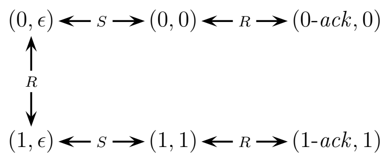
499 Modal and Temporal Logic *Autumn 2007*

Tutorial Exercises 5 (mjs)

SOLUTIONS

Question 1



1. $IS_{\mathrm{b}} \models \textbf{recbit} \rightarrow (\mathsf{K}_R\,(\textbf{bit=0}) \vee \mathsf{K}_R\,(\textbf{bit=1}))$

   recbit is true at global states $(0,0)$, $(1,1)$, $(0\text{-}ack,0)$, $(1\text{-}ack,1)$. So we need to check that $(\mathsf{K}_R\,(\textbf{bit=0}) \vee \mathsf{K}_R\,(\textbf{bit=1}))$ is true at each of these global states.

   To show $IS_{\mathrm{b}},(0,0) \models (\mathsf{K}_R\,(\textbf{bit=0}) \vee \mathsf{K}_R\,(\textbf{bit=1}))$: the global states R-accessible from $(0,0)$ are $(0,0)$ and $(0\text{-}ack,0)$. (Or if you prefer, the global states that are indistinguishable for R from $(0,0)$ are $(0,0)$ and $(0\text{-}ack,0)$.) In both of these global states, **recbit** is true, and so $IS_{\mathrm{b}},(0,0) \models \mathsf{K}_R\,(\textbf{bit=0})$ and $IS_{\mathrm{b}},(0,0) \models (\mathsf{K}_R\,(\textbf{bit=0}) \vee \mathsf{K}_R\,(\textbf{bit=1}))$.

   Now check the calculation for each of the remaining three global states $(1,1)$, $(0\text{-}ack,0)$, $(1\text{-}ack,1)$.

2. $IS_{\mathrm{b}} \models \textbf{recack} \rightarrow \textbf{recbit}$

   recack is true at global states $(0\text{-}ack,0)$, $(1\text{-}ack,1)$. So we need to check that **recbit** is true at each of these global states, which it clearly is.

3. $IS_{\mathrm{b}} \models (\textbf{bit=0}) \rightarrow \mathsf{K}_S\,(\textbf{bit=0})$

   By direct evaluation in the model again. $(\textbf{bit=0})$ is true at global states $(0,\epsilon)$, $(0,0)$, $(0\text{-}ack,0)$. We need to check that $\mathsf{K}_S\,(\textbf{bit=0})$ is true at each of these global states.

   For $(0,\epsilon)$, the S-accessible (or S-indistinguishable) global states are $(0,\epsilon)$ and $(0,0)$. Clearly $(\textbf{bit} = \textbf{0})$ is true at both of these global states.

   For $(0,0)$, the S-accessible (or S-indistinguishable) global states are again $(0,\epsilon)$ and $(0,0)$. Clearly $(\textbf{bit} = \textbf{0})$ is true at both of these global states (we just checked this).

   For $(0\text{-}ack,0)$, the only S-accessible (or S-indistinguishable) global state is $(0\text{-}ack,0)$ itself. Clearly $(\textbf{bit=0})$ is true at this global state.

4. $IS_{\mathrm{b}} \models \textbf{recack} \rightarrow \mathsf{K}_S\,\textbf{recack}$

   As in the previous part, evaluate the formula at every global state in the model. **recack** is true at global states $(0\text{-}ack,0)$ and $(1\text{-}ack,1)$. The only global state S-accessible from $(0\text{-}ack,0)$ is $(0\text{-}ack,0)$ itself, and here **recack** is true. Similarly for $(1\text{-}ack,1)$.

5. $IS_{\mathrm{b}} \not\models \textbf{recack} \rightarrow \mathsf{K}_R\,\textbf{recack}$

   We just need to find one global state at which **recack** is true but $\mathsf{K}_R\,\textbf{recack}$ is false. Consider global state $(0\text{-}ack,0)$: we show $\mathsf{K}_R\,\textbf{recack}$ is false at this global state. The global states R-accessible from $(0\text{-}ack,0)$ are $(0\text{-}ack,0)$ and $(0,0)$. But **recack** is false at $(0,0)$ and so $\mathsf{K}_R\,\textbf{recack}$ is false at $(0\text{-}ack,0)$.

6. $IS_{\mathrm{b}} \models \textbf{recbit} \wedge (\textbf{bit=0}) \rightarrow \mathsf{K}_R\,(\textbf{bit=0})$

   $\textbf{recbit} \wedge (\textbf{bit=0})$ is true at global states $(0,0)$ and $(0\text{-}ack,0)$. The global states R-accessible from $(0,0)$ are $(0,0)$ and $(0\text{-}ack,0)$; the global states R-accessible from $(0,0)$ are also $(0,0)$ and $(0\text{-}ack,0)$. And clearly $(\textbf{bit=0})$ is true at both of those global states.

7. $IS_{\mathrm{b}} \models \textbf{recack} \rightarrow (\mathsf{K}_R\,(\textbf{bit=0}) \vee \mathsf{K}_R\,(\textbf{bit=1}))$

   One can evaluate this directly at each global state in the model. But it is much quicker to notice that it follows immediately by propositional logic from parts (1) and (2).

8. $IS_{\mathrm{b}} \models \textbf{recack} \rightarrow \mathsf{K}_S(\mathsf{K}_R\,(\textbf{bit=0}) \vee \mathsf{K}_R\,(\textbf{bit=1}))$

   Look at the previous part: $IS_{\mathrm{b}} \models \mathsf{K}_S\,\textbf{recack} \rightarrow \mathsf{K}_S(\mathsf{K}_R\,(\textbf{bit=0}) \vee \mathsf{K}_R\,(\textbf{bit=1}))$ follows from it by rule RM. (If $A \rightarrow B$ is valid in model IS_{b} then $\mathsf{K}_S\,A \rightarrow \mathsf{K}_S\,B$ is also valid in model IS_{b}, because that holds for all relational ('Kripke') models, of which IS_{b} is one.)

   Now combine this with part (4) and propositional logic.

9. $IS_{\mathrm{b}} \models \textbf{recack} \wedge (\textbf{bit=0}) \rightarrow \mathsf{K}_S\,\mathsf{K}_R\,(\textbf{bit=0})$

   Parts (2) and (6) by propositional logic give $IS_{\mathrm{b}} \models \textbf{recack} \wedge (\textbf{bit=0}) \rightarrow \mathsf{K}_R\,(\textbf{bit=0})$. By rule RK ($n = 2$) we get $IS_{\mathrm{b}} \models \mathsf{K}_S\,\textbf{recack} \wedge \mathsf{K}_S\,(\textbf{bit=0}) \rightarrow \mathsf{K}_S\,\mathsf{K}_R\,(\textbf{bit=0})$.

   Now parts (4) and (3) by propositional logic give $IS_{\mathrm{b}} \models \textbf{recack} \wedge (\textbf{bit=0}) \rightarrow \mathsf{K}_S\,\textbf{recack} \wedge \mathsf{K}_S\,(\textbf{bit=0})$.

   Propositional logic again gives $IS_{\mathrm{b}} \models \textbf{recack} \wedge (\textbf{bit=0}) \rightarrow \mathsf{K}_S\,\mathsf{K}_R\,(\textbf{bit=0})$.

10. $IS_{\mathrm{b}} \not\models \textbf{recack} \wedge (\textbf{bit=0}) \rightarrow \mathsf{K}_R\,\mathsf{K}_S\,\mathsf{K}_R\,(\textbf{bit} = \textbf{0})$

    $\textbf{recack} \wedge (\textbf{bit=0})$ is true at global state $(0\text{-}ack,0)$ but $IS_{\mathrm{b}},(0\text{-}ack,0) \not\models \mathsf{K}_R\,\mathsf{K}_S\,\mathsf{K}_R\,(\textbf{bit} = \textbf{0})$. Because: the global states R-accessible from $(0\text{-}ack,0)$ are $(0,0)$ and $(0\text{-}ack,0)$, and $\mathsf{K}_S\,\mathsf{K}_R\,(\textbf{bit=0})$ is false at $(0,0)$. This is because the global states S-accessible from $(0,0)$ are $(0,0)$ and $(0,\epsilon)$, and $\mathsf{K}_R\,(\textbf{bit=0})$ is false at $(0,\epsilon)$. And this is because the global states R-accessible from $(0,\epsilon)$ are $(0,\epsilon)$ and $(1,\epsilon)$, and clearly $(\textbf{bit=0})$ is false at $(1,\epsilon)$.

Question 2

Validity of T and 4 in reflexive and transitive frames, respectively, has been shown in lecture notes and previous tutorial exercises. I'll repeat a proof for validity of 4 ($\Box A \longrightarrow \Box\Box A$) in transitive frames for your convenience.

Let $\mathcal{M} = \langle W, R, h \rangle$ be any model in which R is transitive, and let w be any world in W.
Suppose $\mathcal{M}, w \models \Box A$. We have to show $\mathcal{M}, w \models \Box\Box A$.
So consider any world w' such that $w\,R\,w'$. We need to show $\mathcal{M}, w' \models \Box A$.
So consider any world w'' such that $w'\,R\,w''$. We need to show $\mathcal{M}, w'' \models A$.
We have $w\,R\,w'$ and $w'\,R\,w''$, and so also $w\,R\,w''$ because R is transitive. Since $\mathcal{M}, w \models \Box A$, and $w\,R\,w''$, we have $\mathcal{M}, w'' \models A$, as required.

Now validity of 5 ($\Diamond A \longrightarrow \Box\Diamond A$) in symmetric transitive frames.
Let $\mathcal{M} = \langle W, R, h \rangle$ be any model in which R is both symmetric and transitive, and let w be any world in W.
Suppose $\mathcal{M}, w \models \Diamond A$. We have to show $\mathcal{M}, w \models \Box\Diamond A$.
So consider any world w' such that $w\,R\,w'$. We need to show $\mathcal{M}, w' \models \Diamond A$.
¿From $\mathcal{M}, w \models \Diamond A$, we know $\mathcal{M}, w'' \models A$ for some w'' such that $w\,R\,w''$.
We have $w\,R\,w'$ and so also $w'\,R\,w$ because R is symmetric.
But $w'\,R\,w$ and $w\,R\,w''$ implies $w'\,R\,w''$ (R is transitive).
So now we have $w'\,R\,w''$ and $\mathcal{M}, w'' \models A$, and so $\mathcal{M}, w' \models \Diamond A$ as required.

(It is easier to see the argument if you draw a picture.)

Question 3

Solution omitted from this version as Question 3 might be part of the assessed coursework.