499 Modal and Temporal Logic *Autumn 2007*

Tutorial Exercises 5 (mjs)

Question 1 Refer to the formulation of the bit transmission problem in the formalism of interpreted systems (without time) as shown in the lectures. Let IS_{b} name this interpreted system.
For ease of reference, here is the definition of the valuation function for atoms:

$$h(\mathbf{bit{=}0}) = \{(0,\epsilon),(0,0),(0\text{-}ack,0)\}$$
$$h(\mathbf{bit{=}1}) = \{(1,\epsilon),(1,1),(1\text{-}ack,1)\}$$
$$h(\mathbf{recbit}) = \{(0,0),(0\text{-}ack,0),(1,1),(1\text{-}ack,1)\}$$
$$h(\mathbf{recack}) = \{(0\text{-}ack,0),(1\text{-}ack,1)\}$$

Check that each of the following holds:

$$IS_{\mathrm{b}} \models \mathbf{recbit} \to \big(\mathsf{K}_R\,(\mathbf{bit{=}0}) \lor \mathsf{K}_R\,(\mathbf{bit{=}1})\big)$$
$$IS_{\mathrm{b}} \models \mathbf{recack} \to \mathbf{recbit}$$
$$IS_{\mathrm{b}} \models (\mathbf{bit{=}0}) \to \mathsf{K}_S\,(\mathbf{bit{=}0})$$
$$IS_{\mathrm{b}} \models \mathbf{recack} \to \mathsf{K}_S\,\mathbf{recack}$$
$$IS_{\mathrm{b}} \not\models \mathbf{recack} \to \mathsf{K}_R\,\mathbf{recack}$$
$$IS_{\mathrm{b}} \models \mathbf{recbit} \land (\mathbf{bit{=}0}) \to \mathsf{K}_R\,(\mathbf{bit{=}0})$$
$$IS_{\mathrm{b}} \models \mathbf{recack} \to \big(\mathsf{K}_R\,(\mathbf{bit{=}0}) \lor \mathsf{K}_R\,(\mathbf{bit{=}1})\big)$$
$$IS_{\mathrm{b}} \models \mathbf{recack} \to \mathsf{K}_S\big(\mathsf{K}_R\,(\mathbf{bit{=}0}) \lor \mathsf{K}_R\,(\mathbf{bit{=}1})\big)$$
$$IS_{\mathrm{b}} \models \mathbf{recack} \land (\mathbf{bit{=}0}) \to \mathsf{K}_S\,\mathsf{K}_R\,(\mathbf{bit{=}0})$$
$$IS_{\mathrm{b}} \not\models \mathbf{recack} \land (\mathbf{bit{=}0}) \to \mathsf{K}_R\,\mathsf{K}_S\,\mathsf{K}_R\,(\mathbf{bit{=}0})$$

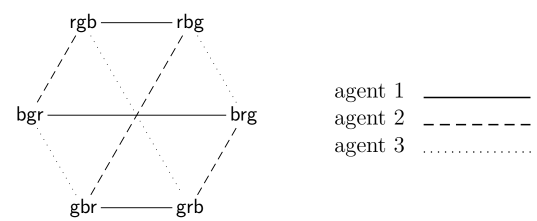
For some of the above, you can shorten the work *very* substantially by making use of properties of the logic of K_R and K_S. Identify clearly any such properties you use. (K_R and K_S are both of type S5. See next question.)

Question 2 Show that the following three schemas are valid in the class of equivalence frames (i.e., frames (W,R) whose relation R is an equivalance relation).

T.	$\Box A \to A$
4.	$\Box A \to \Box\Box A$
5.	$\Diamond A \to \Box\Diamond A$

(You have probably done this several times already, but just in case you haven't, here it is again.)
T and 4 are easy: they are valid in reflexive and transitive frames, respectively, as is shown in the lecture notes and/or previous tutorial exercises. 5 is characteristic of euclidean frames. (A relation R is euclidean iff for all w, w', w'', $w\,R\,w'$ and $w\,R\,w''$ implies $w'\,R\,w''$.)
So you can either: show that every equivalence relation is euclidean, and then show that 5 is valid in the class of euclidean frames; or just show that 5 is valid in equivalence frames directly. (Every symmetric and transitive relation is euclidean.)

Question 3 Card game: there are three cards r, g, and b, and three players 1, 2, and 3. Each player is given one of the cards. Each player can see its own card but not any of the others.
This can be modelled as follows. Represent all the possible states (worlds) by rgb (for player 1 has card r, player 2 has card g, player 3 has card b), rbg, brg, gbr, etc, etc. Represent the epistemic accessibility relations for agents 1, 2 and 3 (i.e., the states indistinguishable for agents 1, 2 and 3) as depicted on the diagram labelled 'model $\mathcal{M}$'. Reflexive arcs have been omitted from the diagram for clarity.
You can also see model $\mathcal{M}$ as an example of the interpreted systems formalism. The possible *local states* for the agents are $L_1 = L_2 = L_3 = \{\mathsf{r},\mathsf{g},\mathsf{b}\}$. The set of *global states* for the system as a whole is $\{\mathsf{rgb},\mathsf{rbg},\mathsf{gbr},\mathsf{grb},\mathsf{brg},\mathsf{bgr}\} \subseteq L_1 \times L_2 \times L_3$. The environment L_E plays no role in this particular example and so is omitted for simplicity. The epistemic accessibility relations for each agent i are defined as usual for interpreted systems, as $g\,R_i\,g'$ iff $l_i(g) = l_i(g')$, where $l_i(g)$ denotes the local state of agent i in global state g.



model $\mathcal{M}$

Now we define the valuation function h for a suitable propositional language. Let propositional atoms r_1, r_2, r_3 represent that players 1, 2, and 3, respectively, hold the r card, atoms g_1, g_2, g_3 that players 1, 2, 3, respectively, hold the g card, and atoms b_1, b_2, b_3 that players 1, 2, 3, respectively, hold the b card. So we define the valuation function h for atoms so that $h(r_1) = \{\mathsf{rgb},\mathsf{rbg}\}$, $h(r_2) = \{\mathsf{grb},\mathsf{brg}\}$, $h(g_1) = \{\mathsf{grb},\mathsf{gbr}\}$, $h(b_3) = \{\mathsf{rgb},\mathsf{grb}\}$, and so on.

Now check the following.

$$\mathcal{M} \models \neg(r_1 \land r_2) \ \land\ \neg(r_2 \land r_3) \ \land\ \neg(r_3 \land r_1)$$
$$\mathcal{M} \models \neg r_i \to (g_i \lor b_i), \quad \text{i.e., } r_i \lor g_i \lor b_i \quad \text{for } i = 1,2,3$$
$$\mathcal{M} \models r_i \to \mathsf{K}_i\, r_i \quad \text{for } i = 1,2,3$$
$$\mathcal{M} \models \mathsf{K}_i\, r_i \to r_i \quad \text{for } i = 1,2,3$$
$$\mathcal{M} \models \neg r_i \to \mathsf{K}_i\, \neg r_i \quad \text{for } i = 1,2,3$$
$$\mathcal{M} \models \mathsf{K}_j(r_i \to \mathsf{K}_i\, r_i) \quad \text{for } i = 1,2,3,\ j = 1,2,3$$
$$\mathcal{M} \not\models r_i \to \mathsf{K}_j\,\mathsf{K}_i\, r_i \quad \text{for } i \neq j$$
$$\mathcal{M} \models r_i \to \mathsf{K}_i\,\mathsf{K}_i\,\neg\,\mathsf{K}_i\, r_j \quad \text{for } i \neq j$$

(By symmetry, it's enough to show the case $i = 1$.)

A more elaborate version of this example will probably be part of the assessed coursework.