499 Modal and Temporal Logic *Autumn 2007*

# Tutorial Exercises 2 (mjs)

## SOLUTIONS

1. We prove the contrapositive. Suppose $\{\Box A_1, \ldots, \Box A_n, \neg B\}$ is S4-inconsistent.

   Then either

   (i) $\vdash_{S4} (\Box A_i \wedge \cdots \wedge \Box A_k) \longrightarrow \bot$

   or (ii) $\vdash_{S4} (\Box A_i \wedge \cdots \wedge \Box A_k \wedge \neg B) \longrightarrow \bot$

   for some $\{\Box A_i, \ldots, \Box A_k\} \subseteq \{\Box A_1, \ldots, \Box A_n\}$.

   If case (i) then $\{\Box A_1, \ldots, \Box A_n, \neg \Box B\}$ is also S4-inconsistent.

   If case (ii) then $\vdash_{S4} (\Box A_i \wedge \cdots \wedge \Box A_k) \longrightarrow B$.

   And so (S4 is normal) $\vdash_{S4} (\Box\Box A_i \wedge \cdots \wedge \Box\Box A_k) \longrightarrow \Box B$.

   But (schema 4, and RPL) $\vdash_{S4} (\Box A_i \wedge \cdots \wedge \Box A_k) \longrightarrow (\Box\Box A_i \wedge \cdots \wedge \Box\Box A_k)$

   and so $\vdash_{S4} (\Box A_i \wedge \cdots \wedge \Box A_k) \longrightarrow \Box B$.

   Hence $\vdash_{S4} (\Box A_i \wedge \cdots \wedge \Box A_k \wedge \neg \Box B) \longrightarrow \bot$ and so $\{\Box A_1, \ldots, \Box A_n, \neg \Box B\}$ is S4-inconsistent.

   (Note that this doesn't use schema T.)

2. This is a theorem in the notes relating deducibility ($\vdash_\Sigma$) with maxiconsistent sets. We need to prove that:

   (a) $\Gamma \vdash_\Sigma A$ iff $A \in \Delta$ for every $\Sigma$-maxi-consistent $\Delta$ such that $\Gamma \subseteq \Delta$.

   (b) $\vdash_\Sigma A$ iff $A \in \Delta$ for every $\Sigma$-maxi-consistent $\Delta$.

   Proof: left to right is easy. Suppose $\Gamma \vdash_\Sigma A$. Suppose $\Gamma \subseteq \Delta$. Then $\Delta \vdash_\Sigma A$ (monotonicity of $\vdash_\Sigma$). For the other half: suppose $\Gamma \nvdash_\Sigma A$. We have to show there is a $\Sigma$-maxi-consistent $\Delta$ such that $\Gamma \subseteq \Delta$ and $A \notin \Delta$. From $\Gamma \nvdash_\Sigma A$, it follows that $\Gamma \cup \{\neg A\}$ is $\Sigma$-consistent. By Lindenbaum's lemma there is therefore a $\Sigma$-maxi-consistent $\Delta$ such that $\Gamma \cup \{\neg A\} \subseteq \Delta$. Because $\{\neg A\} \subseteq \Delta$, i.e., $\neg A \in \Delta$, $A \notin \Delta$ as required.

   Part (b) is just the special case of part (a) where $\Gamma = \emptyset$, and so follows immediately remembering that $\emptyset \vdash_\Sigma A \Leftrightarrow \vdash_\Sigma A$.



3. We want to prove that for any $\Sigma$-maxi-consistent sets $\Gamma$ and $\Gamma'$

   $$\{A \mid \Box A \in \Gamma\} \subseteq \Gamma' \quad \Leftrightarrow \quad \{\Diamond A \mid A \in \Gamma'\} \subseteq \Gamma$$

   or equivalently

   $$\forall A\,[\,\Box A \in \Gamma \Rightarrow A \in \Gamma'\,] \quad \Leftrightarrow \quad \forall A\,[\,A \in \Gamma' \Rightarrow \Diamond A \in \Gamma\,]$$

   Assume LHS. Suppose $A \in \Gamma'$. We need to show $\Diamond A \in \Gamma$.
   Suppose not. Suppose $\Diamond A \notin \Gamma$.

   $$\begin{array}{rcll} \Diamond A \notin \Gamma & \Rightarrow & \neg\Diamond A \in \Gamma & (\Gamma \text{ is maxi}) \\ \neg\Diamond A \in \Gamma & \Rightarrow & \Box\neg A \in \Gamma & \\ \Box\neg A \in \Gamma & \Rightarrow & \neg A \in \Gamma' & (\text{assumed LHS}) \\ \neg A \in \Gamma' & \Rightarrow & A \notin \Gamma' & (\Gamma' \text{ is } \Sigma\text{-consistent}) \\ A \notin \Gamma' & & \multicolumn{2}{l}{\text{Contradiction (we assumed } A \in \Gamma')} \end{array}$$

   The other direction is similar.

   Assume RHS. Suppose $\Box A \in \Gamma$. We need to show $A \in \Gamma'$.
   Suppose not. Suppose $A \notin \Gamma'$.

   $$\begin{array}{rcll} A \notin \Gamma' & \Rightarrow & \neg A \in \Gamma' & (\Gamma' \text{ is maxi}) \\ \neg A \in \Gamma' & \Rightarrow & \Diamond\neg A \in \Gamma & (\text{assumed RHS}) \\ \Diamond\neg A \in \Gamma & \Rightarrow & \neg\Diamond\neg A \notin \Gamma & (\Gamma \text{ is } \Sigma\text{-consistent}) \\ \neg\Diamond\neg A \notin \Gamma & \Rightarrow & \Box A \notin \Gamma & \\ \Box A \notin \Gamma & & \multicolumn{2}{l}{\text{Contradiction (we assumed } \Box A \in \Gamma)} \end{array}$$