Tutorial Exercises 4 (mjs)

SOLUTIONS

1. The example of ν-model in the notes contains a counterexample to validity of each schema. Here are some more (simpler). All we have to do in each case is devise one model with one world w at which one instance of that schema is false. Let $\mathcal{M} = \langle W, \nu, h \rangle$ be the model in each case.

   Contra M: Let $W = \{w_1, w_2\}$, $\nu(w_1) = \{\emptyset\}$ ($\nu(w_2)$ does not matter), and $h(p) = \{w_1\}$, $h(q) = \{w_2\}$. (You might be worrying about $\nu(w_1) = \{\emptyset\}$. Why? It just means $\Box\bot$ is true at w_1.) So in this model $\|p \wedge q\|^{\mathcal{M}} = \|p\|^{\mathcal{M}} \cap \|q\|^{\mathcal{M}} = \emptyset$, and $\|p \wedge q\|^{\mathcal{M}} \in \nu(w_1)$ but $\|p\|^{\mathcal{M}} \notin \nu(w_1)$ and $\|q\|^{\mathcal{M}} \notin \nu(w_1)$. So $\mathcal{M}, w_1 \models \Box(p \wedge q)$ but $\mathcal{M}, w_1 \not\models \Box p$ and $\mathcal{M}, w_1 \not\models \Box q$.

   Contra C: Let $W = \{w_1, w_2\}$, $h(p) = \{w_1\}$, and $h(q) = \{w_2\}$ as above, but now $\nu(w_1) = \{\{w_1\}, \{w_2\}\}$. Then $\|p\|^{\mathcal{M}} \in \nu(w_1)$, $\|q\|^{\mathcal{M}} \in \nu(w_1)$, but $\|p \wedge q\|^{\mathcal{M}} \notin \nu(w_1)$.

   Contra N: Very easy. Let $W = \{w_1\}$, $\nu(w_1) = \emptyset$. The valuation h does not matter. Now $\|\top\|^{\mathcal{M}} = W \notin \nu(w_1)$ so $\mathcal{M}, w_1 \not\models \Box\top$.

2. The proofs are in the lecture notes.

3. We have to show that the following conditions on ν-models are equivalent.

   (m) if $X \cap Y \in \nu(w)$ then $X \in \nu(w)$ and $Y \in \nu(w)$
   (rm) if $X \subseteq Y$ then $X \in \nu(w) \Rightarrow Y \in \nu(w)$.

   Suppose (m) holds. Suppose $X \subseteq Y$, and $X \in \nu(w)$. $X \subseteq Y$ means that $X \cap Y = X$. So $X \in \nu(w)$ implies $X \cap Y \in \nu(w)$, and by condition (m) this implies $Y \in \nu(w)$, which is what we had to prove to show (rm) holds.

   Suppose (rm) holds. Suppose $X \cap Y \in \nu(w)$. Then since $X \cap Y \subseteq X$, by (rm) $X \in \nu(w)$. And similarly $X \cap Y \subseteq Y$ implies $Y \in \nu(w)$ by (rm).

   (These proofs are even shorter if you employ the f notation in place of ν. Try it.)

4. $$\begin{aligned} \nu(w) \neq \emptyset &\Rightarrow X \in \nu(w) \quad \text{for some } X \subseteq W \\ &\Rightarrow W \in \nu(w) \quad \text{by (rm) and } X \subseteq W \\ W \in \nu(w) &\Rightarrow \nu(w) \neq \emptyset, \quad \text{trivially.} \end{aligned}$$

5. Expressed in terms of the function f where $w \in f(X) \Leftrightarrow X \in \nu(w)$ the model conditions are as follows.

   (m_f) $f(X \cap Y) \subseteq f(X) \cap f(Y)$
   (rm_f) $X \subseteq Y \Rightarrow f(X) \subseteq f(Y)$
   (c_f) $f(X) \cap f(Y) \subseteq f(X \cap Y)$
   (n_f) $f(W) = W$

To get these just apply the definition of f and simplify if necessary. For example (m): for all $w \in W$, and all $X, Y \subseteq W$:

$$\begin{aligned} &X \cap Y \in \nu(w) \Rightarrow X \in \nu(w) \;\&\; Y \in \nu(w) \\ &w \in f(X \cap Y) \Rightarrow w \in f(X) \;\&\; w \in f(Y) \\ &w \in f(X \cap Y) \Rightarrow w \in f(X) \cap f(Y) \\ &f(X \cap Y) \subseteq f(X) \cap f(Y) \end{aligned}$$

One more example (n), because it is a little different. (n) says $w \in \nu(W)$ for all w in W. So $w \in f(W)$ for all w in W. Strictly, this just shows $W \subseteq f(W)$. But we already have $f(W) \subseteq W$ because f is a function $W \to \wp(W)$ (the value of every $f(w)$ is a subset of W). So therefore $f(W) = W$, as required.

6. First we show that *EP* $\neq$ *ED* by constructing a model in which P is true somewhere but not D, and then another in which D is true somewhere but not P. (Conditions (p) and (d) identified elsewhere tell us what kind of model to look for.)

   Consider $W = \{w_1, w_2\}$ with $\nu(w_1) = \{\{w_1\}, \{w_2\}\}$ and $\nu(w_2) = \{\{w_2\}\}$. This does the trick. Suppose, for instance, that $h(p) = \{w_1\}$. Then $\mathcal{M}, w_1 \models \Box p \wedge \Box\neg p$, whereas $\mathcal{M}, w_1 \not\models \Box\bot$, i.e., $\mathcal{M}, w_1 \models \neg\Box\bot$.

   Now consider $W = \{w_1\}$, with $\nu(w_1) = \{\emptyset\}$. Now $\mathcal{M}, w_1 \models \Box\bot$. But take $h(p) = \{w_1\}$ again. We have $\mathcal{M}, w_1 \models \Box p \rightarrow \neg\Box\neg p$ (trivially, because $\mathcal{M}, w \not\models \Box p$).

   Now show that every classical *EMD* system contains P but not every *EMP* system contains D. First the second part: we need a model that has M and P true at some world w but does not have D true at w. $W = \{w_1\}$ with $\nu(w_1) = \{\{w_1\}\}$ and $h(p) = \{w_1\}$ does it.

   For the other part, show that P can be derived in *EMD*. Here is one derivation:

1.	$\vdash_{EMD}$	$\bot \rightarrow \top$	*PL*
2.	$\vdash_{EMD}$	$\Box\bot \rightarrow \Box\top$	1, RM
3.	$\vdash_{EMD}$	$\Box\top \rightarrow \neg\Box\neg\top$	D
4.	$\vdash_{EMD}$	$\Box\top \rightarrow \neg\Box\bot$	3, RE (because $(\neg\top \leftrightarrow \bot) \in PL$)
5.	$\vdash_{EMD}$	$\Box\bot \rightarrow \neg\Box\bot$	2, 4, RPL
6.	$\vdash_{EMD}$	$\neg\Box\bot$	5, RPL (because $((A \rightarrow \neg A) \rightarrow \neg A) \in PL$)

   Finally, to show that every *normal* system contains D if and only if it contains P. We already have left-to-right from the previous part: a normal system contains M and so if it also contains D then the previous derivation of P works as well. In other words, if P is shown to be in *EMD*, then it must also be in *EMCND* = *KD*. It just remains to show how D can be derived in *EMCNP*. In fact, *ECP* will do: observe that $(\Box A \wedge \Box\neg A) \rightarrow \Box(A \wedge \neg A)$ is an instance of C. D follows immediately. For the record, here is the derivation in full detail:

1.	$\vdash_{ECP}$	$(A \wedge \neg A) \leftrightarrow \bot$	*PL*
2.	$\vdash_{ECP}$	$\Box(A \wedge \neg A) \leftrightarrow \Box\bot$	1, RE
3.	$\vdash_{ECP}$	$\neg\Box\bot$	P
4.	$\vdash_{ECP}$	$\neg\Box(A \wedge \neg A)$	2, 3, RPL
5.	$\vdash_{ECP}$	$(\Box A \wedge \Box\neg A) \longrightarrow \Box(A \wedge \neg A)$	C
6.	$\vdash_{ECP}$	$\neg(\Box A \wedge \Box\neg A)$	4, 5, RPL
7.	$\vdash_{ECP}$	$\Box A \longrightarrow \neg\Box\neg A$	6, RPL

7. The proofs for model conditions (p), (d), (t), (b) are given in the lecture notes. So here we show that model conditions

   (iv) $X \in \nu(w) \;\Rightarrow\; \{w' \in W : X \in \nu(w')\} \in \nu(w)$
   (v) $X \notin \nu(w) \;\Rightarrow\; \{w' \in W : X \notin \nu(w')\} \in \nu(w)$

   validate the schemas

   4. $\Box A \longrightarrow \Box\Box A$
   5. $\Diamond A \longrightarrow \Box\Diamond A$

   respectively. As with all the other proofs of soundness, you just apply the truth conditions, use the condition (iv) or (v) (as the case may be), and then apply the truth conditions again. It can all be done very much more concisely using the f notation. Here are deriviations in terms of ν, in full detail:

$$\begin{aligned}
\mathcal{M}, w \models \Box A &\Rightarrow \|A\|^{\mathcal{M}} \in \nu(w) \\
&\Rightarrow \{w' \in W : \|A\|^{\mathcal{M}} \in \nu(w')\} \in \nu(w) \quad \text{(condition (iv))} \\
&\Rightarrow \{w' \in W : \mathcal{M}, w' \models \Box A\} \in \nu(w) \\
&\Rightarrow \|\Box A\|^{\mathcal{M}} \in \nu(w) \\
&\Rightarrow \mathcal{M}, w \models \Box\Box A
\end{aligned}$$

   And now for 5:

$$\begin{aligned}
\mathcal{M}, w \models \Diamond A &\Rightarrow \mathcal{M}, w \models \neg\Box\neg A \\
&\Rightarrow w \notin \|\Box\neg A\|^{\mathcal{M}} \\
&\Rightarrow \|\neg A\|^{\mathcal{M}} \notin \nu(w) \\
&\Rightarrow \{w' \in W : \|\neg A\|^{\mathcal{M}} \notin \nu(w')\} \in \nu(w) \quad \text{(condition (v))} \\
&\Rightarrow \{w' \in W : w' \notin \|\Box\neg A\|^{\mathcal{M}}\} \in \nu(w) \\
&\Rightarrow \{w' \in W : w' \in \|\neg\Box\neg A\|^{\mathcal{M}}\} \in \nu(w) \\
&\Rightarrow \|\neg\Box\neg A\|^{\mathcal{M}} \in \nu(w) \\
&\Rightarrow \|\Diamond A\|^{\mathcal{M}} \in \nu(w) \\
&\Rightarrow \mathcal{M}, w \models \Box\Diamond A
\end{aligned}$$

8. In the lecture notes. Here they are again for ease of reference. To derive them, just apply the definition of f and then simplify.

   (p_f) $f(\emptyset) = \emptyset$
   (d_f) $f(X) \subseteq W - f(W - X)$
   (t_f) $f(X) \subseteq X$
   (b_f) $X \subseteq f(W - f(W - X))$
   (iv_f) $f(X) \subseteq f(f(X))$
   (v_f) $W - f(X) \subseteq f(W - f(X))$

   In regard to the last two, note that $\{w' \in W : X \in \nu(w')\}$ is just $f(X)$.
   And $\{w' \in W : X \notin \nu(w')\}$ is just $W - f(X)$. So the model conditions validating 4 and 5 can also be expressed as follows:

   (iv′) $X \in \nu(w) \Rightarrow f(X) \in \nu(w)$
   (v′) $X \notin \nu(w) \Rightarrow (W - f(X)) \in \nu(w)$

   This simplifies the derivations (soundness of 4 and 5) in the previous question.

9. (g_f) $W - f(W - f(X)) \subseteq f(W - f(W - X))$

   Check that it validates G: $\Diamond\Box A \longrightarrow \Box\Diamond A$:

$$\begin{aligned}
\mathcal{M}, w \models \Diamond\Box A &\Rightarrow w \in \|\Diamond\Box A\|^{\mathcal{M}} \\
&\Rightarrow w \in (W - f(W - \|\Box A\|^{\mathcal{M}})) \\
&\Rightarrow w \in (W - f(W - f(\|A\|^{\mathcal{M}}))) \\
&\Rightarrow w \in f(W - f(W - \|A\|^{\mathcal{M}})) \quad \text{(condition } (\mathrm{g}_f)) \\
&\Rightarrow w \in f(W - f(\|\neg A\|^{\mathcal{M}})) \\
&\Rightarrow w \in f(W - \|\Box\neg A\|^{\mathcal{M}}) \\
&\Rightarrow w \in f(\|\neg\Box\neg A\|^{\mathcal{M}}) \\
&\Rightarrow w \in \|\Box\neg\Box\neg A\|^{\mathcal{M}} \\
&\Rightarrow \mathcal{M}, w \models \Box\neg\Box\neg A \\
&\Rightarrow \mathcal{M}, w \models \Box\Diamond A
\end{aligned}$$