Imperial College
London

REVISION NOTES

IMPERIAL COLLEGE LONDON

DEPARTMENT OF COMPUTING

Complex Numbers

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Version: August 1, 2017

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Chapter 1

Complex Numbers

1.1 Introduction

We can see need for complex numbers by looking at the shortcomings of all the simpler (more obvious) number systems that preceded them. In each case the next number system in some sense fixes a perceived problem or omission with the previous one:

$\mathbb{N}$ Natural numbers, for counting, not closed under subtraction

$\mathbb{Z}$ Integers, the natural numbers with 0 and negative numbers, not closed under division

$\mathbb{Q}$ Rational numbers, closed under arithmetic operations but cannot represent the solution of all non-linear equations, e.g., $x^2 = 2$

$\mathbb{R}$ Real numbers, solutions to some quadratic equations with real roots and some higher-order equations, but not all, e.g., $x^2 + 1 = 0$

$\mathbb{C}$ Complex numbers, we require these to represent **all** the roots of all polynomial equations.[1]

Another important use of complex numbers is that often a real problem can be solved by mapping it into complex space, deriving a solution, and mapping back again: a direct solution may not be possible or would be much harder to derive in real space, e.g., finding solutions to integration or summation problems, such as

$$I = \int_0^x e^{a\theta} \cos b\theta \,\mathrm{d}\theta \qquad \text{or} \qquad S = \sum_{k=0}^{n} a^k \cos k\theta \,. \tag{1.1}$$

1.1.1 Applications

Complex numbers are important in many areas. Here are some:

[1] Complex numbers form an algebraically closed field, where any polynomial equation has a root.

- Signal analysis (e.g., Fourier transformation to analyze varying voltages and currents)
- Control theory (e.g., Laplace transformation from time to frequency domain)
- Quantum mechanics is founded on complex numbers (see Schrödinger equation and Heisenberg's matrix mechanics)
- Cryptography (e.g., finding prime numbers).
- Machine learning: Using a pair of uniformly distributed random numbers (x, y), we can generate random numbers in polar form $(r\cos(\theta), r\sin(\theta))$. This can lead to efficient sampling methods like the Box-Muller transform (Box and Muller, 1958).[2] The variant of the Box-Muller transform using complex numbers was proposed by Knop (1969).
- (Tele)communication: digital coding modulations

1.1.2 Imaginary Number

An entity we cannot describe using real numbers are the roots to the equation

$$x^2 + 1 = 0, \tag{1.2}$$

which we will call i and define as

$$i := \sqrt{-1}. \tag{1.3}$$

There is no way of squeezing this into $\mathbb{R}$, it cannot be compared with a real number (in contrast to $\sqrt{2}$ or π, which we can compare with rationals and get arbitrarily accurate approximations in the rationals). We call i the **imaginary number/unit**, orthogonal to the reals.

Properties From the definition of i in (1.3) we get a number of properties for i.

1. $i^2 = -1, \quad i^3 = i^2 i = -i, \quad i^4 = (i^2)^2 = (-1)^2 = 1$ and so on
2. In general $i^{2n} = (i^2)^n = (-1)^n, \quad i^{2n+1} = i^{2n} i = (-1)^n i$ for all $n \in \mathbb{N}$
3. $i^{-1} = \frac{1}{i} = \frac{i}{i^2} = -i$
4. In general $i^{-2n} = \frac{1}{i^{2n}} = \frac{1}{(-1)^n} = (-1)^n, \quad i^{-(2n+1)} = i^{-2n} i^{-1} = (-1)^{n+1} i$ for all $n \in \mathbb{N}$
5. $i^0 = 1$

[2] This is a pseudo-random number sampling method, e.g., for generating pairs of independent, standard, normally distributed (zero mean, unit variance) random numbers, given a source of uniformly distributed random numbers.

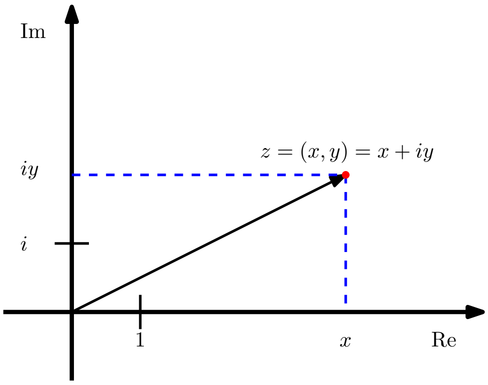


Figure 1.1: Complex plane (Argand diagram). A complex number can be represented in a two-dimensional Cartesian coordinate system with coordinates x and y. x is the real part and y is the imaginary part of a complex number $z = x + iy$.

1.1.3 Complex Numbers as Elements of $\mathbb{R}^2$

It is convenient (and correct[3]) to consider complex numbers

$$\mathbb{C} := \{a + ib : a, b \in \mathbb{R}, i^2 = -1\} \tag{1.4}$$

as the set of tuples $(a, b) \in \mathbb{R}^2$ with the following definition of addition and multiplication:

$$(a, b) + (c, d) = (a + c, b + d), \tag{1.5}$$
$$(a, b) \cdot (c, d) = (ac - bd, ad + bc). \tag{1.6}$$

In this context, the element $i := (0, 1)$ is the **imaginary number/unit**. With the complex multiplication defined in (1.6), we immediately obtain

$$i^2 = (0, 1)^2 = (0, 1)(0, 1) = -1, \tag{1.7}$$

which allows us to factorize the polynomial $z^2 + 1$ fully into $(z - i)(z + i)$.
Since elements of $\mathbb{R}^2$ can be drawn in a plane, we can do the same with complex numbers $z \in \mathbb{C}$. The plane is called **complex plane** or **Argand diagram**, see Figure 1.1.
The Argand diagram allows us to visualize addition and multiplication, which are defined in (1.5)–(1.6).

1.1.4 Closure under Arithmetic Operators

Closing $\mathbb{R} \cup \{i\}$ under the arithmetic operators $+, \cdot$ as defined in (1.5)–(1.6) gives the *complex numbers*, $\mathbb{C}$. To be more specific, if $z_1, z_2 \in \mathbb{C}$, then $z_1 + z_2 \in \mathbb{C}$, $z_1 - z_2 \in \mathbb{C}$, $z_1 \cdot z_2 \in \mathbb{C}$ and $z_1/z_2 \in \mathbb{C}$.

[3]There exists a bijective linear mapping (isomorphism) between $\mathbb{C}$ and $\mathbb{R}^2$. We will briefly discuss this in the Linear Algebra part of the course.

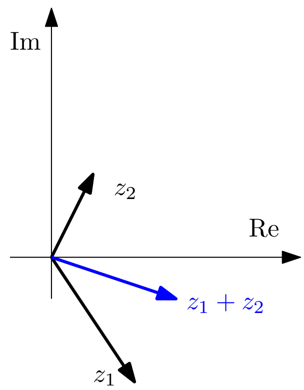


Figure 1.2: Visualization of complex addition. As known from geometry, we simply add the two vectors representing complex numbers.

1.2 Representations of Complex Numbers

In the following, we will discuss three important representations of complex numbers.

1.2.1 Cartesian Coordinates

Every element $z \in \mathbb{C}$ can be decomposed into

$$(x, y) = (x, 0) + (0, y) = (x, 0) + (0, 1)(y, 0) = \underbrace{(x, 0)}_{\in \mathbb{R}} + i \underbrace{(y, 0)}_{\in \mathbb{R}} = x + iy. \tag{1.8}$$

Therefore, every $z = x + iy \in \mathbb{C}$ has a **coordinate representation** (x, y), where x is called the **real part** and y is called the **imaginary part** of z, and we write $x = \mathfrak{Re}(z)$, $y = \mathfrak{Im}(z)$, respectively. $z = x + iy$ is the point (x, y) in the xy-plane (complex plane), which is uniquely determined by its Cartesian coordinates (x, y). An illustration is given in Figure 1.1.

1.2.2 Polar Coordinates

Equivalently, (x, y) can be represented by **polar coordinates**, r, ϕ, where r is the distance of z from the origin 0, and ϕ is the angle between the (positive) x-axis and the direction $\overrightarrow{0z}$. Then,

$$z = r(\cos\phi + i\sin\phi), \quad r \geq 0, \quad 0 \leq \phi < 2\pi \tag{1.9}$$

uniquely determines $z \in \mathbb{C}$. The polar coordinates of z are then

$$r = |z| = \sqrt{x^2 + y^2}, \tag{1.10}$$
$$\phi = \operatorname{Arg} z, \tag{1.11}$$

where r is the length of $\overrightarrow{0z}$ (the distance of z from the origin) and ϕ is the **argument** of z.

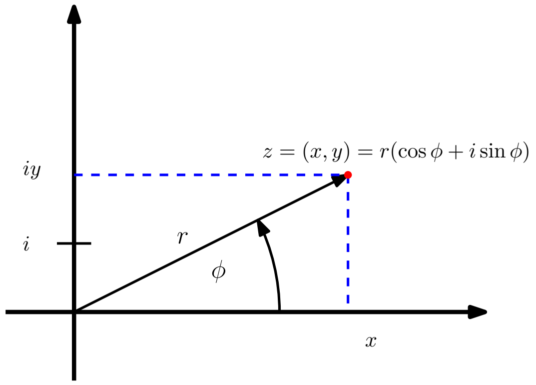


Figure 1.3: Polar coordinates.

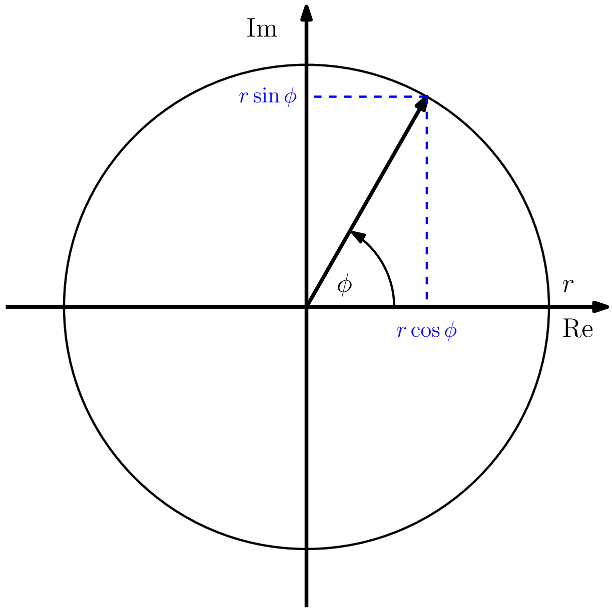


Figure 1.4: Euler representation. In the Euler representation, a complex number $z = r\exp(i\phi)$ "lives" on a circle with radius r around the origin. Therefore, $r\exp(i\phi) = r(\cos\phi + i\sin\phi)$.

1.2.3 Euler Representation

The third representation of complex numbers is the **Euler representation**

$$z = r\exp(i\phi) \tag{1.12}$$

where r and ϕ are the polar coordinates. We already know that $z = r(\cos\phi + i\sin\phi)$, i.e., it must also hold that $r\exp(i\phi) = r(\cos\phi + i\sin\phi)$. This can be proved by looking at the power series expansions of exp, sin, and cos:

$$\exp(i\phi) = \sum_{k=0}^{\infty} \frac{(i\phi)^k}{k!} = 1 + i\phi + \frac{(i\phi)^2}{2!} + \frac{(i\phi)^3}{3!} + \frac{(i\phi)^4}{4!} + \frac{(i\phi)^5}{5!} + \cdots \tag{1.13}$$

$$= 1 + i\phi - \frac{\phi^2}{2!} - \frac{i\phi^3}{3!} + \frac{\phi^4}{4!} + \frac{i\phi^5}{5!} \mp \cdots \tag{1.14}$$

$$= \left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} \mp \cdots\right) + i\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} \mp \cdots\right) \tag{1.15}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \phi^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k \phi^{2k+1}}{(2k+1)!} = \cos\phi + i \sin\phi\,. \tag{1.16}$$

Therefore, $z = \exp(i\phi)$ is a complex number, which lives on the unit circle ($|z| = 1$) and traces out the unit circle in the complex plane as ϕ ranges through the real numbers.

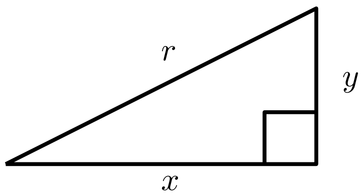
1.2.4 Transformation between Polar and Cartesian Coordinates

Cartesian coordinates $\longleftrightarrow$ **Polar coordinates**

x, y $\quad$ r, ϕ

$x = r\cos\phi$

$y = r\sin\phi$



$r = \sqrt{x^2 + y^2}$

$\tan\phi = \frac{y}{x}$ + quadrant

$$z = x + iy = r(\cos\phi + i\sin\phi)$$

Figure 1.5: Transformation between Cartesian and polar coordinate representations of complex numbers.

Figure 1.5 summarizes the transformation between Cartesian and polar coordinate representations of complex numbers z. We have to pay some attention when computing $\operatorname{Arg}(z)$ when transforming Cartesian coordinates into polar coordinates.

Example: Transformation from Polar to Cartesian Coordinates

Transform the polar representation $z = (r, \phi) = (2, \frac{2\pi}{3})$ into Cartesian coordinates (x, y).

It is always useful to draw the complex number. Figure 1.6(a) shows the setting. We are interested in the blue dots. With $x = r\cos\phi$ and $y = r\sin\phi$, we obtain

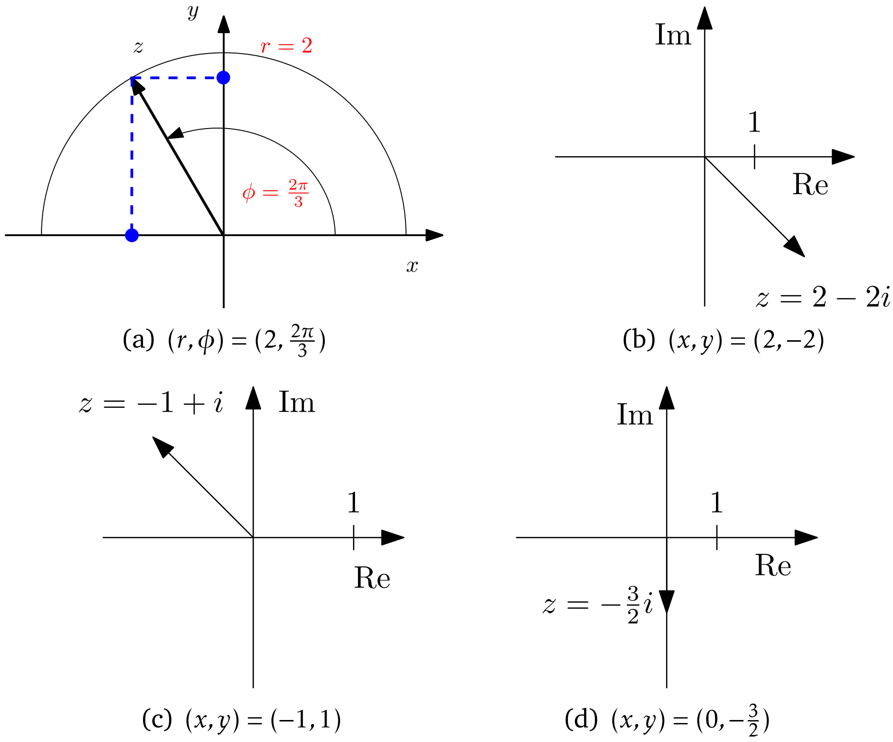
$$x = r\cos(\tfrac{2}{3}\pi) = -1 \tag{1.17}$$

$$y = r\sin(\tfrac{2}{3}\pi) = \sqrt{3}\,. \tag{1.18}$$

Therefore, $z = -1 + i\sqrt{3}$.

Example: Transformation from Cartesian to Polar Coordinates

Getting the Cartesian coordinates from polar coordinates is straightforward. The transformation from Cartesian to polar coordinates is somewhat more difficult because of the argument ϕ. The reason is that tan has a period of π, which means



(a) $(r, \phi) = (2, \frac{2\pi}{3})$ (b) $(x, y) = (2, -2)$

(c) $(x, y) = (-1, 1)$ (d) $(x, y) = (0, -\frac{3}{2})$

Figure 1.6: Coordinate transformations

that y/x has two possible angles, which differ by π, see Figure 1.7. By looking at the quadrant in which the complex number z lives we can resolve this ambiguity. Let us have a look at some examples:

1. $z = 2 - 2i$. We immediately obtain $r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$. For the argument, we obtain $\tan\phi = -\frac{2}{2} = -1$. Therefore, $\phi \in \{\frac{3}{4}\pi, \frac{7}{4}\pi\}$. We identify the correct argument by plotting the complex number and identifying the quadrant. Figure 1.6(b) shows that z lies in the fourth quadrant. Therefore, $\phi = \frac{7}{4}\pi$.

2. $z = -1 + i$.

$$r = \sqrt{1+1} = \sqrt{2} \tag{1.19}$$

$$\tan\phi = \frac{-1}{1} = -1 \quad \Rightarrow \phi \in \{\tfrac{3}{4}\pi, \tfrac{7}{4}\pi\}. \tag{1.20}$$

   Figure 1.6(c) shows that z lies in the second quadrant. Therefore, $\phi = \frac{3}{4}\pi$.

3. $z = -\frac{3}{2}i$.

$$r = \tfrac{3}{2} \tag{1.21}$$

$$\tan\phi = \frac{-\frac{3}{2}}{0} \quad \Rightarrow \phi \in \{\frac{\pi}{2}, \frac{3}{2}\pi\} \tag{1.22}$$

   Figure 1.6(d) shows that z is between the third and fourth quadrant (and not between the first and second). Therefore, $\phi = \frac{3}{2}\pi$

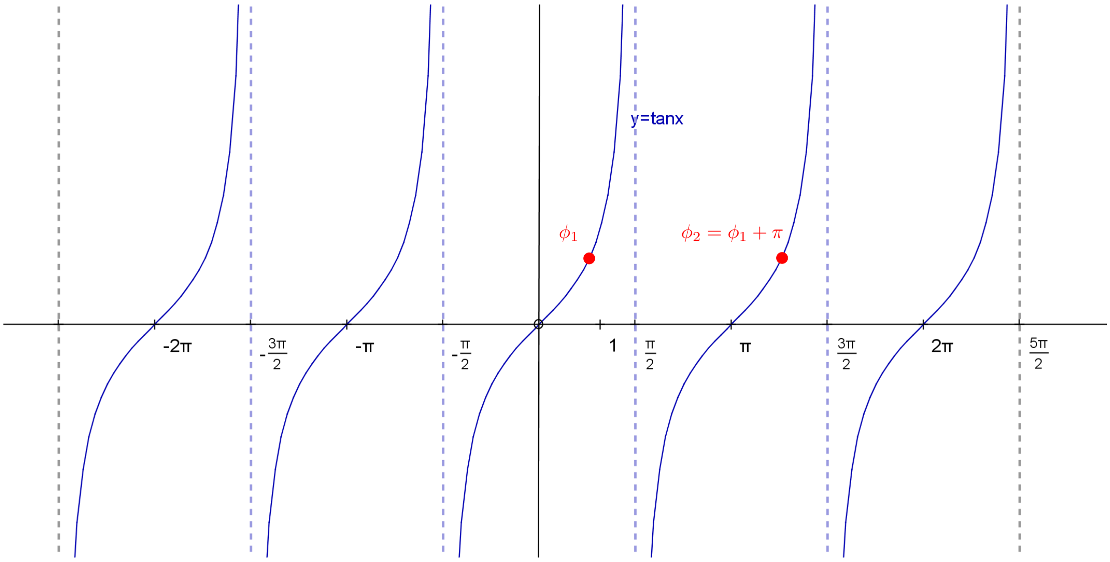


Figure 1.7: Tangens. Since the tangens possesses a period of π, there are two solutions for the argument $0 \leq \phi < 2\pi$ of a complex number, which differ by π.

1.2.5 Geometric Interpretation of the Product of Complex Numbers

Let us now use the polar coordinate representation of complex numbers to geometrically interpret the product $z = z_1 z_2$ of two complex numbers z_1, z_2. For $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ we obtain

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)) \\ &= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)). \end{aligned} \tag{1.23}$$

1. The length $r = |z| = |z_1||z_2|$ is the *product* of the lengths of z_1 and z_2.
2. The argument of z is the *sum* of the arguments of z_1 and z_2.

This means that when we multiply two complex numbers z_1, z_2, the corresponding distances r_1 and r_2 are multiplied while the corresponding arguments ϕ_1, ϕ_2 are summed up. This means, we are now ready to visualize complex multiplication, see Figure 1.8. Overall, multiplying z_1 with z_2 performs two (linear) transformations on z_1: a scaling by r_2 and a rotation by ϕ_2. Similarly, the transformations acting on z_2 are a scaling by r_1 and a rotation by ϕ_1.

1.2.6 Powers of Complex Numbers

We will encounter situations where we need to compute powers of complex numbers of the form z^n. For this, we can use some advantages of some representations of complex numbers. For instance, if we consider the representation using Cartesian coordinates computing $z^n = (x+iy)^n$ for large n will be rather laborious. However, the Euler representation makes our lives a bit easier since

$$z^n = (r\exp(i\phi))^n = r^n\exp(in\phi) \tag{1.24}$$

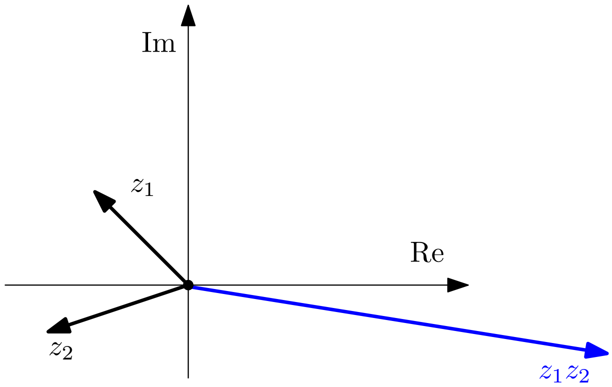


Figure 1.8: Complex multiplication. When we multiply two complex numbers z_1, z_2, the corresponding distances r_1 and r_2 are multiplied while the corresponding arguments ϕ_1, ϕ_2 are summed up.

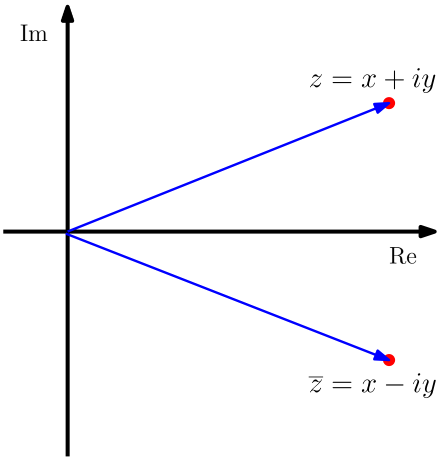


Figure 1.9: The complex conjugate $\bar{z}$ is a reflection of z about the real axis.

can be computed efficiently: The distance r to the origin is simply raised to the power of n and the argument is scaled/multiplied by n. This also immediately gives us the result

$$(r(\cos\phi + i\sin\phi))^n = r^n(\cos(n\phi) + i\sin(n\phi)) \tag{1.25}$$

which will later (Section 1.4) know as de Moivre's theorem.

1.3 Complex Conjugate

The **complex conjugate** of a complex number $z = x + iy$ is $\bar{z} = x - iy$. Some properties of complex conjugates include:

1. $\mathfrak{Re}(\bar{z}) = \mathfrak{Re}(z)$
2. $\mathfrak{Im}(\bar{z}) = -\mathfrak{Im}(z)$
3. $z + \bar{z} = 2x = 2\mathfrak{Re}(z) \in \mathbb{R}$

4. $z - \bar{z} = 2iy = 2i\,\mathrm{Im}(z)$ is purely imaginary

5. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

6. $\overline{z_1 z_2} = \overline{z_1}\;\overline{z_2}$. This can be seen either by noting that the conjugate operation simply changes every occurrence of i to $-i$ or since

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2), \tag{1.26}$$
$$(x_1 - iy_1)(x_2 - iy_2) = (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2), \tag{1.27}$$

which are conjugates. Geometrically, the complex conjugate $\bar{z}$ is a reflection of z where the real axis serves as the axis of reflection. Figure 1.9 illustrates this relationship.

1.3.1 Absolute Value of a Complex Number

The **absolute value (length/modulus)** of $z \in \mathbb{C}$ is $|z| = \sqrt{z\bar{z}}$, where

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 \in \mathbb{R}. \tag{1.28}$$

Notice that the term 'absolute value' is the same as defined for real numbers when $\mathrm{Im}(z) = 0$. In this case, $|z| = |x|$.
The absolute value of the product has the following nice property that matches the product result for real numbers:

$$|z_1 z_2| = |z_1|\,|z_2|. \tag{1.29}$$

This holds since

$$|z_1 z_2|^2 = z_1 z_2 \overline{z_1 z_2} = z_1 z_2 \overline{z_1}\;\overline{z_2} = z_1 \overline{z_1} z_2 \overline{z_2} = |z_1|^2 |z_2|^2. \tag{1.30}$$

1.3.2 Inverse and Division

If $z = x + iy$, its **inverse (reciprocal)** is

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}. \tag{1.31}$$

This can be written $z^{-1} = |z|^{-2}\bar{z}$, using only the complex operators multiply and add, see (1.5) and (1.6), but also real division, which we already know. Complex division is now defined by $z_1/z_2 = z_1 z_2^{-1}$. In practice, we compute the division z_1/z_2 by expanding the fraction by the complex conjugate of the denominator. This ensures that the denominator's imaginary part is 0 (only the real part remains), and the overall fraction can be written as

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2} \tag{1.32}$$

Geometric Interpretation of Division

When we use the Euler representations of two complex numbers $z_1, z_2 \in \mathbb{C}$, we can write the division as

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = r_1 \exp(i\phi_1)\Big(r_2 \exp(i\phi_2)\Big) = \frac{r_1}{r_2} \exp(i(\phi_1 - \phi_2)). \tag{1.33}$$

Geometrically, we divide r_1 by r_2 (equivalently: scale r_1 by $\frac{1}{r_2}$) and rotate z_1 by $-\phi_2$. This is not overly surprising since the division by z_2 does exactly the opposite of a multiplication by r_2. Therefore, looking again at Figure 1.8, if we take $z_1 z_2$ and divide by z_2, we obtain z_1.

Example: Complex Division

Bring the following fraction into the form $x + iy$:

$$z = x + iy = \frac{3 + 2i}{7 - 3i} \tag{1.34}$$

Solution:

$$\frac{3 + 2i}{7 - 3i} = \frac{(3 + 2i)(7 + 3i)}{(7 - 3i)(7 + 3i)} = \frac{15 + 23i}{49 + 9} = \frac{15}{58} + i\frac{23}{58} \tag{1.35}$$

Now, the fraction can be written as $z = x + iy$ with $x = \frac{15}{58}$ and $y = \frac{23}{58}$.

1.4 De Moivre's Theorem

De Moivre's theorem (or formula) is a central result because it connects complex numbers and trigonometry.

Theorem 1 (De Moivre's Theorem)
For any $n \in \mathbb{N}$

$$(\cos\phi + i \sin\phi)^n = \cos n\phi + i \sin n\phi \tag{1.36}$$

The proof is done by induction (which you will see in detail in the course *Reasoning about Programs*). A proof by induction allows you to prove that a property is true for all values of a natural number n. To construct an induction proof, you have to prove that the property, $P(n)$, is true for some base value (say, $n = 1$). A further proof is required to show that if it is true for the parameter $n = k$, then that implies it is also true for the parameter $n = k + 1$: that is $P(k) \Rightarrow P(k + 1)$ for all $k \geq 1$. The two proofs combined allow us to build an arbitrary chain of implication up to some value $n = m$:

$$P(1) \text{ and } (P(1) \Rightarrow P(2) \Rightarrow \cdots \Rightarrow P(m-1) \Rightarrow P(m)) \models P(m)$$

Proof 1
We start the induction proof by checking whether de Moivre's theorem holds for $n = 1$:

$$(\cos\phi + i\sin\phi)^1 = \cos\phi + i\sin\phi \tag{1.37}$$

is trivially true, and we can now make the induction step: We assume that (1.36) *is true for k and show that it also holds for $k+1$.*
Assuming

$$(\cos\phi + i\sin\phi)^k = \cos k\phi + i\sin k\phi \tag{1.38}$$

we can write

$$\begin{aligned}
(\cos\phi + i\sin\phi)^{k+1} &= (\cos\phi + i\sin\phi)(\cos\phi + i\sin\phi)^k \\
&= (\cos\phi + i\sin\phi)(\cos k\phi + i\sin k\phi) && \textit{using assumption } (1.38) \\
&= (\cos(k+1)\phi + i\sin(k+1)\phi) && \textit{using complex product } (1.23)
\end{aligned}$$

which concludes the proof.

1.4.1 Integer Extension to De Moivre's Theorem

We can extend de Moivre to include negative numbers, $n \in \mathbb{Z}$

$$(\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi$$

We have tackled the case for $n > 0$ already, $n = 0$ can be shown individually. So we take the case $n < 0$. We let $n = -m$ for $m > 0$.

$$\begin{aligned}
(\cos\phi + i\sin\phi)^n &= \frac{1}{(\cos\phi + i\sin\phi)^m} \\
&= \frac{1}{\cos m\phi + i\sin m\phi} && \text{by de Moivre's theorem} \\
&= \frac{\cos m\phi - i\sin m\phi}{\cos^2 m\phi + \sin^2 m\phi} \\
&= \cos(-m\phi) + i\sin(-m\phi) && \text{Trig. identity: } \cos^2 m\phi + \sin^2 m\phi = 1 \\
&= \cos n\phi + i\sin n\phi
\end{aligned}$$

1.4.2 Rational Extension to De Moivre's Theorem

Finally, for our purposes, we will show that if $n \in \mathbb{Q}$, one value of $(\cos\phi + i\sin\phi)^n$ is $\cos n\phi + i\sin n\phi$. Take $n = p/q$ for $p, q \in \mathbb{Z}$ and $q \neq 0$. We will use both de Moivre's theorems in the following:

$$\left(\cos\frac{p}{q}\phi + i\sin\frac{p}{q}\phi\right)^q = \cos p\phi + i\sin p\phi \tag{1.39}$$

$$= (\cos\phi + i\sin\phi)^p \tag{1.40}$$

Hence $\cos\frac{p}{q}\phi + i\sin\frac{p}{q}\phi$ is *one* of the qth roots of $(cos\phi + i\sin\phi)^p$.
The qth roots of $\cos\phi + i\sin\phi$ are easily obtained. We need to use the fact that (repeatedly) adding 2π to the argument of a complex number does not change the complex number.

$$(\cos\phi + i\sin\phi)^{\frac{1}{q}} = (\cos(\phi + 2n\pi) + i\sin(\phi + 2n\pi))^{\frac{1}{q}} \tag{1.41}$$

$$= \cos\frac{\phi + 2n\pi}{q} + i\sin\frac{\phi + 2n\pi}{q} \quad \text{for } 0 \le n < q \tag{1.42}$$

We will use this later to calculate roots of complex numbers.
Finally, the full set of values for $(\cos + i\sin\phi)^n$ for $n = p/q \in \mathbb{Q}$ is:

$$\cos\frac{p\phi + 2n\pi}{q} + i\sin\frac{p\phi + 2n\pi}{q} \quad \text{for } 0 \le n < q \tag{1.43}$$

Example: Multiplication using Complex Products

We require the result of:

$$(3 + 3i)(1 + i)^3$$

We could expand $(1+i)^3$ and multiply by $3+3i$ using real and imaginary components. Alternatively, we could tackle this in polar form $(\cos\phi + i\sin\phi)$ using the complex product of (1.23) and de Moivre's theorem.

$$(1 + i)^3 = [2^{1/2}(\cos\pi/4 + i\sin\pi/4)]^3$$
$$= 2^{3/2}(\cos 3\pi/4 + i\sin 3\pi/4)$$

by de Moivre's theorem. $3 + 3i = 18^{1/2}(\cos\pi/4 + i\sin\pi/4)$ and so the result is

$$18^{1/2}2^{3/2}(\cos\pi + i\sin\pi) = -12$$

Geometrically, we just observe that the Arg of the second number is 3 times that of $1+i$, i.e., $3\pi/4$ (or $3\cdot 45°$ in degrees). The first number has the same Arg, so the Arg of the result is π.
Similarly, the absolute values (lengths) of the numbers multiplied are $\sqrt{18}$ and $\sqrt{2^3}$, so the product has absolute value 12. The result is therefore -12.

1.5 Triangle Inequality for Complex Numbers

The triangle inequality for complex numbers is as follows:

$$\forall z_1, z_2 \in \mathbb{C} : |z_1 + z_2| \le |z_1| + |z_2| \tag{1.44}$$

An alternative form, with $w_1 = z_1$ and $w_2 = z_1 + z_2$ is $|w_2| - |w_1| \le |w_2 - w_1|$ and, switching w_1, w_2, $|w_1| - |w_2| \le |w_2 - w_1|$. Thus, relabelling back to z_1, z_2:

$$\forall z_1, z_2 \in \mathbb{C} : \Big||z_1| - |z_2|\Big| \le |z_2 - z_1| \tag{1.45}$$

In the Argand diagram, this just says that "In the triangle with vertices at $0, z_1, z_2$, the length of side $z_1 z_2$ is not less than the difference between the lengths of the other two sides".

Proof 2
Let $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$*. Squaring the left-hand side of* (1.45) *yields*

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 = |z_1|^2 + |z_2|^2 + 2(x_1x_2 + y_1y_2), \tag{1.46}$$

and the square of the right-hand side is

$$|z_1|^2 + |z_2|^2 + 2|z_1||z_2| \tag{1.47}$$

It is required to prove $x_1x_2 + y_1y_2 \le |z_1||z_2|$*. We continue by squaring this inequality*

$$x_1x_2 + y_1y_2 \le |z_1||z_2| \tag{1.48}$$
$$\Leftrightarrow (x_1x_2 + y_1y_2)^2 \le |z_1|^2|z_2|^2 \tag{1.49}$$
$$\Leftrightarrow x_1^2x_2^2 + y_1^2y_2^2 + 2x_1x_2y_1y_2 \le x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + y_1^2x_2^2 \tag{1.50}$$
$$\Leftrightarrow 0 \le (x_1y_2 - y_1x_2)^2, \tag{1.51}$$

which concludes the proof.

The geometrical argument via the Argand diagram is a good way to understand the triangle inequality.

1.6 Fundamental Theorem of Algebra

Theorem 2 (Fundamental Theorem of Algebra)
Any polynomial of degree n *of the form*

$$p(z) = \sum_{k=0}^{n} a_k z^k, \quad a_k \in \mathbb{C}, \quad a_n \neq 0 \tag{1.52}$$

possesses, counted with multiplicity, exactly n *roots in* $\mathbb{C}$*.*

A root z_* of $p(z)$ satisfies $p(z_*) = 0$. Bear in mind that complex roots include all real roots as the real numbers are a subset of the complex numbers. Also some of the roots might be coincident, e.g., for $z^2 = 0$. Finally, we also know that if ω is a root and $\omega \in \mathbb{C}\backslash\mathbb{R}$, then $\overline{\omega}$ is also a root. So all truly complex roots occur in complex conjugate pairs.

1.6.1 nth Roots of Unity

In the following, we consider the equation

$$z^n = 1, \quad n \in \mathbb{N}, \tag{1.53}$$

for which we want to determine the roots. The fundamental theorem of algebra tells us that there exist exactly n roots, one of which is $z = 1$.

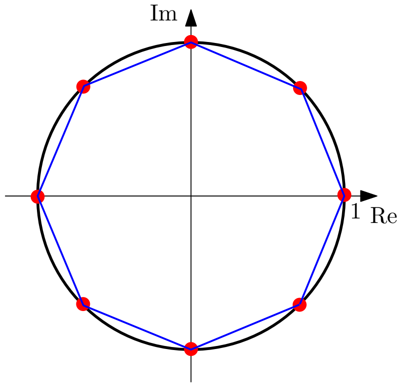


Figure 1.10: Then nth roots of $z^n = 1$ lie on the unit circle and form a regular polygon. Here, we show this for $n = 8$.

To find the other solutions, we write (1.53) in a slightly different form using the Euler representation:

$$z^n = 1 = e^{ik2\pi}, \quad \forall k \in \mathbb{Z}. \tag{1.54}$$

Then the solutions are $z = e^{i2k\pi/n}$ for $k = 0, 1, 2, \dots, n-1$.[4]
Geometrically, all n roots lie on the unit circle, and they form a regular polygon with n corners where the roots are $360°/n$ apart, see an example in Figure 1.10. Therefore, if we know a single root and the total number of roots, we could even geometrically find all other roots.

Example: Cube Roots of Unity

The 3rd roots of 1 are $z = e^{2k\pi i/3}$ for $k = 0, 1, 2$, i.e., $1, e^{2\pi i/3}, e^{4\pi i/3}$. These are often referred to as ω_1 ω_1 and ω_3, and simplify to

$$\begin{aligned}
\omega_1 &= 1 \\
\omega_2 &= \cos 2\pi/3 + i \sin 2\pi/3 = (-1 + i\sqrt{3})/2, \\
\omega_3 &= \cos 4\pi/3 + i \sin 4\pi/3 = (-1 - i\sqrt{3})/2.
\end{aligned}$$

Try cubing each solution directly to validate that they are indeed cubic roots.

1.6.2 Solution of $z^n = a + ib$

Finding the n roots of $z^n = a + ib$ is similar to the approach discussed above: Let $a + ib = re^{i\phi}$ in polar form. Then, for $k = 0, 1, \dots, n-1$,

$$z^n = (a + ib)e^{2\pi ki} = re^{(\phi + 2\pi k)i} \tag{1.55}$$

$$\Rightarrow z_k = r^{\frac{1}{n}} e^{\frac{(\phi + 2\pi k)}{n} i}, \quad k = 0, \dots, n-1. \tag{1.56}$$

[4] Note that the solutions repeat when $k = n, n+1, \dots$

Example

Determine the cube roots of $1 - i$.

1. The polar coordinates of $1 - i$ are $r = \sqrt{2}$, $\phi = \frac{7}{4}\pi$, and the corresponding Euler representation is

$$z = \sqrt{2}\exp(i\tfrac{7\pi}{4}). \tag{1.57}$$

2. Using (1.56), the cube roots of z are

$$z_1 = 2^{\frac{1}{6}}(\cos\tfrac{7\pi}{12} + i\sin\tfrac{7\pi}{12}) = 2^{\frac{1}{6}}\exp(i\tfrac{7\pi}{12}) \tag{1.58}$$
$$z_2 = 2^{\frac{1}{6}}(\cos\tfrac{15\pi}{12} + i\sin\tfrac{15\pi}{12}) = 2^{\frac{1}{6}}(\cos\tfrac{5\pi}{4} + i\sin\tfrac{5\pi}{4}) = 2^{\frac{1}{6}}\exp(i\tfrac{5\pi}{4}) \tag{1.59}$$
$$z_3 = 2^{\frac{1}{6}}(\cos\tfrac{23\pi}{12} + i\sin\tfrac{23\pi}{12}) = 2^{\frac{1}{6}}\exp(i\tfrac{23\pi}{12}). \tag{1.60}$$

1.7 Complex Sequences and Series*

A substantial part of the theory that we have developed for convergence of sequences and series of real numbers also applies to complex numbers. We will not reproduce all the results here, there is no need; we will highlight a couple of key concepts instead.

1.7.1 Limits of a Complex Sequence

For a sequence of complex numbers $z_1, z_2, z_3, \ldots$, we can define limits of convergence, $z_n \to l$ as $n \to \infty$ where $z_n, l \in \mathbb{C}$. This means that for all $\epsilon > 0$ we can find a natural number N, such that

$$\forall n > N : |z_n - l| < \epsilon. \tag{1.61}$$

The only distinction here is the meaning of $|z_n - l|$, which refers to the complex absolute value and not the absolute real value.

Example of complex sequence convergence Prove that the complex sequence $z_n = \frac{1}{n+i}$ converges to 0 as $n \to \infty$. Straight to the limit inequality:

$$\left|\frac{1}{n+i}\right| < \epsilon \tag{1.62}$$
$$\Leftrightarrow \frac{|n-i|}{n^2+1} < \epsilon \tag{1.63}$$
$$\Leftrightarrow \frac{\sqrt{(n-i)(n+i)}}{n^2+1} < \epsilon \tag{1.64}$$
$$\Leftrightarrow \frac{1}{\sqrt{n^2+1}} < \epsilon \tag{1.65}$$

$$\Rightarrow n > \sqrt{\frac{1}{\epsilon^2} - 1} \qquad \text{for } \epsilon \leq 1 \tag{1.66}$$

Thus, we can set

$$N(\epsilon) = \begin{cases} \left\lceil \sqrt{\frac{1}{\epsilon^2} - 1} \right\rceil & \epsilon \leq 1 \\ 1 & \text{otherwise} \end{cases} \tag{1.67}$$

We have to be a tiny bit careful as $N(\epsilon)$ needs to be defined *for all* $\epsilon > 0$ and the penultimate line of the limit inequality is true for all $n > 0$ if $\epsilon > 1$. In essence this was no different in structure from the normal sequence convergence proof. The only difference was how we treated the absolute value.

Absolute Convergence

Similarly, a complex series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |z_n|$ converges. Again the $|z_n|$ refers to the complex absolute value.

Complex Ratio Test

A complex series $\sum_{n=1}^{\infty} z_n$ converges if

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| < 1 \tag{1.68}$$

and diverges if

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| > 1 \,. \tag{1.69}$$

Example of Complex Series Convergence

Let us take a general variant of the geometric series:

$$S = \sum_{n=1}^{\infty} az^{n-1} \tag{1.70}$$

We can prove that this will converge for some values of $z \in \mathbb{C}$ in the same way we could for the real-valued series. Applying the complex ratio test, we get $\lim_{n\to\infty} |\frac{az^n}{az^{n-1}}| = |z|$. We apply the standard condition and get that $|z| < 1$ for this series to converge. The radius of convergence is still 1 (and is an actual radius of a circle in the complex plane). What is different here is that now any z-point taken from within the circle centred on the origin with radius 1 will make the series converge, not just on the real interval $(-1, 1)$.
For your information, the limit of this series is $\frac{a}{1-z}$, which you can show using Maclaurin as usual, discussed below.

1.8 Complex Power Series

We can expand functions as power series in a complex variable, usually z, in the same way as we could with real-valued functions. The same expansions hold in $\mathbb{C}$ because the functions below (at any rate) are differentiable in the complex domain. Therefore, Maclaurin's series applies and yields

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \tag{1.71}$$

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \tag{1.72}$$

$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \tag{1.73}$$

1.8.1 A Generalized Euler Formula

A more general form of Euler's formula (1.12) is

$$\forall z \in \mathbb{C}, n \in \mathbb{Z} : z = re^{i(\phi+2n\pi)} \tag{1.74}$$

since $e^{i2n\pi} = \cos 2n\pi + i \sin 2n\pi = 1$. This is the same general form we used in the rational extension to De Moivres theorem to access the many roots of a complex number.
In terms of the Argand diagram, the points $e^{i(\phi+2n\pi)}$ for $i \geq 1$ lie on top of each other, each corresponding to one more revolution (through 2π).
The complex conjugate of $e^{i\phi}$ is $e^{-i\phi} = \cos\phi - i\sin\phi$. This allows us to get useful expressions for $\sin\phi$ and $\cos\phi$:

$$\cos\phi = (e^{i\phi} + e^{-i\phi})/2 \tag{1.75}$$

$$\sin\phi = (e^{i\phi} - e^{-i\phi})/2i. \tag{1.76}$$

We will be able to use these relationships to create trigonometric identities.

1.9 Applications of Complex Numbers

1.9.1 Trigonometric Multiple Angle Formulae

How can we calculate $\cos n\phi$ in terms of $\cos\phi$ and $\sin\phi$? We can use de Moivre's theorem to expand $e^{in\phi}$ and equate real and imaginary parts: e.g., for $n = 5$, by the Binomial theorem,

$$\begin{aligned}(\cos\phi + i\sin\phi)^5 = \cos^5\phi + i5\cos^4\phi\sin\phi - 10\cos^3\phi\sin^2\phi \\ - i10\cos^2\phi\sin^3\phi + 5\cos\phi\sin^4\phi + i\sin^5\phi\end{aligned} \tag{1.77}$$

Comparing real and imaginary parts now gives

$$\cos 5\phi = \cos^5\phi - 10\cos^3\phi\sin^2\phi + 5\cos\phi\sin^4\phi \tag{1.78}$$

and

$$\sin 5\phi = 5\cos^4\phi\sin\phi - 10\cos^2\phi\sin^3\phi + \sin^5\phi \tag{1.79}$$

Trigonometric Power Formulae

We can also calculate $\cos^n\phi$ in terms of $\cos m\phi$ and $\sin m\phi$ for $m \in \mathbb{N}$: Let $z = e^{i\phi}$ so that $z + z^{-1} = z + \bar{z} = 2\cos\phi$. Similarly, $z^m + z^{-m} = 2\cos m\phi$ by de Moivre's theorem. Hence by the Binomial theorem, e.g., for $n = 5$,

$$(z+z^{-1})^5 = (z^5+z^{-5}) + 5(z^3+z^{-3}) + 10(z+z^{-1}) \tag{1.80}$$

$$2^5\cos^5\phi = 2(\cos 5\phi + 5\cos 3\phi + 10\cos\phi) \tag{1.81}$$

Similarly, $z - z^{-1} = 2i\sin\phi$ gives $\sin^n\phi$
When n is even, we get an extra term in the binomial expansion, which is *constant*. For example, for $n = 6$, we obtain

$$(z+z^{-1})^6 = (z^6+z^{-6}) + 6(z^4+z^{-4}) + 15(z^2+z^{-2}) + 20 \tag{1.82}$$

$$2^6\cos^6\phi = 2(\cos 6\phi + 6\cos 4\phi + 15\cos 2\phi + 10) \tag{1.83}$$

and, therefore,

$$\cos^6\phi = \frac{1}{32}(\cos 6\phi + 6\cos 4\phi + 15\cos 2\phi + 10). \tag{1.84}$$

1.9.2 Summation of Series

Some series with sines and cosines can be summed similarly, e.g.,

$$C = \sum_{k=0}^{n} a^k \cos k\phi \tag{1.85}$$

Let $S = \sum_{k=1}^{n} a^k \sin k\phi$. Then,

$$C + iS = \sum_{k=0}^{n} a^k e^{ik\phi} = \frac{1 - (ae^{i\phi})^{n+1}}{1 - ae^{i\phi}}. \tag{1.86}$$

Hence,

$$C + iS = \frac{(1-(ae^{i\phi})^{n+1})(1-ae^{-i\phi})}{(1-ae^{i\phi})(1-ae^{-i\phi})} \tag{1.87}$$

$$= \frac{1 - ae^{-i\phi} - a^{n+1}e^{i(n+1)\phi} + a^{n+2}e^{in\phi}}{1 - 2a\cos\phi + a^2}. \tag{1.88}$$

Equating real and imaginary parts, the cosine series is

$$C = \frac{1 - a\cos\phi - a^{n+1}\cos(n+1)\phi + a^{n+2}\cos n\phi}{1 - 2a\cos\phi + a^2}, \tag{1.89}$$

and the sine series is

$$S = \frac{a\sin\phi - a^{n+1}\sin(n+1)\phi + a^{n+2}\sin n\phi}{1 - 2a\cos\phi + a^2} \tag{1.90}$$

1.9.3 Integrals

We can determine integrals

$$C = \int_0^x e^{a\phi}\cos b\phi d\phi, \tag{1.91}$$

$$S = \int_0^x e^{a\phi}\sin b\phi d\phi \tag{1.92}$$

by looking at the sum[5]

$$C + iS = \int_0^x e^{(a+ib)\phi} d\phi \tag{1.93}$$

$$= \frac{e^{(a+ib)x} - 1}{a + ib} = \frac{(e^{ax}e^{ibx} - 1)(a - ib)}{a^2 + b^2} \tag{1.94}$$

$$= \frac{(e^{ax}\cos bx - 1 + ie^{ax}\sin bx)(a - ib)}{a^2 + b^2} \tag{1.95}$$

The result is therefore

$$C + iS = \frac{e^{ax}(a\cos bx + b\sin bx) - a + i(e^{ax}(a\sin bx - b\cos bx) + b)}{a^2 + b^2} \tag{1.96}$$

and so we get

$$C = \frac{e^{ax}(a\cos bx + b\sin bx - a)}{a^2 + b^2}, \tag{1.97}$$

$$S = \frac{e^{ax}(a\sin bx - b\cos bx) + b}{a^2 + b^2} \tag{1.98}$$

as the solutions to the integrals we were seeking.

[5]The reduction formula would require a and b to be integers.

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