Tutorial Exercises 3 (mjs)
(Canonical models for normal systems)

1. A relation R is 'serially reflexive' when $w\,R\,w'$ implies $w'\,R\,w'$ for all w, w'. Show that the logic $K \cup \{\Box(\Box A \longrightarrow A)\}$ is determined by the class of serially reflexive frames.

2. Suppose the logic Σ has box operators K_a and K_b and is interpreted on frames $\langle W, R_a, R_b \rangle$. Show that if
$$\vdash_\Sigma \mathsf{K}_b p \longrightarrow \mathsf{K}_a \neg \mathsf{K}_b \neg p$$
then the canonical frame $\langle W^\Sigma, R_a^\Sigma, R_b^\Sigma \rangle$ for Σ has the property that
$$w\,R_a^\Sigma\,w' \text{ and } w''\,R_b^\Sigma\,w' \text{ implies } w\,R_b^\Sigma\,w''$$
for all w, w', w''.

3. As in previous question, but show that
$$\vdash_\Sigma \mathsf{K}_a(\mathsf{K}_b A \longrightarrow \mathsf{K}_a \mathsf{K}_b A)$$
implies the canonical frame has the property:
$$u\,R_a^\Sigma\,w \text{ and } w\,R_a^\Sigma\,w' \text{ and } w'\,R_b^\Sigma\,w'' \text{ implies } w\,R_b^\Sigma\,w''$$
for all u, w, w', w''.

4. Prove that the normal modal logic *KT5* is determined by the class of equivalence frames.

   ($KT5 = KT45 = KTB5 = KTB4$ is the logic S5.)

5. S5 is also determined by the class of universal frames. (A relation R is universal when $w\,R\,w'$ for all worlds w, w'.) Show however that the canonical relation for S5 is not universal.

   Hint: consider either $\{p\}$ or $\{\Box p\}$ (p any atom), and observe that both these sets are obviously S5-consistent.

6. *From the 2003 exam*:

   The system $S4.2$ is a normal modal logic of type $KT4G$, i.e., the smallest normal system containing the schemas T and 4 and the following schema:

   G. $\Diamond\Box A \longrightarrow \Box\Diamond A$

   A relation R is said to be *strongly convergent* when, for all w, w' there exists a v such that $w\,R\,v$ and $w'\,R\,v$.

   Using the canonical model, show that $S4.2$=$KT4G$ is complete with respect to the class of reflexive, transitive, strongly convergent Kripke models.

   You may assume without proof that
$$\{A \mid \Box A \in \Gamma\} \cup \{A \mid \Box A \in \Gamma'\}$$
   is $KT4G$-consistent for any maximal $KT4G$-consistent sets Γ and Γ'.

   (But see the comment in the next question.)

7. *Harder* Actually, $\{A \mid \Box A \in \Gamma\} \cup \{A \mid \Box A \in \Gamma'\}$ in the previous question is not necessarily $KT4G$-consistent. That doesn't affect the argument in the previous question — it was simplified to make a short exam question. In fact, $KT4G$ is determined by the class of reflexive, transitive frames which satisfy the property ('incestual' or 'Church-Rosser') that, for all u, w, w' such that $u\,R\,w$ and $u\,R\,w'$ there exists a v such that $w\,R\,v$ and $w'\,R\,v$.

   Modify the argument in the previous question to show that this is so.