499 Modal and Temporal Logic *Autumn 2007*

Tutorial Exercises 1 (mjs)

SOLUTIONS

1. Suppose that Σ is closed under RM.

Suppose first that Σ contains C. A derivation of K:

1.	$\vdash_\Sigma (A \land (A \to B)) \to B$	*PL*
2.	$\vdash_\Sigma \Box(A \land (A \to B)) \to \Box B$	1, RM
3.	$\vdash_\Sigma (\Box A \land \Box(A \to B)) \to \Box(A \land (A \to B))$	C
4.	$\vdash_\Sigma (\Box A \land \Box(A \to B)) \to \Box B$	2, 3, RPL
5.	$\vdash_\Sigma \Box(A \to B) \to (\Box A \to \Box B)$	4, RPL

Suppose now that Σ contains K. A derivation of C:

1.	$\vdash_\Sigma A \to (B \to (A \land B))$	*PL*
2.	$\vdash_\Sigma \Box A \to \Box(B \to (A \land B))$	1, RM
3.	$\vdash_\Sigma \Box(B \to (A \land B)) \to (\Box B \to \Box(A \land B))$	K
4.	$\vdash_\Sigma \Box A \to (\Box B \to \Box(A \land B))$	2, 3, RPL
5.	$\vdash_\Sigma (\Box A \land \Box B) \to \Box(A \land B)$	4, RPL

2. Let Σ be closed under RM:

(a)

1.	$\vdash_\Sigma A \to (B \to A)$	*PL*
2.	$\vdash_\Sigma \Box A \to \Box(B \to A)$	1, RM

(b)

1.	$\vdash_\Sigma \neg A \to (A \to B)$	*PL*
2.	$\vdash_\Sigma \Box\neg A \to \Box(A \to B)$	1, RM

(c)

1.	$\vdash_\Sigma A \to (A \lor B)$	*PL*
2.	$\vdash_\Sigma \Box A \to \Box(A \lor B)$	1, RM
3.	$\vdash_\Sigma B \to (A \lor B)$	*PL*
4.	$\vdash_\Sigma \Box B \to \Box(A \lor B)$	3, RM
5.	$\vdash_\Sigma (\Box A \lor \Box B) \to \Box(A \lor B)$	2, 4, RPL

(d)

1.	$\vdash_\Sigma (\Box\neg A \lor \Box\neg B) \to \Box(\neg A \lor \neg B)$	part (c)
2.	$\vdash_\Sigma (\neg\Diamond A \lor \neg\Diamond B) \to \Box(\neg A \lor \neg B)$	1, Df$\Diamond$
3.	$\vdash_\Sigma (\neg\Diamond A \lor \neg\Diamond B) \to \Box\neg(A \land B)$	2, RPL, RE
4.	$\vdash_\Sigma (\neg\Diamond A \lor \neg\Diamond B) \to \neg\Diamond(A \land B)$	3, Df$\Diamond$
5.	$\vdash_\Sigma \Diamond(A \land B) \to (\Diamond A \land \Diamond B)$	4, RPL

It is not necessary to be so long-winded. I am showing all the steps in detail.

(e)

1.	$\vdash_\Sigma (A \to B) \lor (B \to A)$	*PL*
2.	$\vdash_\Sigma \neg(A \to B) \to (B \to A)$	1, RPL
3.	$\vdash_\Sigma \Box\neg(A \to B) \to \Box(B \to A)$	2, RM
4.	$\vdash_\Sigma \neg\Diamond(A \to B) \to \Box(B \to A)$	3, Df$\Diamond$
5.	$\vdash_\Sigma \Diamond(A \to B) \lor \Box(B \to A)$	4, RPL

Can also use part (b) and the instance $\Box\neg(A \to B) \to \Box((A \to B) \to A)$.

(f) I will show the contrapositive. Here it is presented in a different style, informally, to make the chain of reasoning shorter and clearer. (There are other possible derivations.)

$$\begin{array}{rcll} \vdash_\Sigma \neg\Diamond(A \to B) & \to & \Box\neg(A \to B) & \text{Df}\Diamond \\ & \to & \Box(A \land \neg B) & \text{RPL, RE} \\ & \to & (\Box A \land \Box\neg B) & \text{M} \\ & \to & (\Box A \land \neg\Diamond B) & \text{Df}\Diamond \\ & \to & \neg(\Box A \to \Diamond B) & \text{RPL} \end{array}$$

(g) Follows from part (f). Here is an instance of (f):

$$\begin{array}{rcll} \vdash_\Sigma (\Box A \to \Diamond A) & \to & \Diamond(A \to A) & \\ & \to & \Diamond\top & \text{RPL, RE} \end{array}$$

(h) There are other ways to do it, but again it is easier (for me) to prove the contrapositive, and to see the chain of reasoning when presented informally as follows:

$$\begin{array}{rcll} \vdash_\Sigma \neg\Box(A \to B) & \to & \Diamond\neg(A \to B) & \text{Df}\Diamond \\ & \to & \Diamond(A \land \neg B) & \text{RPL, RE} \\ & \to & (\Diamond A \land \Diamond\neg B) & \text{part (d)} \\ & \to & (\Diamond A \land \neg\Box B) & \text{Df}\Diamond \\ & \to & \neg(\Diamond A \to \Box B) & \text{RPL} \end{array}$$

3. The lecture notes contain proofs for parts (i) and (ii) and a sketch of how to prove closure under uniform substitution (US) in part (iii). The remaining tasks in part (iii) are to prove that Σ_{F}, the set of formulas valid in a class F of frames, contains *PL* and is closed under modus ponens (the rule MP).

The first is very easy. Suppose A is a tautology, i.e., A is an element of *PL*. Then by definition A is true under any assignment of truth values to atoms in A, and so true at every world in every model, including the models belonging to class F.

Closure under modus ponens: we have to show that if $A \in \Sigma_{\mathsf{F}}$ and $A \to B \in \Sigma_{\mathsf{F}}$ then $B \in \Sigma_{\mathsf{F}}$. So suppose $\mathcal{M}$ is a model in Σ_{F} and w is a world in $\mathcal{M}$. Since $A \to B$ is valid in the class F, $\mathcal{M}, w \models A \to B$, i.e. $\mathcal{M}, w \models A \Rightarrow \mathcal{M}, w \models B$. But $A \in \Sigma_{\mathsf{F}}$ and so $\mathcal{M} \models A$ also. So $\mathcal{M}, w \models A$, which implies that $\mathcal{M}, w \models B$, as required.

Here is the same argument using the truth set notation: $A \to B \in \Sigma_{\mathsf{F}}$ means that $\mathcal{M} \models A \to B$ for any model $\mathcal{M}$ in F, i.e., if $\mathcal{M} \models A$ then $\mathcal{M} \models B$. In truth set notation this is $\|A\|^{\mathcal{M}} \subseteq \|B\|^{\mathcal{M}}$. $A \in \Sigma_{\mathsf{F}}$ means that $\|A\|^{\mathcal{M}} = W$ where W is the set of worlds in $\mathcal{M}$. So $W \subseteq \|B\|^{\mathcal{M}}$, and this must mean $\|B\|^{\mathcal{M}} = W$, i.e. $\mathcal{M} \models B$, as required.

4. We already know (question above) that each of these is a system of modal logic. So we just have to show that (a) they contain schema K as theorems, and (b) that they are closed under necessitation (rule RN).

(i) The inconsistent logic contains all formulas and so obviously (a) contains all instances of schema K, and (b) is closed under RN.

(ii) *PL* is not a normal logic: instances of K are not tautologies (theorems of *PL*), and nor is *PL* closed under RN. (For example, $A \rightarrow A$ is a tautology, but $\Box(A \rightarrow A)$ is not.)

(iii) If $\{\Sigma_i \mid i \in I\}$ is a collection of normal logics, then each Σ_i contains all instances of K and is closed under RN. So all instances of K are also in the intersection. And suppose A is in the intersection. Then A is in every Σ_i, and so $\Box A$ is in every Σ_i, and $\Box A$ is in the intersection.

(iv) If F is any class of frames then Σ_{F}, the set of formulas valid on F, is a normal logic. We have to show (a) that schema K is in Σ_{F}, i.e. that K is valid on F, and (b) that if A is valid on F then $\Box A$ is valid on F.

(a) I think this was shown in Ian Hodkinson's notes. But for your convenience, here is a proof: consider any world w in any model $\mathcal{M}$ whose relation R is in F, and any formulas A and B. We show $\mathcal{M}, w \models \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$. It is more convenient to show the equivalent: $\mathcal{M}, w \models (\Box(A \rightarrow B) \wedge \Box A) \rightarrow \Box B$.

$$\begin{aligned}
\mathcal{M}, w \models \Box(A \rightarrow B) \wedge \Box A \;&\Rightarrow\; \mathcal{M}, w \models \Box(A \rightarrow B) \;\&\; \mathcal{M}, w \models \Box A \\
&\Rightarrow\; \forall w' \,(w\, R\, w' \Rightarrow \mathcal{M}, w' \models A \rightarrow B) \;\& \\
&\;\; \forall w' \,(w\, R\, w' \Rightarrow \mathcal{M}, w' \models A) \\
&\Rightarrow\; \forall w' \,(w\, R\, w' \Rightarrow \mathcal{M}, w' \models A \rightarrow B \;\&\; \mathcal{M}, w' \models A) \\
&\Rightarrow\; \forall w' \,(w\, R\, w' \Rightarrow \mathcal{M}, w' \models B) \\
&\Rightarrow\; \mathcal{M}, w \models \Box B
\end{aligned}$$

(b) Consider any model $\mathcal{M}$ whose relation R is in F, and any formula A. We need to show that if A is true at every world w of $\mathcal{M}$, then so is $\Box A$. So suppose A is true at every world w of $\mathcal{M}$. Suppose t is a world in $\mathcal{M}$. $\mathcal{M}, t \models \Box A$ when A is true at every world R-accessible from t. But A is true at all worlds, and so true at the worlds R-accessible from t. So $\Box A$ is true at t. Here is the argument symbolically:

$$\begin{aligned}
\forall w \in \mathcal{M}\, (\mathcal{M}, w \models A) \;&\Rightarrow\; \forall t \in \mathcal{M}, \forall w \in \mathcal{M}\, (t\, R\, w \Rightarrow \mathcal{M}, w \models A) \\
&\Rightarrow\; \forall t \in \mathcal{M}\, (\mathcal{M}, t \models \Box A)
\end{aligned}$$

Or again in another notation: $\mathcal{M}, t \models \Box A$ iff $R[t] \subseteq \|A\|^{\mathcal{M}}$. But when A is valid on $\mathcal{M}$, $\|A\|^{\mathcal{M}} = W$, and so $R[t] \subseteq W$ for every t in $\mathcal{M}$.

5. Let Σ be a normal logic.

Derivation of rule RM$\Diamond$

1.	$\vdash_\Sigma A \rightarrow B$	*ass.*
2.	$\vdash_\Sigma \neg B \rightarrow \neg A$	1, RPL
3.	$\vdash_\Sigma \Box\neg B \rightarrow \Box\neg A$	2, RM
4.	$\vdash_\Sigma \neg\Box\neg A \rightarrow \neg\Box\neg B$	3, RPL
5.	$\vdash_\Sigma \Diamond A \rightarrow \Diamond B$	Def$\Diamond$, and 4, RPL

Derivation of schema N$\Diamond$

1.	$\vdash_\Sigma \top$	*PL*
2.	$\vdash_\Sigma \Box\top$	1, RN
3.	$\vdash_\Sigma \neg\Diamond\neg\top$	2, Def$\Diamond$
4.	$\vdash_\Sigma \neg\top \leftrightarrow \bot$	*PL*
5.	$\vdash_\Sigma \neg\Diamond\bot$	3, 4, RE$\Diamond$

RE$\Diamond$ is the rule $\dfrac{A \leftrightarrow B}{\Diamond A \leftrightarrow \Diamond B}$ (which we can easily get, e.g. from rule RM$\Diamond$ and RPL).

The following would be perfectly acceptable:

1.	$\vdash_\Sigma \Box\top$	N (in every normal modal logic — check)
2.	$\vdash_\Sigma \neg\Diamond\neg\top$	1, Def$\Diamond$
3.	$\vdash_\Sigma \neg\Diamond\bot$	2, RPL and RE$\Diamond$

Derivation of schema MC$\Diamond$ We can derive this in two parts: $\Diamond(A \vee B) \rightarrow (\Diamond A \vee \Diamond B)$ (which is the schema C$\Diamond$), and $(\Diamond A \vee \Diamond B) \rightarrow \Diamond(A \vee B)$ (which is the schema M$\Diamond$).

1.	$\vdash_\Sigma (\Box\neg A \wedge \Box\neg B) \rightarrow \Box(\neg A \wedge \neg B)$	C. (in every normal logic — check!)
2.	$\vdash_\Sigma (\Box\neg A \wedge \Box\neg B) \rightarrow \Box\neg(A \vee B)$	1, RPL, RE
3.	$\vdash_\Sigma \neg\Box\neg(A \vee B) \rightarrow \neg(\Box\neg A \wedge \Box\neg B)$	2, RPL
4.	$\vdash_\Sigma \Diamond(A \vee B) \rightarrow \neg(\neg\Diamond A \wedge \neg\Diamond B)$	3, Def$\Diamond$, RPL, RE$\Diamond$
5.	$\vdash_\Sigma \Diamond(A \vee B) \rightarrow (\Diamond A \vee \Diamond B)$	4, RPL

For the other half (M$\Diamond$):

1.	$\vdash_\Sigma A \rightarrow (A \vee B)$	*PL*
2.	$\vdash_\Sigma \Diamond A \rightarrow \Diamond(A \vee B)$	1, RM$\Diamond$
3.	$\vdash_\Sigma B \rightarrow (A \vee B)$	*PL*
4.	$\vdash_\Sigma \Diamond B \rightarrow \Diamond(A \vee B)$	3, RM$\Diamond$
5.	$\vdash_\Sigma (\Diamond A \vee \Diamond B) \rightarrow \Diamond(A \vee B)$	2, 4, RPL

M$\Diamond$ can also be derived directly from schema M.

Alternatively, derive MC$\Diamond$ from schema MC:

1.	$\vdash_\Sigma (\Box\neg A \wedge \Box\neg B) \leftrightarrow \Box(\neg A \wedge \neg B)$	MC. (in every normal logic — check!)
	$\vdash_\Sigma \qquad \vdots$	(as for C$\Diamond$ above)
5.	$\vdash_\Sigma \Diamond(A \vee B) \leftrightarrow (\Diamond A \vee \Diamond B)$	4, RPL

6. To show that the normal logic *KT5* is the same as the normal logic *KT45*, we show that 4 is already in *KT5*. (I mean: all instances of schema 4 are theorems of *KT5*.)

Semantically, one might argue like this. We know that T ($\Box A \rightarrow A$) is valid in reflexive frames, and 5 ($\Diamond A \longrightarrow \Box\Diamond A$) is valid in euclidean frames. Recall that a relation R is euclidean iff, for all w, w', w'' we have $w\,R\,w'$ & $w\,R\,w'' \;\Rightarrow\; w'\,R\,w''$.

Now observe: if a relation R is reflexive and euclidean then it is also symmetric. (Check: $w\,R\,w'$ & $w\,R\,w \;\Rightarrow\; w'\,R\,w$.)

Further, a symmetric relation is euclidean iff it is transitive. (Check: $w\,R\,w'$ & $w'\,R\,w'' \Rightarrow$ (symmetric) $w'\,R\,w$ & $w'\,R\,w'' \;\Rightarrow$ (euclidean) $w\,R\,w''$.)

Symmetric frames validate schema B ($A \longrightarrow \Box\Diamond A$) and transitive frames validate 4 ($\Box A \longrightarrow \Box\Box A$).

We cannot use this semantic argument directly (we haven't proved any soundness and completeness results for normal modal logics, yet) but it suggests a strategy for deriving schema 4 from schemas T and 5, via schema B. Here goes. I will present it in fragments rather than as one long single derivation.

First:

1. $\vdash_{KT5} \Box\neg A \longrightarrow \neg A$ instances of T
2. $\vdash_{KT5} A \longrightarrow \neg\Box\neg A$ 1, RPL
3. $\vdash_{KT5} A \longrightarrow \Diamond A$ 2, Def.$\Diamond$

The schema $A \longrightarrow \Diamond A$ is often called the 'dual schema' of T; call it T$\Diamond$.

For future reference, we can also derive the 'dual schema' 5$\Diamond$ of schema 5 by a similar argument:

1. $\vdash_{KT5} \Diamond\neg A \longrightarrow \Box\Diamond\neg A$ instances of schema 5
2. $\vdash_{KT5} \neg\Box\Diamond\neg A \longrightarrow \neg\Diamond\neg A$ 1, RPL
3. $\vdash_{KT5} \neg\Box\Diamond\neg A \longrightarrow \Box A$ 2, Def.$\Diamond$
4. $\vdash_{KT5} \Diamond\Box A \longrightarrow \Box A$ 3, Def.$\Diamond$, rule RE

Next:

1. $\vdash_{KT5} A \longrightarrow \Diamond A$ T$\Diamond$
2. $\vdash_{KT5} \Diamond A \longrightarrow \Box\Diamond A$ schema 5
3. $\vdash_{KT5} A \longrightarrow \Box\Diamond A$ 1, 2, RPL

So now we have proved that *KT5* contains all instances of schema B as theorems. (Cf. reflexive and euclidean implies symmetric.)

Now the last steps:

1. $\vdash_{KT5} \Diamond\Box\ A \longrightarrow \Box A$ schema 5$\Diamond$
2. $\vdash_{KT5} \Box\Diamond\Box A \longrightarrow \Box\Box A$ 1, rule RM
3. $\vdash_{KT5} \Box A \longrightarrow \Box\Diamond\Box A$ instances of schema B
4. $\vdash_{KT5} \Box A \longrightarrow \Box\Box A$ 3, 2, RPL

Notice that since we have also shown above that all instances of schema B are theorems of *KT5*, we have actually shown:

$$\text{S5} = \mathit{KT5} = \mathit{KT45} = \mathit{KTB5} = \mathit{KTB45}$$

7.

noN. $\vdash \text{Oblig}\,\top \;\leftrightarrow\; \text{O}\top \wedge \neg\Box\top$
$\leftrightarrow\; \text{O}\top \wedge \bot$ (because $\Box$ is normal and hence $\vdash \Box\top$)
$\leftrightarrow\; \bot$ (by propositional logic)

So we have, by propositional logic, $\vdash \neg\,\text{Oblig}\,\top$.

D. $\vdash \text{Oblig}\,A \wedge \text{Oblig}\,\neg A \;\leftrightarrow\; \text{O}A \wedge \neg\Box A \wedge \text{O}\neg A \wedge \neg\Box\neg A$
$\leftrightarrow\; \text{O}A \wedge \text{O}\neg A \wedge \neg\Box A \wedge \neg\Box\neg A$
$\leftrightarrow\; \bot \wedge \neg\Box A \wedge \neg\Box\neg A$ (because O is of type KD)
$\leftrightarrow\; \bot$

So we have, by propositional logic, $\vdash \neg(\text{Oblig}\,A \wedge \text{Oblig}\,\neg A)$, i.e. (by propositional logic) $\vdash \text{Oblig}\,A \longrightarrow \neg\,\text{Oblig}\,\neg A$.

C. $\vdash \text{Oblig}\,A \wedge \text{Oblig}\,B \;\leftrightarrow\; \text{O}A \wedge \neg\Box A \wedge \text{O}B \wedge \neg\Box B$
$\leftrightarrow\; \text{O}A \wedge \text{O}B \wedge \neg\Box A \wedge \neg\Box B$
$\leftrightarrow\; \text{O}(A \wedge B) \wedge \neg\Box A \wedge \neg\Box B$
(because O is normal: $\vdash (\text{O}A \wedge \text{O}B) \longrightarrow \text{O}(A \wedge B)$)
$\leftrightarrow\; \text{O}(A \wedge B) \wedge \neg\Box(A \wedge B)$
(because $\Box$ is normal: see below)
$\leftrightarrow\; \text{Oblig}(A \wedge B)$

The missing step: because $\Box$ is normal we have $\vdash \Box(A \wedge B) \longrightarrow \Box A$, and so (contra-positive) $\vdash \neg\Box A \longrightarrow \neg\Box(A \wedge B)$. Similarly, $\vdash \neg\Box B \longrightarrow \neg\Box(A \wedge B)$. And so (by propositional logic) $\vdash (\neg\Box A \wedge \neg\Box B) \longrightarrow \neg\Box(A \wedge B)$.

8. Derivation of P is very easy if you notice that the schema T ($\Box A \longrightarrow A$) is a theorem iff its dual schema ($A \longrightarrow \Diamond A$) is a theorem. Here is the full derivation in case you did not notice that:

1. $\vdash_{ET5} \Box\neg A \longrightarrow \neg A$ T
2. $\vdash_{ET5} A \longrightarrow \neg\Box\neg A$ 1, RPL
3. $\vdash_{ET5} A \longrightarrow \Diamond A$ 2, Df$\Diamond$
4. $\vdash_{ET5} \top \longrightarrow \Diamond\top$ instance of 3
5. $\vdash_{ET5} \top$ *PL*
6. $\vdash_{ET5} \Diamond\top$ 4, 5, MP

To derive schema N using 5 ($\Diamond A \rightarrow \Box\Diamond A$) requires some inspiration. Here is a derivation:

1.	$\vdash_{ET5} \Diamond\top$	from 6 above
2.	$\vdash_{ET5} \Diamond\top \leftrightarrow \top$	1, RPL
3.	$\vdash_{ET5} \Box\Diamond\top \leftrightarrow \Box\top$	2, RE
4.	$\vdash_{ET5} \Diamond\top \rightarrow \Box\Diamond\top$	schema 5
5.	$\vdash_{ET5} \Diamond\top \rightarrow \Box\top$	4, 3, RPL
6.	$\vdash_{ET5} \Box\top$	1, 5, MP

9. For ease of reference, these are the 'reduction laws' that we need to show are theorems of the normal system S4 (=*KT4*):

$$\Box A \leftrightarrow \Box\Box A \qquad \Diamond A \leftrightarrow \Diamond\Diamond A$$

$$\Diamond\Box A \leftrightarrow \Diamond\Box\Diamond\Box A \qquad \Box\Diamond A \leftrightarrow \Box\Diamond\Box\Diamond A$$

Schema T is $\Box A \rightarrow A$ and its dual schema T$\Diamond$ is $A \rightarrow \Diamond A$. Schema 4 is $\Box A \rightarrow \Box\Box A$ and its dual schema 4$\Diamond$ is $\Diamond\Diamond A \rightarrow \Diamond A$.

For the reduction laws: first note that those on the right are dual schemas of those on the left. That is to say:

$$\Diamond A \;\leftrightarrow\; \neg\Box\neg A \;\leftrightarrow\; \neg\Box\Box\neg A \;\leftrightarrow\; \Diamond\neg\Box\neg A \;\leftrightarrow\; \Diamond\Diamond A$$

$$\Box\Diamond A \;\leftrightarrow\; \neg\Diamond\neg\neg\Box\neg A \;\leftrightarrow\; \neg\Diamond\Box\neg A \;\leftrightarrow\; \neg\Diamond\Box\Diamond\Box\neg A \;\leftrightarrow\; \Box\Diamond\Box\Diamond A$$

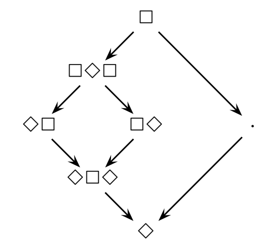
Check the last step: $\neg\Diamond\Box\Diamond\Box\neg A \leftrightarrow \neg\Diamond\Box\Diamond\neg\Diamond A \leftrightarrow \neg\Diamond\Box\neg\Box\Diamond A \leftrightarrow \neg\Diamond\neg\Diamond\Box\Diamond A \leftrightarrow \neg\neg\Box\Diamond\Box\Diamond A \;\leftrightarrow\; \Box\Diamond\Box\Diamond A$.

So we only need to check that the formulas on the left are theorems of *KT4* (S4). The first one on the left is immediate: $\Box A \rightarrow \Box\Box A$ is just schema 4, and $\Box\Box A \rightarrow \Box A$ is a special case of T.

For the bottom one on the left, there are various ways to do it. For example, we can get left-to-right by showing that $\Box A \rightarrow \Box\Diamond\Box A$ is a theorem, because then $\Diamond\Box A \rightarrow \Diamond\Box\Diamond\Box A$ follows by rule RM$\Diamond$. And we have this, because $\Box A \rightarrow \Diamond\Box A$ is a special case of T$\Diamond$, from which follows $\Box\Box A \rightarrow \Box\Diamond\Box A$ by rule RM. And $\Box A \rightarrow \Box\Box A$ (schema 4) gives us what we need.

We can get right-to-left by showing that $\Diamond\Box\Diamond A \rightarrow \Diamond A$ is a theorem, because then $\Diamond\Box\Diamond(\Box A) \rightarrow \Diamond(\Box A)$ is a special case. (The brackets are to aid readability.) We also have this, because (e.g.) $\Box\Diamond A \rightarrow \Diamond A$ is a special case of schema T, from which follows $\Diamond\Box\Diamond A \rightarrow \Diamond\Diamond A$ by rule RM$\Diamond$. And $\Diamond\Diamond A \rightarrow \Diamond A$ is just schema 4$\Diamond$.

These various implications are summarised in the following diagram (Chellas, p149).



Now let's see how many modalities there are in *KT4* (S4). A modality is any sequence ϕ of $\neg$, $\Box$, $\Diamond$ in any combination, including the empty sequence, denoted $\cdot$. Within a system of modal logic, two modalities ϕ and ψ are *equivalent* when, for every formula A, the expression $\phi A \leftrightarrow \psi A$ is a theorem.

First notice that, by moving all negations to the outside, interchanging $\Diamond$ and $\Box$ as necessary, and then replacing all occurrences of $\Diamond\ldots\Diamond$ by $\Diamond$ and $\Box\ldots\Box$ by $\Box$, every modality ψ must be equivalent in S4 to one of the form ϕ or $\neg\phi$ where ϕ is one of

$$(\Box\Diamond)^n \qquad (\Box\Diamond)^n\Box \qquad (\Diamond\Box)^n \qquad (\Diamond\Box)^n\Diamond \qquad\qquad (n \geq 0)$$

or: $\cdot \quad \Box \quad \Diamond \quad (\Box\Diamond)^n \quad (\Box\Diamond)^n\Box \quad (\Diamond\Box)^n \quad (\Diamond\Box)^n\Diamond \quad (n \geq 1)$.

Now, by means of the reduction laws (for $n \geq 1$):

$$\begin{aligned}
&(\Box\Diamond)^n A \leftrightarrow \Box\Diamond A\\
&(\Box\Diamond)^n\Box A \leftrightarrow \Box\Diamond\Box A\\
&(\Diamond\Box)^n A \leftrightarrow \Diamond\Box A\\
&(\Diamond\Box)^n\Diamond A \leftrightarrow \Diamond\Box\Diamond A
\end{aligned}$$

So every modality is equivalent in S4 to one of

$$\cdot \qquad \Box \qquad \Diamond \qquad \Box\Diamond \qquad \Diamond\Box \qquad \Box\Diamond\Box \qquad \Diamond\Box\Diamond$$

or one of their negations: $\neg \quad \neg\Box \quad \neg\Diamond \quad \neg\Box\Diamond \quad \neg\Diamond\Box \quad \neg\Box\Diamond\Box \quad \neg\Diamond\Box\Diamond$. So we've proved that *KT4* (S4) has at most 14 modalities. It still remains to prove that these 14 are distinct, i.e., that S4 does not have fewer than these 14 modalities.

To show this we have to show $\nvdash_{KT4} \phi A \leftrightarrow \psi A$ for all pairs ϕ and ψ of the 14 modalities. How? One way is by constructing a suitable countermodel. For suppose we know that a system Σ is sound with respect to some class C of models, i.e. $\vdash_\Sigma A \Rightarrow \models_{\mathsf{C}} A$. Then $\not\models_{\mathsf{C}} A \Rightarrow \nvdash_\Sigma A$. So one way of showing $\nvdash_\Sigma A$ is to find a model $\mathcal{M}$ in class C such that $\mathcal{M}, w \not\models A$ for some world w in $\mathcal{M}$ (a 'countermodel'). So, for example, the model $\mathcal{M}$ with $W = \{w_1, w_2\}$, $R = \{(w_1,w_1), (w_1,w_2), (w_2,w_2)\}$, and $h(p) = \{w_2\}$ is reflexive and transitive. $\mathcal{M}, w_1 \models \Diamond p$ but $\mathcal{M}, w \not\models p$, so $\nvdash_{KT4} \Diamond A \leftrightarrow A$. Construction of suitable countermodels to show all the 14 modalities are distinct in *KT4* (S4) requires some ingenuity. You can use techniques such as Salqvist's to identify what kind of countermodels to look for. Details omitted.

10. Now let's do the same exercise for the logic S5 (= *KT5* = *KT45*). We know that S5 is a *KT4* system (we showed that in an earlier question). So we can proceed by investigating what further reduction laws are available in in *KT45* (S5) besides those present in *KT4* (S4).

For ease of reference, schema 5 is $\Diamond A \longrightarrow \Box\Diamond A$ and its dual schema 5$\Diamond$ is $\Diamond\Box A \longrightarrow \Box A$.

Notice that the converse of schema 5 ($\Box\Diamond A \longrightarrow \Diamond A$) is a special case of schema T, and the converse of schema 5$\Diamond$ is a special case of schema T$\Diamond$. So in S5 we have the following pair of further reduction laws:

$$\Box\Diamond A \leftrightarrow \Diamond A \qquad \Diamond\Box A \leftrightarrow \Box A$$

Check what has happened to the bottom pair of S4 reduction laws ($\Diamond\Box A \leftrightarrow \Diamond\Box\Diamond\Box A$).

What of the modalities? Let's check the S4 modalities. In S5:

$$\begin{aligned}
&\Box\Diamond A \leftrightarrow \Diamond A\\
&\Diamond\Box A \leftrightarrow \Box A\\
&\Box\Diamond\Box A \leftrightarrow \Diamond\Box A \leftrightarrow \Box A\\
&\Diamond\Box\Diamond A \leftrightarrow \Box\Diamond A \leftrightarrow \Diamond A
\end{aligned}$$

So we are left with just three modalities in S5:

$$\cdot \qquad \Box \qquad \Diamond$$

and their negations: $\neg \quad \neg\Box \quad \neg\Diamond$.

$$\Box \longrightarrow \cdot \longrightarrow \Diamond$$

Alternatively, to determine the modalities in S5 from the reduction laws, follow the method used for determining the S4 modalities: every modality ψ must be equivalent in S5 to one of the form ϕ or $\neg\phi$ where ϕ is one of

$$(\Box\Diamond)^n \qquad (\Box\Diamond)^n\Box \qquad (\Diamond\Box)^n \qquad (\Diamond\Box)^n\Diamond \qquad\qquad (n \geq 0)$$

Now, by means of the reduction laws:

$$\begin{aligned}
&(\Box\Diamond)^n A \leftrightarrow (\Diamond)^n A \leftrightarrow \Diamond A\\
&(\Box\Diamond)^n\Box A \leftrightarrow (\Diamond)^n\Box A \leftrightarrow \Diamond\Box A \leftrightarrow \Box A\\
&(\Diamond\Box)^n A \leftrightarrow (\Box)^n A \leftrightarrow \Box A\\
&(\Diamond\Box)^n\Diamond A \leftrightarrow (\Box)^n\Diamond A \leftrightarrow \Box\Diamond A \leftrightarrow \Diamond A
\end{aligned}$$