499 Modal and Temporal Logic *Autumn 2007*

Tutorial Exercises 3 (mjs)

SOLUTIONS

1. A relation R is 'serially reflexive' when $w\,R\,w'$ implies $w'\,R\,w'$ for all w, w'.

   Soundness is easy (as usual): $\mathcal{M}, w \models \Box(\Box A \longrightarrow A)$ if $\mathcal{M}, w' \models (\Box A \longrightarrow A)$ for all w' such that $w\,R\,w'$.

   Suppose $w\,R\,w'$. Then $w'\,R\,w'$ (R is serially reflexive), and so $\mathcal{M}, w' \models \Box A$ implies $\mathcal{M}, w' \models A$, and $\mathcal{M}, w' \models (\Box A \longrightarrow A)$ as required.

   Completeness: by showing that the canonical relation R^Σ is serially reflexive.

   Suppose that $w\,R^\Sigma\,w'$. Show $w'\,R^\Sigma\,w'$, i.e. $\forall A\,[\Box A \in w' \Rightarrow A \in w']$.

   Suppose $\Box A \in w'$. Show $A \in w'$.

   First: $\Box(\Box A \longrightarrow A) \in w$ (axiom, and w is a maxi-consistent set.)

   $$\Box(\Box A \longrightarrow A) \in w \;\Rightarrow\; \Box A \longrightarrow A \in w' \quad (w\,R^\Sigma\,w').$$

   Now $\Box A \in w'$ (assumption) and $\Box A \longrightarrow A \in w'$ together imply $A \in w'$ (because w' is a maxiconsistent set and so closed under modus ponens).

2. Suppose $w\,R_a^\Sigma\,w'$ and $w''\,R_b^\Sigma\,w'$.

   We need to show $w\,R_b^\Sigma\,w''$, i.e., $\forall A\,[\mathsf{K}_b A \in w \Rightarrow A \in w'']$.

   Suppose $\mathsf{K}_b A \in w$. Show $A \in w''$.

   $$\mathsf{K}_b A \in w \;\Rightarrow\; \mathsf{K}_a\neg\mathsf{K}_b\neg A \in w \quad \text{(axiom, and } w \text{ is a maxi-consistent set)}$$
   $$\mathsf{K}_a\neg\mathsf{K}_b\neg A \in w \;\Rightarrow\; \neg\mathsf{K}_b\neg A \in w' \quad (w\,R_a^\Sigma\,w')$$
   $$\neg\mathsf{K}_b\neg A \in w' \;\Rightarrow\; A \in w'' \quad (w''\,R_b^\Sigma\,w').$$

   (Last step because $w''\,R_b^\Sigma\,w'$ iff $\forall A\,[\neg\mathsf{K}_b\neg A \in w' \;\Rightarrow\; A \in w'']$ is the definition of $w''\,R_b^\Sigma\,w'$ in terms of the dual of K_b.)

3. Suppose $u\,R_a^\Sigma\,w$, $w\,R_a^\Sigma\,w'$ and $w'\,R_b^\Sigma\,w''$.

   We need to show $w\,R_b^\Sigma\,w''$, i.e., $\forall A\,[\mathsf{K}_b A \in w \Rightarrow A \in w'']$.

   Suppose $\mathsf{K}_b A \in w$. Show $A \in w''$.

   First, we have: $\mathsf{K}_a(\mathsf{K}_b A \longrightarrow \mathsf{K}_a\mathsf{K}_b A) \in u$ (axiom, and u is a maxi-consistent set).

   $$\mathsf{K}_a(\mathsf{K}_b A \longrightarrow \mathsf{K}_a\mathsf{K}_b A) \in u \;\Rightarrow\; \mathsf{K}_b A \longrightarrow \mathsf{K}_a\mathsf{K}_b A \in w \quad (u\,R_a^\Sigma\,w)$$

   $\mathsf{K}_b A \in w$ (assumption) and $\mathsf{K}_b A \longrightarrow \mathsf{K}_a\mathsf{K}_b A \in w$ imply $\mathsf{K}_a\mathsf{K}_b A \in w$ (w is a maxi-consistent set, so closed under modus ponens).

   $$\mathsf{K}_a\mathsf{K}_b A \in w \;\Rightarrow\; \mathsf{K}_b A \in w' \quad (w\,R_a^\Sigma\,w')$$
   $$\mathsf{K}_b A \in w' \;\Rightarrow\; A \in w'' \quad (w'\,R_b^\Sigma\,w'')$$

4. *Soundness* We have to show that schemas T ($\Box A \longrightarrow A$) and 5 ($\Diamond A \longrightarrow \Box\Diamond A$) are both valid in the class of equivalence (reflexive, symmetric, transitive) frames. I omit T and reflexive: you've seen it a thousand times.

   For 5: suppose $\mathcal{M}, w \models \Diamond A$ for some model $\mathcal{M}$ whose relation R is an equivalence relation. Then $\mathcal{M}, t \models A$ for some world t such that $w\,R\,t$. To show $\mathcal{M}, w \models \Box\Diamond A$ we have to show that $\mathcal{M}, u \models \Diamond A$ for every world u such that $w\,R\,u$. So suppose $w\,R\,u$. We have to show there exists a world v such that $u\,R\,v$ and $\mathcal{M}, v \models A$. We will do this by showing that $u\,R\,t$, as follows. R is symmetric, so from $w\,R\,u$ it follows that $u\,R\,w$. And R is transitive, so from $u\,R\,w$ and $w\,R\,t$ it follows that $u\,R\,t$, as required.

   Completeness We will use the canonical model method. We show that the canonical relation R^{KT5} is reflexive, symmetric, and transitive. The required completeness result then follows by the usual argument.

   There's various ways to do it. The easiest way (depending on your point of view) is to observe that T and 5 are canonical for reflexive and euclidean frames, respectively.

   (A relation R is euclidean if, for all $w, w'w''$ we have $w\,R\,w'$ and $w\,R\,w''$ implies $w'\,R\,w''$.

   Now observe that *any* reflexive and euclidean relation is symmetric and transitive (and hence an equivalence relation).

   So since R^{KT5} is reflexive and euclidean, it must also be symmetric and transitive. Done.

   The proof that T is canonical for reflexive relations is in the lecture notes (and is very easy). I thought that the proof that 5 is canonical for euclidean relations was also in the notes but I see that it is not. Here is a proof that, for any normal logic Σ containing the schema 5, the canonical relation R^Σ is euclidean.

   Suppose $w\,R^\Sigma\,w'$ and $w\,R^\Sigma\,w''$. We show $w'\,R^\Sigma\,w''$, i.e., that $\forall A\,[A \in w'' \Rightarrow \Diamond A \in w']$.

   Suppose $A \in w''$. We show $\Diamond A \in w'$.

   $$A \in w'' \;\Rightarrow\; \Diamond A \in w \quad (w\,R^\Sigma\,w'')$$
   $$\Diamond A \in w \;\Rightarrow\; \Box\Diamond A \in w \quad \text{(schema 5, and } w \text{ is a maxi-consistent set of } \Sigma)$$
   $$\Box\Diamond A \in w \;\Rightarrow\; \Diamond A \in w' \quad (w\,R^\Sigma\,w')$$

   Back to $KT5$. We know that R^{KT5} is reflexive. Another way to show it is symmetric and transitive is to observe that we can derive (syntactically) schemas B and 4 in $KT5$ (which is an exercise on tutorial sheet 1). Since B and 4 are canonical for symmetry and transitivity respectively (in the lecture notes) that does it.

5. Suppose the canonical relation R^{S5} is universal.

Any atom, say p, is S5-consistent. By Lindenbaum's lemma there must be some maxi-consistent set, say w, such that $p \in w$.

Now if R^{S5} were universal, $p \in w$ implies $\Diamond p \in w'$ for all maxi-consistent sets w'. And this means that $\vdash_{\mathrm{S5}} \Diamond p$.

So if R^{S5} were universal, we would have $\vdash_{\mathrm{S5}} \Diamond p$ for all atoms p, which is obviously not the case.

Or, similarly: $\{\Box p\}$ is S5-consistent, so by Lindenbaum's lemma there must be a maxi-consistent set, say w, such that $\Box p \in w$. If R^{S5} were universal, $\Box p \in w$ would imply $p \in w'$ for all maxi-consistent sets w'. And this would mean that $\vdash_{\mathrm{S5}} p$.

6. We show that the canonical relation

$$w\, R^{KT4G}\, w' \Leftrightarrow \{\, A \mid \Box A \in w \,\} \subseteq w'$$

for S4.2=$KT4G$ is reflexive, transitive, and strongly convergent. The arguments for reflexive and transitive were done in the lecture notes, and in a earlier question on this sheet.

For strongly convergent we need to show that for all $KT4G$-maxi-consistent sets w, w' there exists a $KT4G$-maxi-consistent set v such that:

$$\{\, A \mid \Box A \in w \,\} \subseteq v \text{ and } \{\, A \mid \Box A \in w' \,\} \subseteq v$$

This is equivalent to showing that there exists a $KT4G$-maxi-consistent set v such that:

$$\{\, A \mid \Box A \in w \,\} \cup \{\, A \mid \Box A \in w' \,\} \subseteq v$$

Since v is a $KT4G$-maxi-consistent set, by Lindenbaum's lemma it is sufficient to show that

$$\{\, A \mid \Box A \in w \,\} \cup \{\, A \mid \Box A \in w' \,\}$$

is $KT4G$-consistent. This is given in the question, since w and w' are $KT4G$-maxi-consistent sets.

7. We show that the canonical relation

$$w\, R^{KT4G}\, w' \Leftrightarrow \{\, A \mid \Box A \in w \,\} \subseteq w'$$

for S4.2=$KT4G$ is 'incestual'/'Church-Rosser'.

Suppose $u\, R^{KT4G}\, w$ and $u\, R^{KT4G}\, w'$. We need to show that there exists a v such that $w\, R^{KT4G}\, v$ and $w'\, R^{KT4G}\, v$. By the same argument as in the previous question, it is sufficient to show that

$$\{\, A \mid \Box A \in w \,\} \cup \{\, B \mid \Box B \in w' \,\}$$

is $KT4G$-consistent.

Suppose it is not. Then

$$\vdash_{KT4G} A_1 \wedge \cdots \wedge A_m \wedge B_1 \wedge \cdots \wedge B_n \rightarrow \bot \quad \text{and hence} \quad \vdash_{KT4G} A_1 \wedge \cdots \wedge A_m \rightarrow \neg(B_1 \wedge \cdots \wedge B_n)$$

for some $\{\Box A_1, \ldots, \Box A_m\} \subseteq w$ and $\{\Box B_1, \ldots, \Box B_n\} \subseteq w'$. By the rule RK (which we have in any normal system), we get

$$\vdash_{KT4G} \Box A_1 \wedge \cdots \wedge \Box A_m \rightarrow \Box\neg(B_1 \wedge \cdots \wedge B_n)$$

Now $\{\Box A_1, \ldots, \Box A_m\} \subseteq w$ and w is a maxi-consistent set of $KT4G$, so $\Box\neg(B_1 \wedge \cdots \wedge B_n) \in w$.

- $\Box\neg(B_1 \wedge \cdots \wedge B_n) \in w \;\Rightarrow\; \Diamond\Box\neg(B_1 \wedge \cdots \wedge B_n) \in u$ ($u\, R^{KT4G}\, w$ by assumption)
- $\Diamond\Box\neg(B_1 \wedge \cdots \wedge B_n) \in u \;\Rightarrow\; \Box\Diamond\neg(B_1 \wedge \cdots \wedge B_n) \in u$ (axiom G, and u is a maxi-consistent set of $KT4G$)
- $\Box\Diamond\neg(B_1 \wedge \cdots \wedge B_n) \in u \;\Rightarrow\; \Diamond\neg(B_1 \wedge \cdots \wedge B_n) \in w'$ ($u\, R^{KT4G}\, w'$ by assumption)
- $\Diamond\neg(B_1 \wedge \cdots \wedge B_n) \in w' \;\Rightarrow\; \neg\Box(B_1 \wedge \cdots \wedge B_n) \in w'$ (w' is a maxi-consistent set of $KT4G$)

So now we have $\{\Box B_1, \ldots, \Box B_n\} \subseteq w'$ and $\neg\Box(B_1 \wedge \cdots \wedge B_n) \in w'$, which obviously contradicts the assumption that w' is a $KT4G$ maxi-consistent set.