suppose each R_i is *reflexive*. Then $\mathsf{K}_i A \to A$ is valid for each $i = a, b, c, \ldots$.
There may also be *bridging properties*, say:

$$\mathsf{K}_b A \to \mathsf{K}_a A$$

which is valid in frames in which $R_a \subseteq R_b$.
We will prove that later.

One can also have modal operators with arity 2, 3, … (even arity 0). Example: 'since' and 'until', which are *binary*. I may include some further examples later, for various forms of *conditionals*, if time allows.

Modal logics: syntactic characterisations

But now we are going to look at **syntactic characterisations** of modal logics — axioms, rules of inference, systems, theorems, deducibility, etc.

Of particular interest are so-called *normal* systems of modal logics. These are the logics of *relational* ('Kripke') frames.

There are also weaker *non-normal* modal logics. They don't have a relational ('Kripke') semantics.

Systems of modal logic

In common with most modern approaches, we will define systems of modal logic ('modal logics' or just 'logics' for short) in rather abstract terms — a system of modal logic is just a set of formulas satisfying certain closure conditions. A formula A is a theorem of the system Σ simply when $A \in \Sigma$. Which closure conditions? See below.

Systems of modal logic can also be defined (syntactically) in other ways, usually by reference to some kind of proof system. For example:

- Hilbert systems: given a set of formulas called *axioms* and a set of *rules of proof*, a formula A is a theorem of the system when it is the last formula of a sequence of formulas each of which is either an axiom or obtained from its predecessor by applying one of the rules of proof.

There are also natural deduction proof systems, tableau-style proof systems, etc for modal logics. We shall *not* be covering them in this course.
In this course, a system of modal logic is just a set of formulas satisfying certain closure conditions (defined below).
The *theorems* of a logic Σ are just the formulas in Σ. We write $\vdash_\Sigma A$ to mean that A is a theorem of Σ.

Definition $\vdash_\Sigma A$ iff $A \in \Sigma$

Schemas

A schema is a set of formulas of a particular form. Example:

5. $\Diamond A \to \Box\Diamond A$

stands for the set of all formulas of this form where A is any formula. '5' here is just a label for the schema, for reference.

PL denotes the set of all propositional tautologies (including formulas with modal operators, such as $\Box p \vee \neg\Box p$, which are tautologies).

Rules of inference

In general, a rule of inference has the form

$$\frac{A_1, \ldots, A_n}{A} \quad (n \geq 0)$$

A set of formulas Σ is *closed under* (or sometimes just *has*) a rule of inference just in case whenever the set Σ contains all of $A_1, \ldots, A_n$ it contains also A, in other words

$$\text{if } \{A_1, \ldots, A_n\} \subseteq \Sigma \text{ then } A \in \Sigma$$

One can use rule schemas too.

Example

Some modal logics have (are closed under) the rule

RN. $$\frac{A}{\Box A}$$

If $\vdash_\Sigma A$ then $\vdash_\Sigma \Box A$

Example

Many modal logics (the 'classical' systems) have (are closed under) the rule

RE. $$\frac{A \leftrightarrow B}{\Box(A \leftrightarrow B)}$$

If $\vdash_\Sigma A \leftrightarrow B$ then $\vdash_\Sigma \Box(A \leftrightarrow B)$

Modus ponens

MP. $$\frac{A \to B, A}{B}$$

If $\vdash_\Sigma A \to B$ and $\vdash_\Sigma A$ then $\vdash_\Sigma B$.

Uniform substitution

US. $$\frac{A}{B}$$
where B is obtained from A by uniformly replacing propositional atoms in A by arbitrary formulas.

If $\vdash_\Sigma A$ then $\vdash_\Sigma B$ where B is obtained from A by uniformly replacing propositional atoms in A by arbitrary formulas.

Tautological consequence (propositional consequence)

RPL. $$\frac{A_1, \ldots, A_n}{A} \quad (n \geq 0),$$
where A is a tautological consequence of $A_1, \ldots, A_n$.

(A is a tautological consequence of $A_1, \ldots, A_n$ when A follows from $A_1, \ldots, A_n$ in ordinary propositional logic; that is to say, when $(A_1 \wedge \cdots \wedge A_n) \to A$ is a tautology.)

Systems of modal logic — definition

Definition 3 (System of modal logic) *A set of formulas* Σ *is a* system of modal logic *iff it contains all propositional tautologies (PL) and is closed under modus ponens (MP) and uniform substitution (US).*

Equivalently, a system of modal logic is any set of formulas closed under the rules RPL and uniform substitution (US).

The point is: a system of modal logic is any set of formulas that is closed with respect to all propositionally correct modes of interence.
We will usually just say 'logic' or sometimes 'system' instead of 'system of modal logic'.

The *theorems* of a logic are just the formulas in it. We write $\vdash_\Sigma A$ to mean that A is a theorem of Σ.

Definition 4 $\vdash_\Sigma A$ *iff* $A \in \Sigma$

Example 5 (Blackburn et al, p190)

(i) The set of all formulas $\mathcal{L}$ is a system of modal logic, the *inconsistent logic*.
(ii) If $\{\Sigma_i \mid i \in I\}$ is a collection of logics, then $\bigcap_{i \in I} \Sigma_i$ is a logic.
(iii) Define Σ_F to be the set of formulas valid on a class F of frames. Σ_F is a logic.
(iv) Define Σ_C to be the set of formulas valid on a class C of models. Σ_C need not be a logic. (Consider a class C consisting of models $\mathcal{M}$ in which p is true at all worlds but q is not. Then $\models_\mathsf{C} p$, but $\not\models_\mathsf{C} q$. So Σ_C is not closed under uniform substitution.)

Exercise: Prove the above statements (i) to (iii).

(i) Trivial. We need to show that $\mathcal{L}$ contains *PL* and is closed under MP and US. This is trivial, because $\mathcal{L}$ is the set of all formulas.
(ii) Easy. *PL* is a subset of every Σ_i, so also a subset of the intersection. To show the intersection is closed under MP: suppose A and $A \to B$ are formulas in the intersection. Then both formulas belong to every Σ_i too, and since every Σ_i is closed under MP, B must belong to every Σ_i. So B is in the intersection also, so the intersection is closed under MP. A similar argument works for uniform substitution US.
(iii) We have to show that Σ_F contains *PL* and is closed under MP and US.

The first two are straightforward and are left as an exercise (tutorial sheet). To show closure under US is not difficult but is rather long and fiddly so details omitted here. The basic idea is simple enough. Blackburn et al put it like this: validity on a frame abstracts away from the effects of particular assignments: if a formula is valid on a frame, this cannot be because of the particular values assigned to its propositional atoms. So we should be free to replace the atoms in the formula with any other formula, as long as we do this uniformly (i.e., as long as we replace all occurrences of an atom p by the same formula).

Note: For those looking at the book by Chellas. The definition of a system of modal logic used by Chellas is very slightly different. Chellas's definition (2.11, p46) requires only that the set of formulas is closed under propositional consequence (RPL, or equivalently, contains all tautologies *PL* and is closed under modus ponens (MP)) — there is no mention of uniform substitution. In the presentation of actual systems of interest it makes no difference because these are usually presented in terms of schemas, and schemas build in the effect of uniform substitution indirectly. But be aware that there is a technical difference between these definitions. We have already seen an example: the Σ_C defined in the last part of the previous example.

Because logics are simply sets of formulas, their relative strength is measured in terms of set inclusion: a logic Σ is at least as strong as a logic Σ' when $\Sigma \supseteq \Sigma'$.

Definition 6 *A system of modal logic is a* Σ-system *when it contains every theorem of* Σ.

So Σ is always itself a Σ-system. And every system of modal logic is a *PL*-system.

Theorem 7 *[Chellas Thm 2.13, p46]*

(1) *PL is a system of modal logic.*
(2) *Every system of modal logic is a PL-system.*
(3) *PL is the smallest (set inclusion) system of modal logic.*

Proof Exercise. (Very easy — apply the definitions.)

How to define a system of modal logic Σ?

Various ways, but one common way is this: given a set of formulas Γ and a set of rules of inference R, define Σ to be the smallest system of modal logic containing Γ and closed under R. (Or equivalently, given the definition of 'system of modal logic', the smallest set of formulas containing *PL* and Γ, and closed under R, modus ponens (MP), and uniform substitution (US).)

One then says that the system Σ is *generated by* or sometimes *axiomatized by* $\langle \Gamma, R \rangle$. Γ and R are sometimes called 'axioms' of Σ.

Note that the same system may be generated by different sets of formulas and rules.

Example The modal logic S4 (which is the normal modal logic *KT4* in the standard classification, as explained later) is generated by (is the smallest system of modal logic containing) the following schemas (the labels are standard — see later):

K.	$\Box(A \to B) \to (\Box A \to \Box B)$
T.	$\Box A \to A$
4.	$\Box A \to \Box\Box A$

and closed under the following rule of inference:

RN. $\dfrac{A}{\Box A}$

Because of uniform substitution, some authors prefer to write the above with propositional atoms instead of arbitrary formulas (A, B etc) and schemas. So they would write that S4 is generated by the following set of formulas

K.	$\Box(p \to q) \to (\Box p \to \Box q)$
T.	$\Box p \to p$
4.	$\Box p \to \Box\Box p$

and the following rule of inference:

RN. $\dfrac{p}{\Box p}$

My personal preference is to use schemas, but it's a trivial point.

The same system S4 can be defined in other ways. For example it is also generated by the following schemas and rules:

RM.	$\dfrac{A \to B}{\Box A \to \Box B}$
N.	$\Box\top$
C.	$(\Box A \wedge \Box B) \to \Box(A \wedge B)$
T.	$\Box p \to p$
4.	$\Box p \to \Box\Box p$

Exercise: justify the above claim. (The answer is contained in Theorem 12 below.)

One typical problem is to determine whether $\langle \Gamma, R \rangle$ generates the same system of modal logic as $\langle \Gamma', R' \rangle$. One way of doing that is to show that Γ' and R' can be derived from $\langle \Gamma, R \rangle$ (and that Γ and R can be derived from $\langle \Gamma', R' \rangle$). Another way is to show that $\langle \Gamma, R \rangle$ and $\langle \Gamma', R' \rangle$ are sound and complete with respect to the same class of semantic structures (models or frames). We will look at how to do that presently.

Example

Suppose Σ is a system of modal logic containing the following schema

K. $\Box(A \to B) \to (\Box A \to \Box B)$

and closed under the following rule of inference:

RN. $\dfrac{A}{\Box A}$

(Σ is then by definition a 'normal modal logic', but ignore that for now.)

Show that Σ contains all instances of the following schema as theorems:

$$(\Box A \wedge \Box B) \to \Box(A \wedge B)$$

1.	$\vdash_\Sigma A \to (B \to (A \wedge B))$	*PL*
2.	$\vdash_\Sigma \Box(A \to (B \to (A \wedge B)))$	1, RN
3.	$\vdash_\Sigma \Box(A \to (B \to (A \wedge B))) \to (\Box A \to \Box(B \to (A \wedge B)))$	K (an instance of K)
4.	$\vdash_\Sigma \Box A \to \Box(B \to (A \wedge B))$	2, 3, MP
5.	$\vdash_\Sigma \Box(B \to (A \wedge B)) \to (\Box B \to \Box(A \wedge B))$	K (an instance of K)
6.	$\vdash_\Sigma \Box A \to (\Box B \to \Box(A \wedge B))$	4, 5, RPL
7.	$\vdash_\Sigma (\Box A \wedge \Box B) \to \Box(A \wedge B)$	6, RPL

Note:

Step 6 is a bit casual. You can read it as '6 follows from 4 and 5 by propositional logic'. Officially, there are a couple of steps omitted here.

The same holds for step 7: read it as 'follows from 6 by propositional logic'.

Steps 2 and 3 above are a bit long-winded. They could be combined, like this:

2.	$\vdash_\Sigma \Box(A \to (B \to (A \wedge B)))$	
3′.	$\vdash_\Sigma \Box A \to \Box(B \to (A \wedge B))$	2, K, MP

This is probably easier to read.

Similarly, one could combine steps 4 and 5, like this:

4.	$\vdash_\Sigma \Box A \to \Box(B \to (A \wedge B))$	
5′.	$\vdash_\Sigma \Box A \to (\Box B \to \Box(A \wedge B))$	4, K, RPL

Example

Suppose, as in the previous example, that Σ is a modal logic containing schema K and closed under the rule RN. Show that Σ is closed under the following rule RM:

RM. $$\frac{A \to B}{\Box A \to \Box B}$$

1. $\vdash_\Sigma A \to B$ *assumption*
2. $\vdash_\Sigma \Box(A \to B)$ 1, RN
3. $\vdash_\Sigma \Box A \to \Box B$ 2, K, MP

With step 2 written out in full detail, the derivation looks like this

1. $\vdash_\Sigma A \to B$ *assumption*
2. $\vdash_\Sigma \Box(A \to B)$ 1, RN
3. $\vdash_\Sigma \Box(A \to B) \to (\Box A \to \Box B)$ K (an instance of K)
4. $\vdash_\Sigma \Box A \to \Box B$ 2, 3, MP

Normal modal logics

Normal systems of modal logic are defined in terms of the schema

K. $$\Box(A \to B) \to (\Box A \to \Box B)$$

and the rule of inference ('necessitation')

RN. $$\frac{A}{\Box A}$$

(or equivalently, in terms of the rule RK, see below).

$\Diamond$ can be treated as an abbreviation for $\neg\Box\neg$ or as a primitive of the language. We will follow usual practice and treat it as a primitive. So we need another schema:

Df$\Diamond$. $$\Diamond A \leftrightarrow \neg\Box\neg A$$

Definition 8 (Normal system) *A system of modal logic is* normal *iff it contains* Df$\Diamond$ *and* K *and is closed under* RN.

Equivalently ...

Theorem 9 *A system of modal logic is* normal *iff it contains* Df$\Diamond$ *and is closed under* RK.

RK. $$\frac{(A_1 \land \ldots \land A_n) \to A}{(\Box A_1 \land \ldots \land \Box A_n) \to \Box A} \quad (n \geq 0)$$

This is a special case of Theorem 12 later.

Example 10 (Blackburn et al, p192)

(i) The inconsistent logic is a normal logic.
(ii) *PL* is not a normal logic.
(iii) If $\{\Sigma_i \mid i \in I\}$ is a collection of normal logics, then $\bigcap_{i \in I} \Sigma_i$ is a normal logic.
(iv) If F is any class of frames then Σ_{F}, the set of formulas valid on F, is a normal logic.

Exercise: Prove the above statements. (In the tutorial exercises.)

The smallest normal modal logic is called *K*. It is therefore the smallest modal logic containing K and Df$\Diamond$ and closed under RN. It is also the logic of formulas valid on the class of all relational ('Kripke') frames.

Normal systems

The smallest normal modal logic is called K. To name normal systems it is usual to write

$$K\xi_1 \dots \xi_n$$

for the normal modal logic that results when the schemas $\xi_1, \dots, \xi_n$ are taken as theorems; i.e., $K\xi_1 \dots \xi_n$ is the smallest normal system of modal logic containing (every instance of) the schemas $\xi_1, \dots, \xi_n$.
Some common axiom schemas:

D.	$\Box A \to \Diamond A$
T.	$\Box A \to A$
B.	$A \to \Box\Diamond A$
4.	$\Box A \to \Box\Box A$
5.	$\Diamond A \to \Box\Diamond A$

And some that are a bit less common (but still well-known):

H.	$(\Box(A \lor B) \land \Box(\Box A \lor B) \land \Box(A \lor \Box B)) \to (\Box A \lor \Box B)$
L.	$\Box(\Box A \to A) \to \Box A$

(Obviously you are not expected to memorise these axioms schemas. Those in the first group come up so frequently that you probably will remember them anyway.)

Some systems also have historical names (T, B, S4, S4.2, S4.3, S5, ...).

K	the class of all frames ('Kripke frames')
K4	the class of transitive frames
T = *KT*	the class of reflexive frames
B = *KB*	the class of symmetric frames
KD	the class of serial frames
KD45	the class of serial, transitive, euclidean frames
S4 = *KT4*	the class of reflexive, transitive frames
S5 = *KT5*	the class of frames whose relation is an equivalence relation
	the class of frames whose relation is a universal relation
S4.3 =*KT4H*	the class of reflexive, transitive frames with no branching to the right
KL	the class of finite transitive trees

Some Soundness and Completeness Results

Theorem 11 *[Chellas Thm 4.2, p114] Every normal system of modal logic has the following rules of inference and theorems.*

RN.	$\dfrac{A}{\Box A}$
RM.	$\dfrac{A \to B}{\Box A \to \Box B}$
RR.	$\dfrac{(A \land B) \to C}{(\Box A \land \Box B) \to \Box C}$
RK.	$\dfrac{(A_1 \land \dots \land A_n) \to A}{(\Box A_1 \land \dots \land \Box A_n) \to \Box A} \quad (n \geq 0)$
RE.	$\dfrac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$
N.	$\Box\top$
M.	$\Box(A \land B) \to (\Box A \land \Box B)$
C.	$(\Box A \land \Box B) \to \Box(A \land B)$
R. (or MC.)	$\Box(A \land B) \leftrightarrow (\Box A \land \Box B)$
K.	$\Box(A \to B) \to (\Box A \to \Box B)$

Note: There is an important difference between the *system* K (the smallest normal system of modal logic), the *schema* K, and the *rule of inference* RK. All normal systems, including the smallest such, K, are closed under the rule RK. All systems closed under RK are normal. But there are *non-normal* systems that contain the schema K: the *schema* K by itself does not guarantee that a system is normal. The schema K together with the 'rule of necessitation' RN are equivalent to the rule RK characteristic of normal systems.

Proofs of theorem 11

RN: By definition of normal system.

RM: This is the special case of RK for $n = 1$. But here is a direct proof:

1.	$\vdash_\Sigma A \to B$	ass.
2.	$\vdash_\Sigma \Box(A \to B)$	1, RN
3.	$\vdash_\Sigma \Box(A \to B) \to (\Box A \to \Box B)$	K
4.	$\vdash_\Sigma \Box A \to \Box B$	2, 3, MP

RR: This is the special case of RK for $n = 2$. But it might be helpful to see a direct proof:

1.	$\vdash_\Sigma (A \wedge B) \to C$	ass.
2.	$\vdash_\Sigma A \to (B \to C)$	1, RPL
3.	$\vdash_\Sigma \Box A \to \Box(B \to C)$	2, RM
4.	$\vdash_\Sigma \Box(B \to C) \to (\Box B \to \Box C)$	K
5.	$\vdash_\Sigma \Box A \to (\Box B \to \Box C)$	3, 4, RPL
6.	$\vdash_\Sigma (\Box A \wedge \Box B) \to \Box C$	5, RPL

RK: This is a generalisation of the previous proof RR, by induction on n. The base case $n = 0$ is just RN. For the inductive step, assume RK holds for some $n \geq 0$ and consider the case $n + 1$:

1.	$\vdash_\Sigma (A_1 \wedge \cdots \wedge A_n \wedge A_{n+1}) \to A$	ass.
2.	$\vdash_\Sigma (A_1 \wedge \cdots \wedge A_n) \to (A_{n+1} \to A)$	1, RPL
3.	$\vdash_\Sigma (\Box A_1 \wedge \cdots \wedge \Box A_n) \to \Box(A_{n+1} \to A)$	2, Ind.Hyp.
4.	$\vdash_\Sigma \Box(A_{n+1} \to A) \to (\Box A_{n+1} \to \Box A)$	K
5.	$\vdash_\Sigma (\Box A_1 \wedge \cdots \wedge \Box A_n) \to (\Box A_{n+1} \to \Box A)$	3, 4, RPL
6.	$\vdash_\Sigma (\Box A_1 \wedge \cdots \wedge \Box A_n \wedge \Box A_{n+1}) \to \Box A$	5, RPL

RE: Follows immediately from RM. $A \leftrightarrow B$ is the conjunction of $A \to B$ and $B \to A$. Apply RM to both, and then form the conjunction to get the biconditional $\Box A \leftrightarrow \Box B$.

N: Just apply RN:

1.	$\vdash_\Sigma \top$	*PL*
2.	$\vdash_\Sigma \Box\top$	1, RN

M: Follows from RM:

1.	$\vdash_\Sigma (A \wedge B) \to A$	*PL*
2.	$\vdash_\Sigma \Box(A \wedge B) \to \Box A$	1, RM
3.	$\vdash_\Sigma (A \wedge B) \to B$	*PL*
4.	$\vdash_\Sigma \Box(A \wedge B) \to \Box B$	3, RM
5.	$\vdash_\Sigma \Box(A \wedge B) \to (\Box A \wedge \Box B)$	2, 4, RPL

C: Follows immediately by RR, i.e. by RK for the case $n = 2$:

1.	$\vdash_\Sigma (A \wedge B) \to (A \wedge B)$	*PL*
2.	$\vdash_\Sigma (\Box A \wedge \Box B) \to \Box(A \wedge B)$	1, RK($n = 2$)

K: By definition of normal system.

Theorem 12 *[Chellas Thm 4.3, p115] Let Σ be a system of modal logic containing* Df$\Diamond$. *Then:*

(1) Σ *is normal iff it is closed under* RK.
(2) Σ *is normal iff it contains* N *and is closed under* RR.
(3) Σ *is normal iff it contains* N *and* C *and is closed under* RM.
(4) Σ *is normal iff it contains* N, C, *and* M *and is closed under* RE.

(The last of these (4) will be particularly useful when we look at non-normal systems later.)

Proofs of theorem 12

We only need to prove the 'only if' parts. The 'if' parts are theorem 11.

(1) RN is the special case of RK for case $n = 0$. To derive K: notice K is logically equivalent (in *PL*) to $(\,\Box A \wedge \Box(A \to B)\,) \to \Box B$. Which leads me to start a derivation as follows:

1.	$\vdash_\Sigma (\,A \wedge (A \to B)\,) \to B$	*PL*
2.	$\vdash_\Sigma (\,\Box A \wedge \Box(A \to B)\,) \to \Box B$	1, RK($n = 2$)
3.	$\vdash_\Sigma \Box(A \to B) \to (\Box A \to \Box B)$	2, RPL

(2) To derive K, the same derivation works as in part (1) since it uses only RK($n = 2$) which is RR. To derive RN:

1.	$\vdash_\Sigma A$	ass.
2.	$\vdash_\Sigma (\top \wedge \top) \to A$	1, RPL
3.	$\vdash_\Sigma (\Box\top \wedge \Box\top) \to \Box A$	2, RR
4.	$\vdash_\Sigma \Box\top$	N
5.	$\vdash_\Sigma \Box A$	3, 4, RPL

(3) Given part (2), it is sufficient to derive RR.

1.	$\vdash_\Sigma (A \wedge B) \to C$	ass.
2.	$\vdash_\Sigma \Box(A \wedge B) \to \Box C$	1, RM
3.	$\vdash_\Sigma (\Box A \wedge \Box B) \to \Box(A \wedge B)$	C
4.	$\vdash_\Sigma (\Box A \wedge \Box B) \to \Box C$	2, 3, RPL

(4) Given part (3), it is sufficient to derive RM.

1.	$\vdash_\Sigma A \to B$	ass.
2.	$\vdash_\Sigma (A \wedge B) \leftrightarrow A$	1, RPL
3.	$\vdash_\Sigma \Box(A \wedge B) \leftrightarrow \Box A$	2, RE
4.	$\vdash_\Sigma \Box(A \wedge B) \to (\Box A \wedge \Box B)$	M
5.	$\vdash_\Sigma \Box A \to \Box B$	3, 4, RPL

Some authors (e.g., Blackburn et al) prefer to present systems, schemas and rules primarily in terms of the possibility operator $\Diamond$ rather than the necessity operator $\Box$. This is just a matter of personal preference.

Theorem 13 *[Chellas Thm 4.4, p116] Every normal system of modal logic has the following rules and theorems.*

K$\Diamond$.	$(\neg\Diamond A \wedge \Diamond B) \rightarrow \Diamond(\neg A \wedge B)$
RN$\Diamond$.	$\dfrac{\neg A}{\neg\Diamond A}$
Df$\Box$.	$\Box A \leftrightarrow \neg\Diamond\neg A$
RK$\Diamond$.	$\dfrac{A \rightarrow (A_1 \vee \cdots \vee A_n)}{\Diamond A \rightarrow (\Diamond A_1 \vee \cdots \vee \Diamond A_n)} \quad (n \geq 0)$
RM$\Diamond$.	$\dfrac{A \rightarrow B}{\Diamond A \rightarrow \Diamond B}$
RE$\Diamond$.	$\dfrac{A \leftrightarrow B}{\Diamond A \leftrightarrow \Diamond B}$
N$\Diamond$.	$\neg\Diamond\bot$
M$\Diamond$.	$(\Diamond A \vee \Diamond B) \rightarrow \Diamond(A \vee B)$
C$\Diamond$.	$\Diamond(A \vee B) \rightarrow (\Diamond A \vee \Diamond B)$
R$\Diamond$ (or MC$\Diamond$).	$\Diamond(A \vee B) \rightarrow (\Diamond A \vee \Diamond B)$

Proof For example, for RK$\Diamond$:

1.	$\vdash_\Sigma A \rightarrow (A_1 \vee \cdots \vee A_n)$	*ass.*
2.	$\vdash_\Sigma (\neg A_1 \wedge \cdots \wedge \neg A_n) \rightarrow \neg A$	1, RPL
3.	$\vdash_\Sigma (\Box\neg A_1 \wedge \cdots \wedge \Box\neg A_n) \rightarrow \Box\neg A$	2, RK
4.	$\vdash_\Sigma \neg\Box\neg A \rightarrow (\neg\Box\neg A_1 \vee \cdots \vee \neg\Box\neg A_n)$	3, RPL
5.	$\vdash_\Sigma \Diamond A \rightarrow (\Diamond A_1 \vee \cdots \vee \Diamond A_n)$	4, Df$\Box$ and RPL

The other parts are left as exercises. They all proceed similarly.

Theorem 14 *Let Σ be a system of modal logic containing* Df$\Box$. *Then Σ is normal iff:*

(1) it contains K *and is closed under* RN$\Diamond$*;*

(2) it contains K$\Diamond$ *and is closed under* RN$\Diamond$*;*

(3) it is closed under RK$\Diamond$.

Proof Easy exercise.

Other characterisations can be given. (Cf. Theorem 12.)

Summary: Normal modal logics

Three equivalent characterisations:

Definition [Normal system] A system of modal logic is *normal* iff it contains Df$\Diamond$ and K and is closed under RN.

Equivalently ...

Theorem A system of modal logic is *normal* iff it contains Df$\Diamond$ and is closed under RK.

RK. $$\frac{(A_1 \wedge \ldots \wedge A_n) \rightarrow A}{(\Box A_1 \wedge \ldots \wedge \Box A_n) \rightarrow \Box A} \quad (n \geq 0)$$

Equivalently again ...

Theorem A system of modal logic is *normal* iff it contains Df$\Diamond$, is closed under RE

RE. $$\frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$$

and contains the following schemas:

M.	$\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$
C.	$(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$
N.	$\Box\top$

(Personally I like the last characterisation best.)

Remember: There is an important difference between the *system K* (the smallest normal system of modal logic), the *schema* K, and the *rule of inference* RK. All normal systems, including the smallest such, *K*, are closed under the rule RK. All systems closed under RK are normal. But there are *non-normal* systems that contain the schema K: the *schema* K by itself does not guarantee that a system is normal. The schema K together with the 'rule of necessitation' RN are equivalent to the rule RK characteristic of normal systems.

Classical systems of modal logic

(See Chellas [1980], Ch. 7–9.)
The smallest classical modal logic is called E. To name classical systems we write

$$E\xi_1 \dots \xi_n$$

for the classical modal logic that results when the schemas $\xi_1, \dots, \xi_n$ are taken as theorems; i.e., $E\xi_1 \dots \xi_n$ is the smallest classical system of modal logic containing (every instance of) the schemas $\xi_1, \dots, \xi_n$.
Classical systems of modal logic are defined in terms of the schema

Df$\Diamond$. $\qquad \Diamond A \leftrightarrow \neg\Box\neg A$

and the rule of inference

RE. $\qquad \dfrac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$

Definition [Classical system] A system of modal logic is *classical* iff it contains Df$\Diamond$ and is closed under RE.

Notice: every normal system is classical but not every classical system is normal.

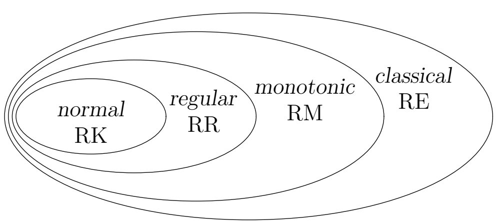
Classical systems are sometimes classified further. (You don't need to remember the names!!)

RE. $\qquad \dfrac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$

RM. $\qquad \dfrac{A \to B}{\Box A \to \Box B}$

RR. $\qquad \dfrac{(A \wedge B) \to C}{(\Box A \wedge \Box B) \to \Box C}$

RK. $\qquad \dfrac{(A_1 \wedge \dots \wedge A_n) \to A}{(\Box A_1 \wedge \dots \wedge \Box A_n) \to \Box A} \quad (n \geq 0)$



RM "=" RE + M
RR "=" RE + MC
RK "=" RE + MCN

Other schemas

The schemas P, D, T, B, 4, 5 also come up frequently.

P. $\qquad \neg\Box\bot$

D. $\qquad \Box A \to \Diamond A$

T. $\qquad \Box A \to A$

B. $\qquad A \to \Box\Diamond A$

4. $\qquad \Box A \to \Box\Box A$

5. $\qquad \Diamond A \to \Box\Diamond A$

We will look at some of their properties later.

Note that for a *normal* system Σ, schema P is in Σ iff D is in Σ. That is *not the case* for non-normal systems in general.

Are there any systems of modal logic that are not classical? YES — but they are very weak and are not studied much.

How do classical non-normal logics come up?

- when a language is interpreted on model structures that are not *relational*;
- sometimes a 'box' operator defined in terms of two or more normal modalities comes out non-normal;
- we give an axiomatic characterisation of some concept represented as a modal operator, and these axioms make the operator non-normal (example below);
- and perhaps lots of other reasons.

Example

Let $\mathrm{E}_x A$ represent that agent x is responsible for, is the direct cause of, A is the case.
So for example we have an axiom $\mathrm{E}_x A \to A$.
But clearly no agent x can be responsible for, can be the direct cause of, logical truth. So we add an axiom

$$\neg\mathrm{E}_x\top$$

Now, every normal system contains $\mathrm{E}_x\top$.
So either this logic is not normal, or it is the inconsistent logic (which is, trivially, a normal logic).

We will look at classical systems of modal logic in a bit more detail later.