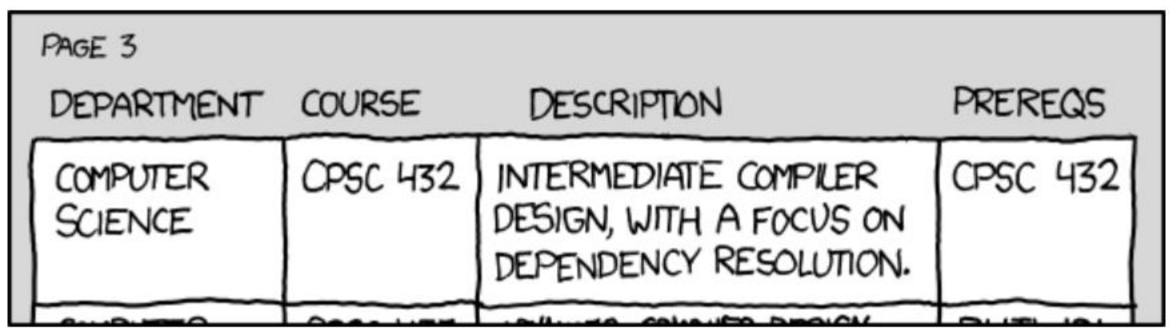
Compilers - Chapter 8:

Loop scheduling optimisations

Part 1: Why mess with the order of loop execution?

- Lecturer:
  - Paul Kelly (p.kelly@imperial.ac.uk)



(It turns out that this cartoon is good for almost any compilers topic, definitely this one) https://xkcd.com/754/

“Restructuring” compilers

- The optimisations we have studied so far reduce the number of instructions that need to be executed at runtime
  - This is fundamentally a good idea!
- But sometimes we can get a performance improvement by thinking about the order in which loops are executed
- Why might that be?

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    - So different iterations of a loop might be assigned to different threads running on different CPUs

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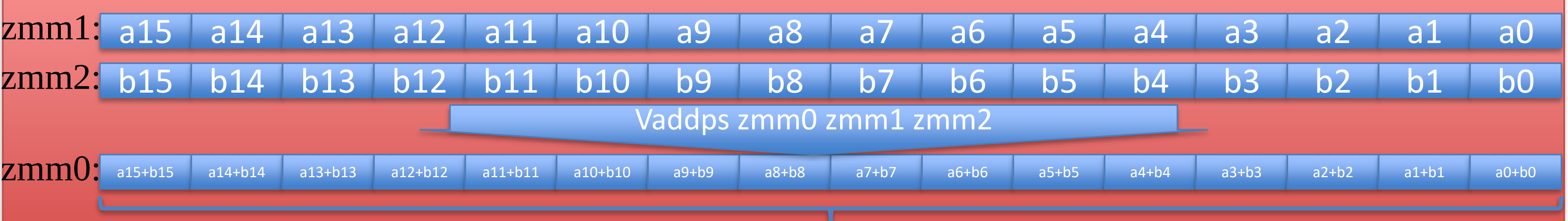
    - So different iterations of a loop are being executed at the same time
  - We might be able to use multiple cores
    - So different iterations of a loop might be assigned to different threads running on different CPUs
  - We might be able to improve how the cache is used
    - We will come to this later!

Vector instruction set extensions

- Example: Intel's AVX512
- Extended registers ZMM0-ZMM31, 512 bits wide
  - Can be used to store 8 doubles, 16 floats, 32 shorts, 64 bytes
  - So instructions are executed in parallel in 64,32,16 or 8 "lanes"

Vector instruction set extensions

- Example: Intel’s AVX512
- Extended registers ZMM0-ZMM31, 512 bits wide
  - Can be used to store 8 doubles, 16 floats, 32 shorts, 64 bytes
- Example: `vaddps zmm0 zmm1 zmm2`
  - “Add Packed Single Precision Floating-Point Values”



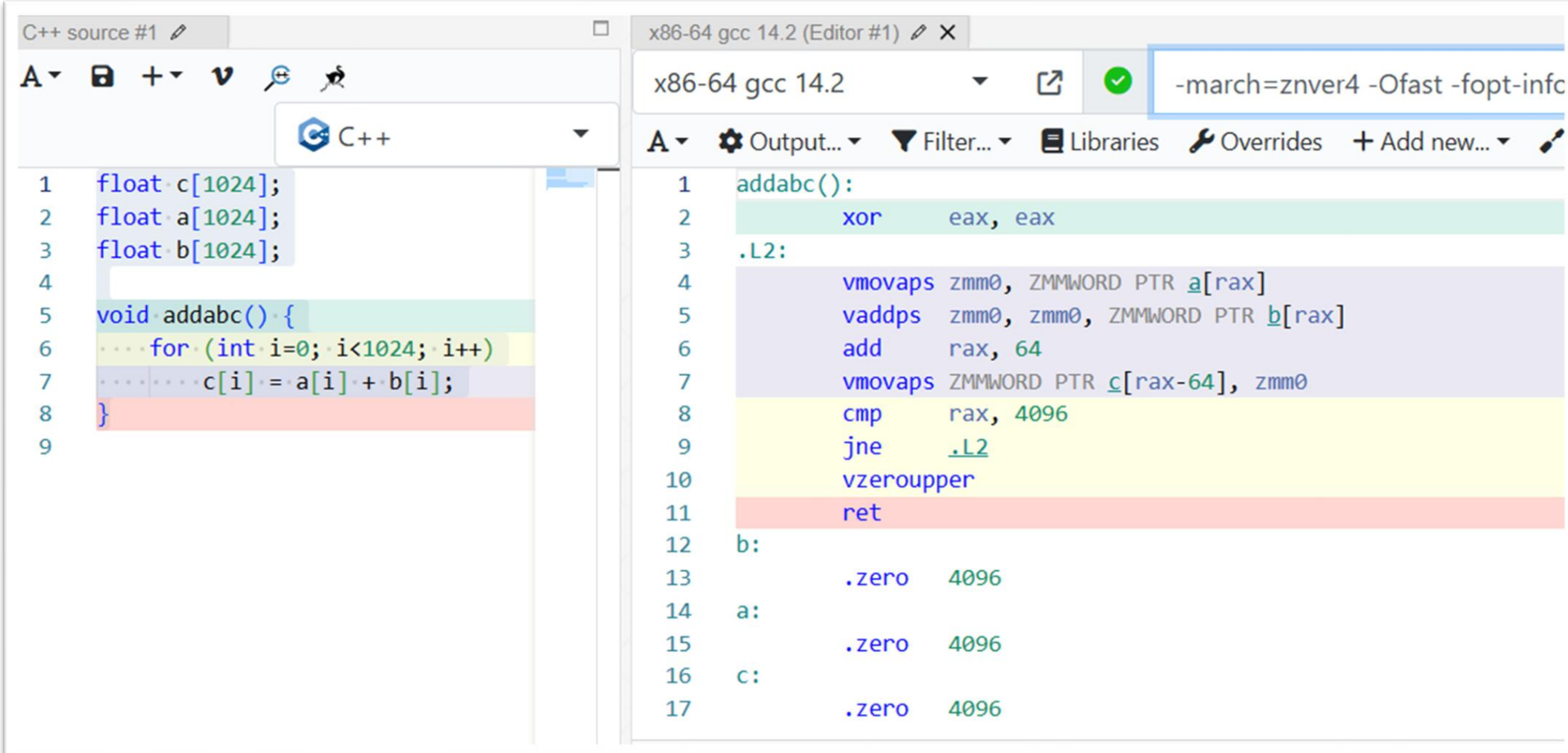
In *one instruction* we add 16 32-bit floating point values from zmm1 and 16 32-bit values from zmm2

A stretch of Mumbai to Pune expressway near Lonavala.

Dubai
https://www.worldatlas.com/articles/countries-with-the-best-roads.html


Lane 0
Lane 1
Lane 2
Lane 3
https://blog.playo.co/cycling-work-insane-idea-boon/

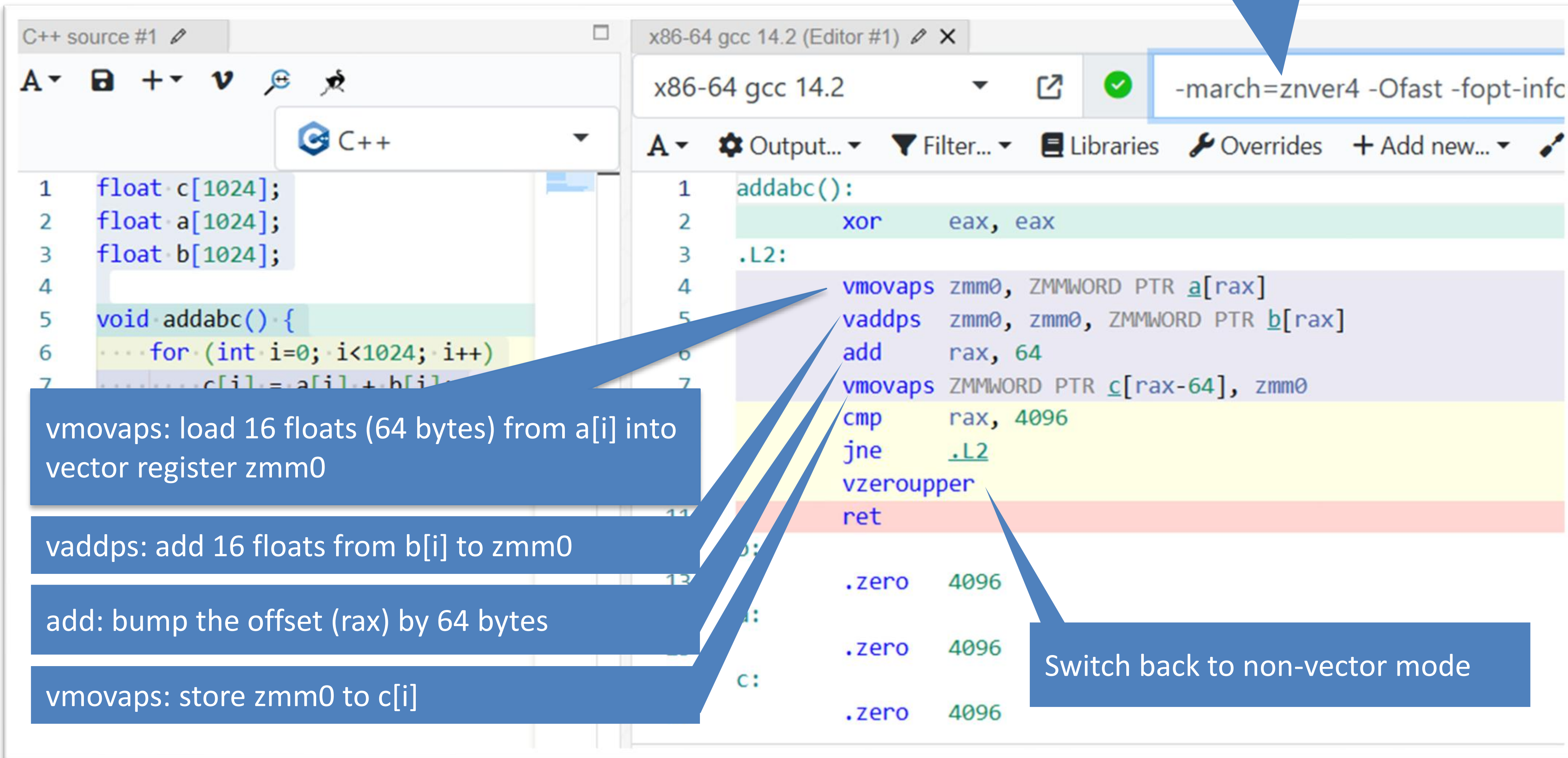
Can we get the compiler to vectorise?



In sufficiently simple cases, no problem:
Gcc reports: **addcba.c:6:20: optimized: loop vectorized using 64 byte vectors**

Can we get the compiler to vectorise?

Tell the compiler to generate code for AMD Zen 4 which has AVX512



In sufficiently simple cases, no problem:
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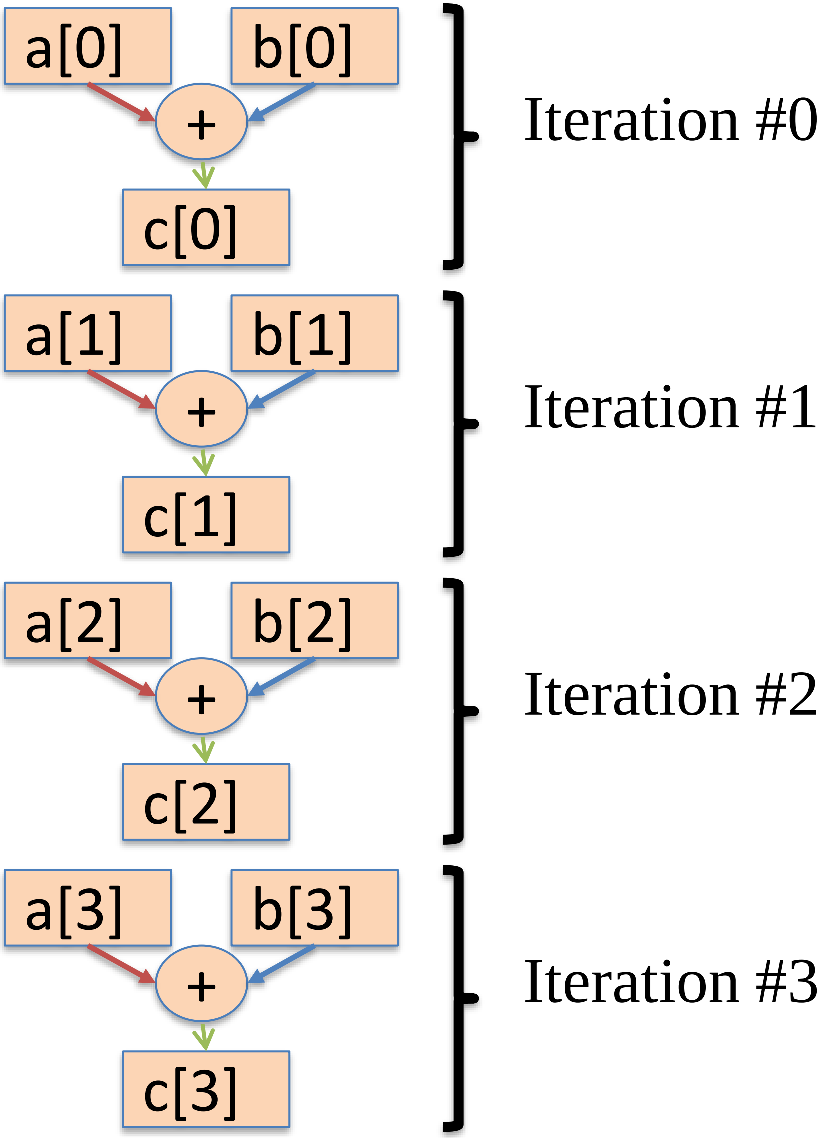
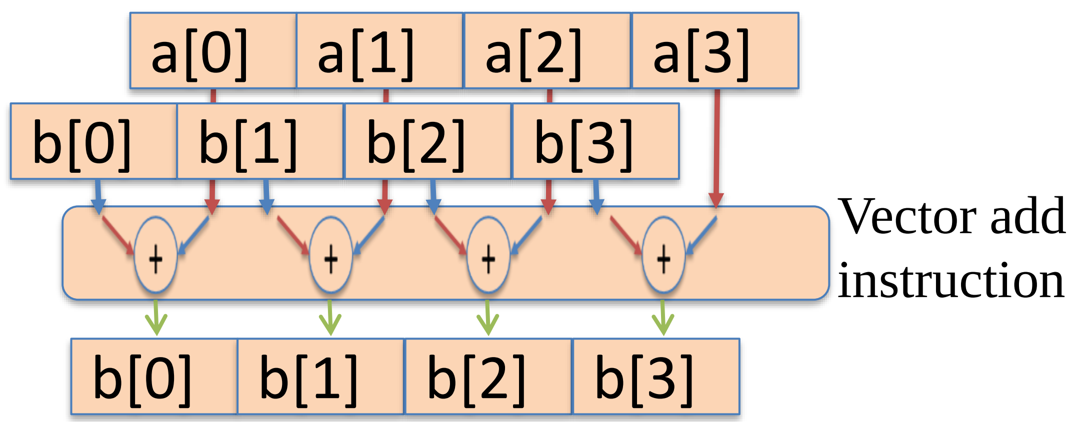
How do we know a loop is parallel?



- To use vector instructions, we need to verify that different iterations of the loop are truly parallel

- In this case we can easily see that the dependence arrows do not cross iteration boundaries



- This case was easy:

```
for (int i=0; i<4; i++)
  c[i] = a[i] + b[i];
```

P

- So we can use a vector add

How much does it help?
First: *without* vectorisation

- Source code:

```
for (int i=0; i<size; i++)
        c[i] = a[i] + b[i];
```

- Processor: AMD Ryzen 9 7940HS (“maple10”)
- Compiler command line:

```
gcc -O1 addcba-perf.c
```

- Generated code – not vectorised:

```
.L3:
        movss   (%rsi,%rax), %xmm0
        addss   (%rcx,%rax), %xmm0
        movss   %xmm0, (%rdi,%rax)
        addq    $4, %rax
        cmpq    %rdx, %rax
        jne     .L3
```

- Performance: 4.8 GFLOPS (4.8*10^9 single precision floating-point operations/second)
- Time per loop iteration: 0.21ns (one clock cycle at 4.8GHz, 1 result per iteration)

How much does it help?
This time *with* vectorisation

- Source code:

```
for (int i=0; i<size; i++)
        c[i] = a[i] + b[i];
```

- Processor: AMD Ryzen 9 7940HS ("maple10")
- Compiler command line:

```
gcc -Ofast -march=znver4 addcba-perf.c
```

- Generated code:

```
.L4:
        vmovaps (%r8,%rax), %zmm1
        vaddps  (%rdi,%rax), %zmm1, %zmm0
        vmovaps %zmm0, (%rsi,%rax)
        addq    $64, %rax
        cmpq    %rax, %rdx
        jne     .L4
```

- Performance: 34.8 GFLOPS (single precision)
- Time per loop iteration: 0.45ns (two clock cycles, 16 results per iteration)

How much does it help?
This time *with* vectorisation

- Source code:

```
for (int i=0; i<size; i++)
        c[i] = a[i] + b[i];
```

- Processor: AMD Ryzen 9 7940HS ("maple10")
- Compiler command line:

```
gcc -Ofast -march=znver4 addcba-perf.c
```

- Generated code:

```
.L4:
        vmovaps (%r8,%rax), %zmm1
        vaddps  (%rdi,%rax), %zmm1, %zmm0
        vmovaps %zmm0, (%rsi,%rax)
        addq    $64, %rax
        cmpq    %rax, %rdx
        jne     .L4
```

Speed of light is 30cm/ns.

So this machine completes about three iterations in the time it takes the light to get from your computer screen to your eyes

- Performance: 34.8 GFLOPS (single precision)
- Time per loop iteration: 0.45ns (two clock cycles, 16 results per iteration)

Compilers - Chapter 8:

Loop scheduling optimisations

Part 2: Determining whether a loop can be executed in parallel

- Lecturer:
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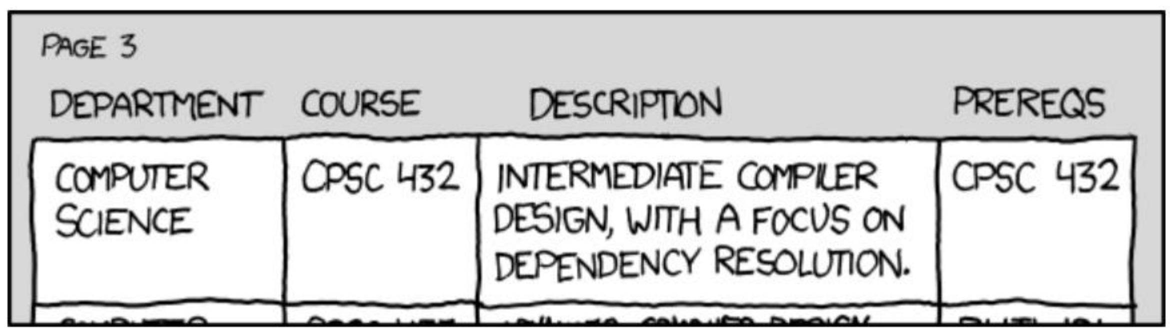

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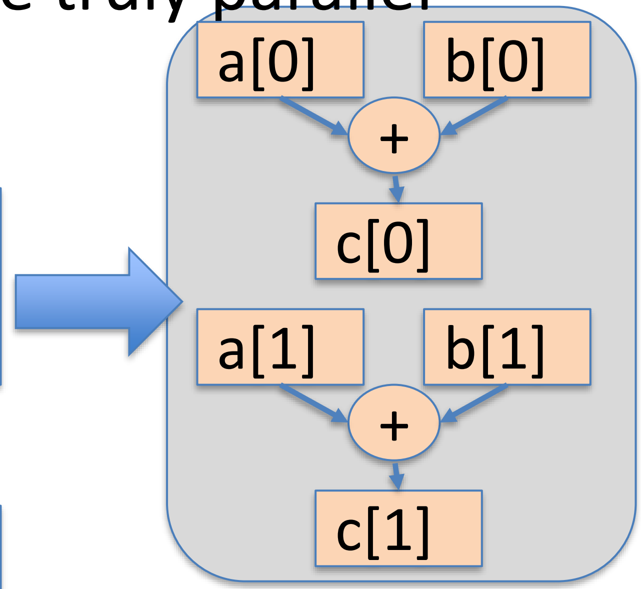
But that example was *obviously* parallel?

- To use vector instructions, we need to verify that different iterations of the loop are truly parallel

- This case was easy:

```
for (int i=0; i<1024; i++)
  c[i] = a[i] + b[i];
```

P



- How about this one?

```
for (int i=0; i<1024; i++)
  c[i] = c[i-1] + b[i];
```

Q

But that example was *obviously* parallel?

- To use vector instructions, we need to verify that different iterations of the loop are truly parallel

- This case was easy:

```
for (int i=0; i<1024; i++)        P
  c[i] = a[i] + b[i];
```

- How about this one?

```
for (int i=0; i<1024; i++)        Q
  c[i] = c[i-1] + b[i];
```

- And this?

```
for (int i=0; i<1024; i+=2)       R
  c[i] = c[i-1] + b[i];
```

"Loop-carried dependence"

- Consider this example:

```
for (int i=1; i<8; i++)
  c[i] = c[i-1] + b[i];
```

Q

- What does it *do*?

“Loop-carried dependence”

- Consider this example:

```
for (int i=1; i<8; i++)
  c[i] = c[i-1] + b[i];
```

Q

- When executed we get:

```
c[1] = c[0] + b[1];
c[2] = c[1] + b[2];
c[3] = c[2] + b[3];
c[4] = c[3] + b[4];
c[5] = c[4] + b[5];
c[6] = c[5] + b[6];
c[7] = c[6] + b[7];
```

“Loop-carried dependence”

- Consider this example:

```
for (int i=1; i<8; i++)
  c[i] = c[i-1] + b[i];
```

Q

Each iteration produces a value that is used in the next iteration

- When executed we get:

```
c[1] = c[0] + b[1];
c[2] = c[1] + b[2];
c[3] = c[2] + b[3];
c[4] = c[3] + b[4];
c[5] = c[4] + b[5];
c[6] = c[5] + b[6];
c[7] = c[6] + b[7];
```

The dependence arrows go from one iteration to the next

The dependence is *carried* by the loop

"Loop-carried dependence"

- Consider this example:

```
for (int i=1; i<8; i++)
  c[i] = c[i-1] + b[i];
```

Q

- When executed we get:

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c[1] = c[0] + b[1];
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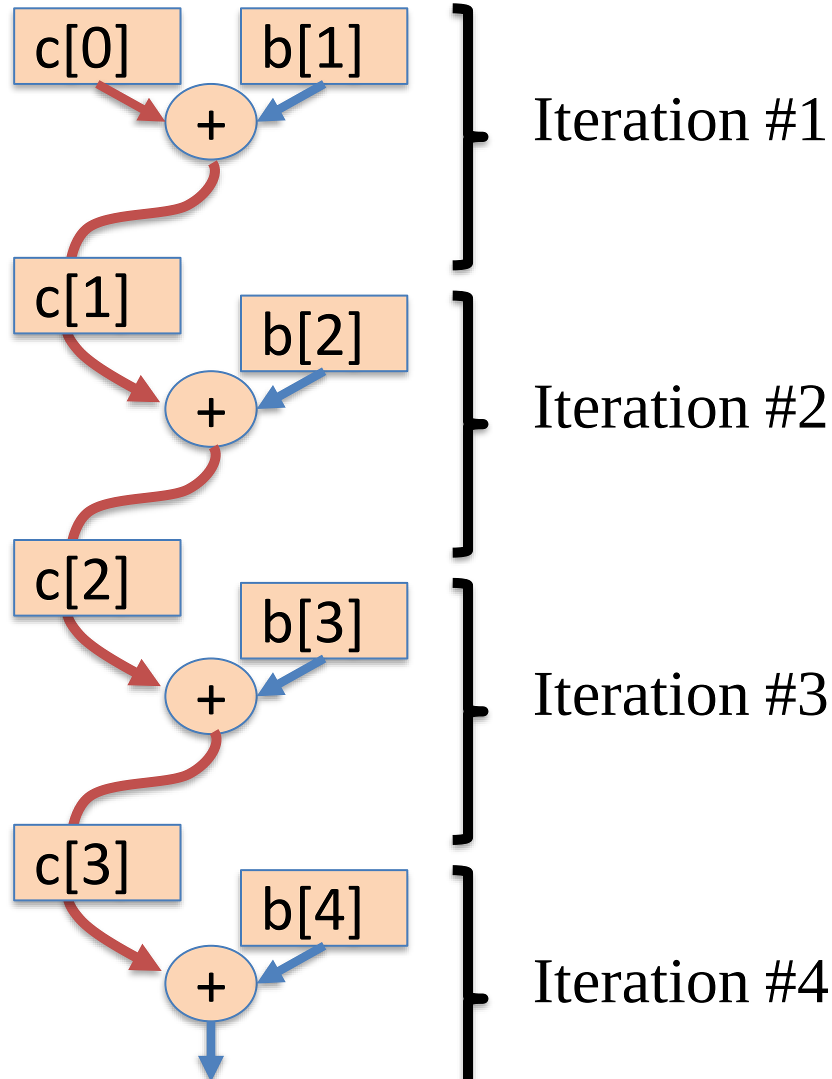
```



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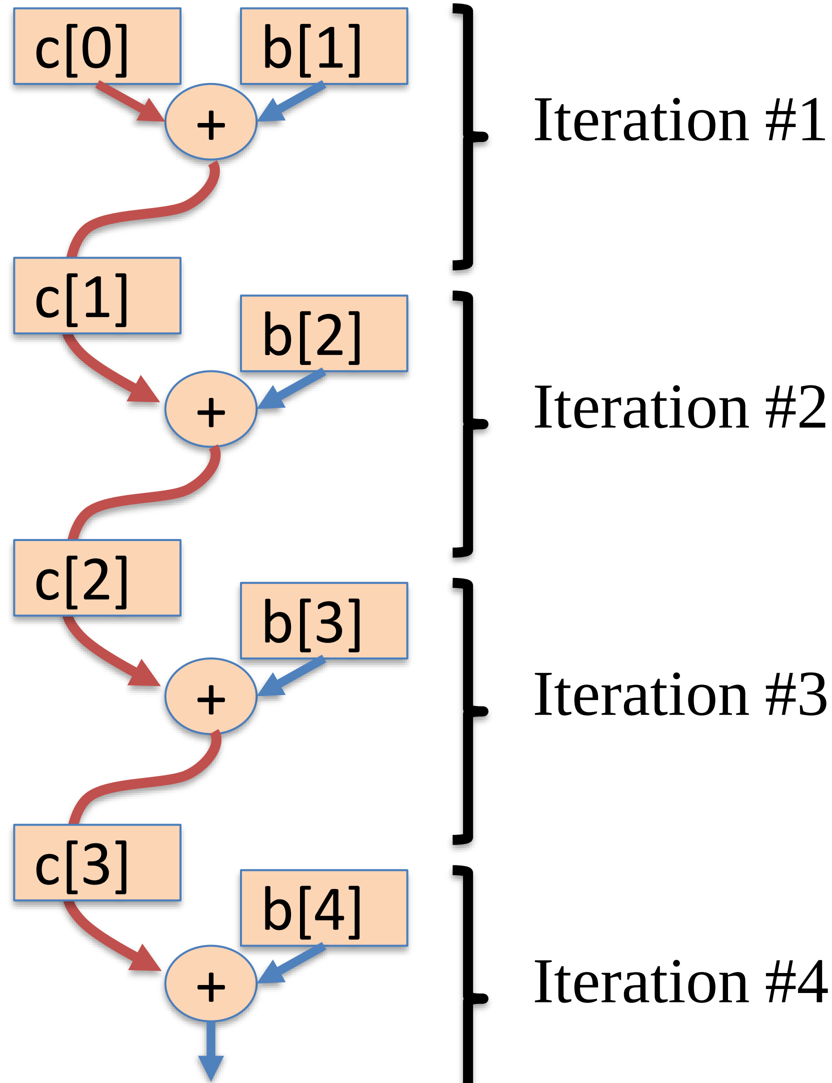

There is a chain of dependence from iteration to iteration

"Loop-carried dependence"

- Consider this example:

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for (int i=1; i<8; i++)
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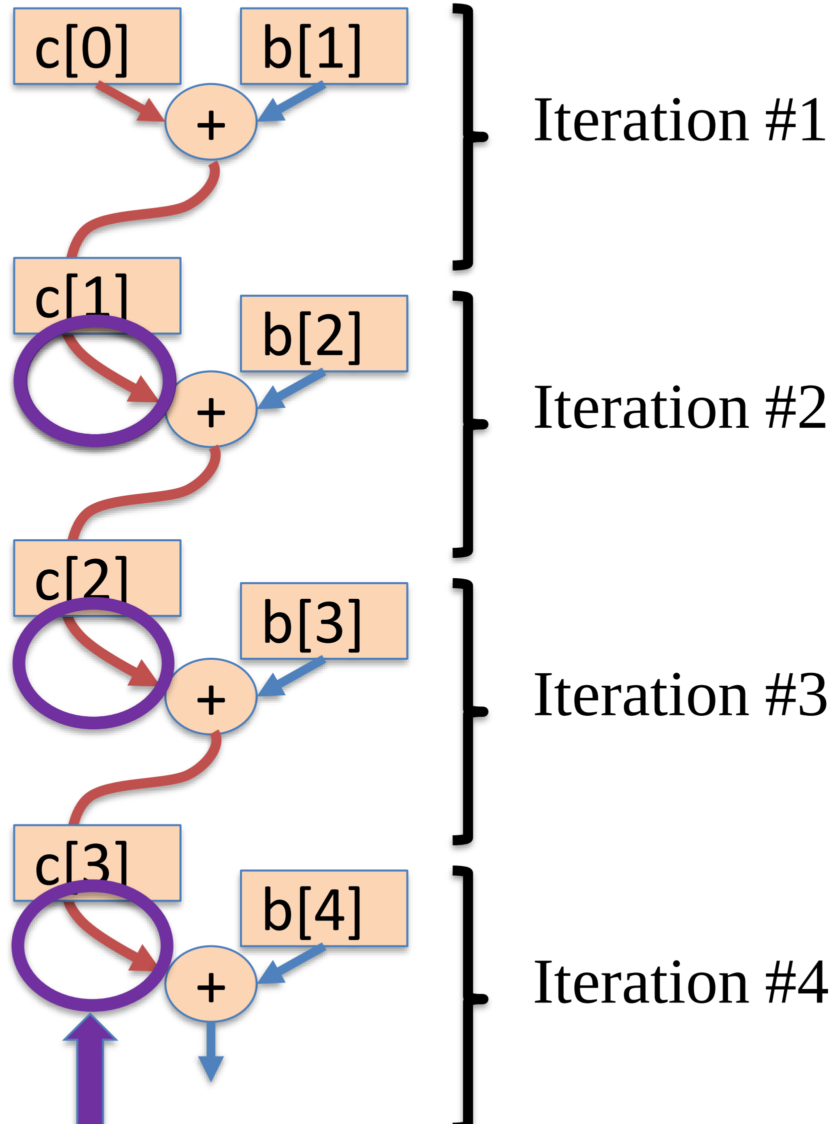

These are the *loop-carried* dependences

So we need a compiler algorithm

- To determine whether there is a loop-carried dependence
- To distinguish, for example, **P**, **Q** and **R**:

P

```
for (int i=0; i<1024; i++)
  c[i] = a[i] + b[i];
```

- No loop-carried dependence
- So iterations can be executed in parallel
- **So vectorisable**

Q

```
for (int i=0; i<1024; i++)
  c[i] = c[i-1] + b[i];
```

- loop-carried dependence
- So iterations cannot be executed in parallel
- **So *not* vectorisable**

R

```
for (int i=0; i<1024; i+=2)
  c[i] = c[i-1] + b[i];
```

- No loop-carried dependence
- So iterations can be executed in parallel
- **So vectorisable**

So we need a compiler algorithm

- To determine whether there is a loop-carried dependence
- To distinguish, for example, **P**, **Q** and **R**:

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for (int i=0; i<1024; i++)
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```

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- No loop-carried dependence
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for (int i=0; i<1024; i++)
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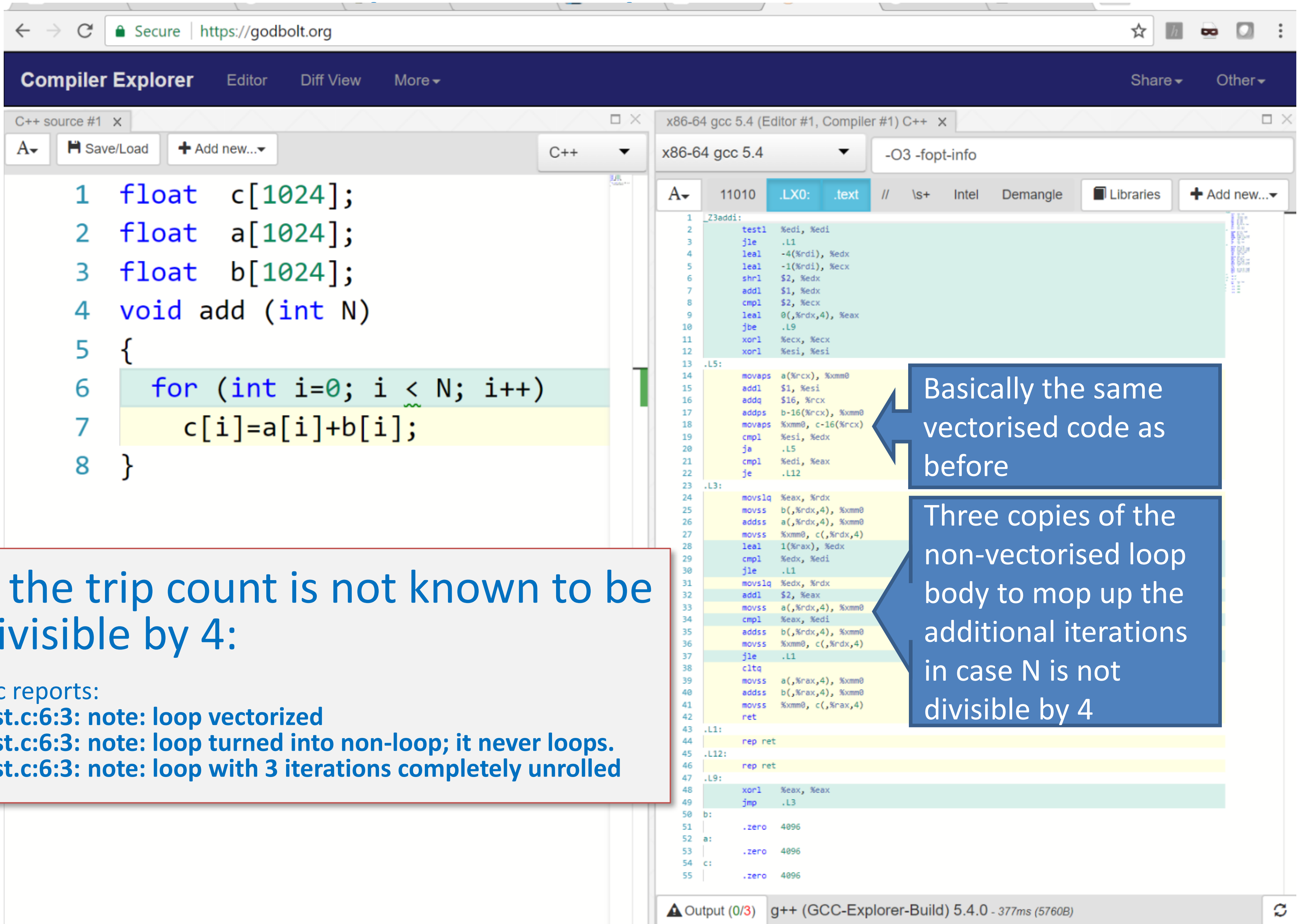
Q

- loop-carried dependence
- So iterations cannot be executed in parallel
- **So *not* vectorisable**

```
for (int i=0; i<1024; i+=2)
  c[i] = c[i-1] + b[i];
```

R

- No loop-carried dependence
- So iterations can be executed in parallel
- **So vectorisable**
- **(though actually generating efficient vector code for this might be a bit tricky?)**



Basically the same vectorised code as before

Three copies of the non-vectorised loop body to mop up the additional iterations in case N is not divisible by 4

If the trip count is not known to be divisible by 4:

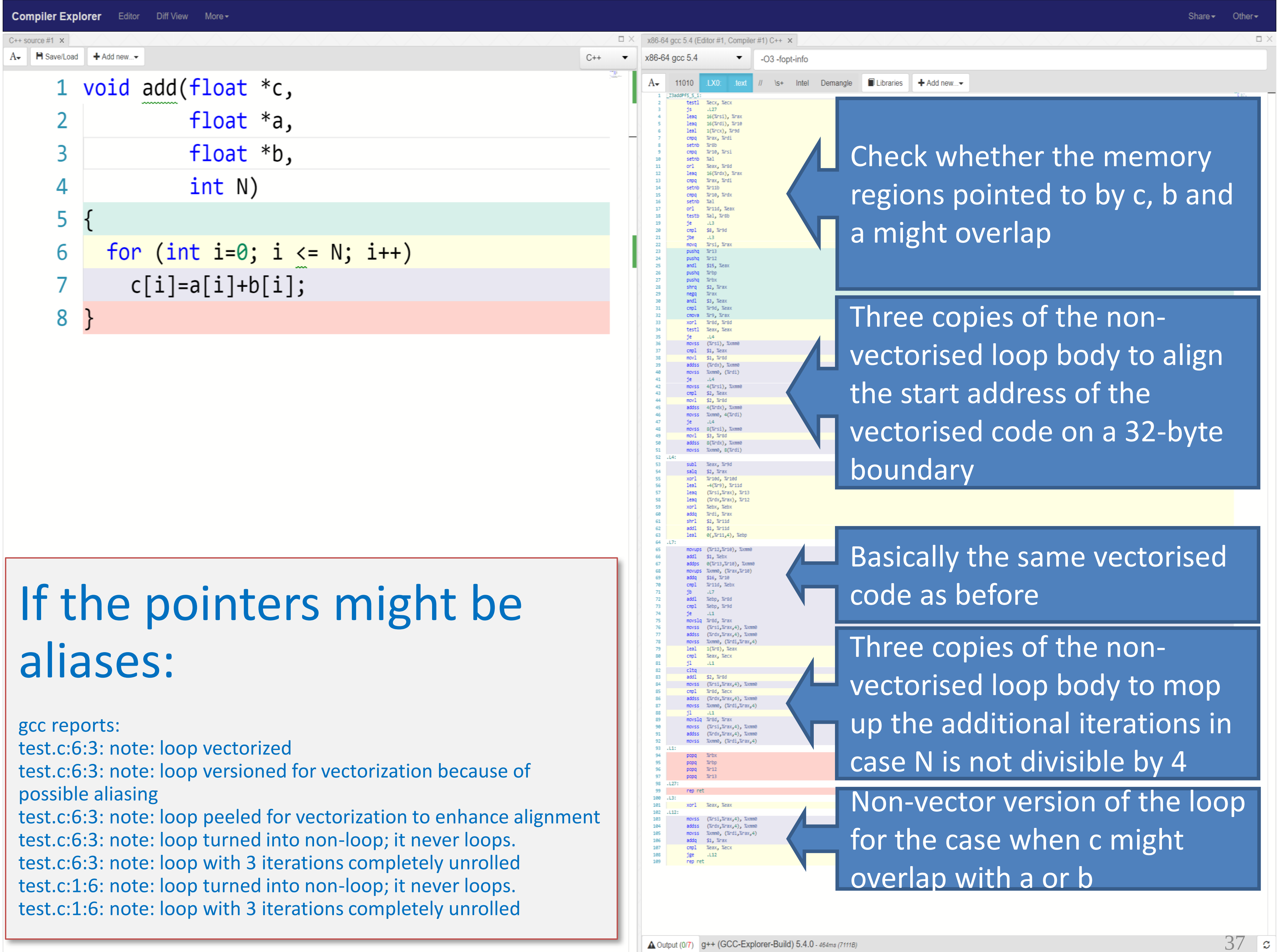
gcc reports:

test.c:6:3: note: loop vectorized
test.c:6:3: note: loop turned into non-loop; it never loops.
test.c:6:3: note: loop with 3 iterations completely unrolled

```
void add(float *__restrict__ c,
         float *__restrict__ a,
         float *__restrict__ b,
         int N)
{
  for (int i=0; i <= N; i++)
    c[i]=a[i]+b[i];
}
```

If the alignment of the operand pointers is not known:

gcc reports:
test.c:6:3: note: loop vectorized
test.c:6:3: note: loop peeled for vectorization to enhance alignment
test.c:6:3: note: loop turned into non-loop; it never loops.
test.c:6:3: note: loop with 3 iterations completely unrolled
test.c:1:6: note: loop turned into non-loop; it never loops.
test.c:1:6: note: loop with 4 iterations completely unrolled



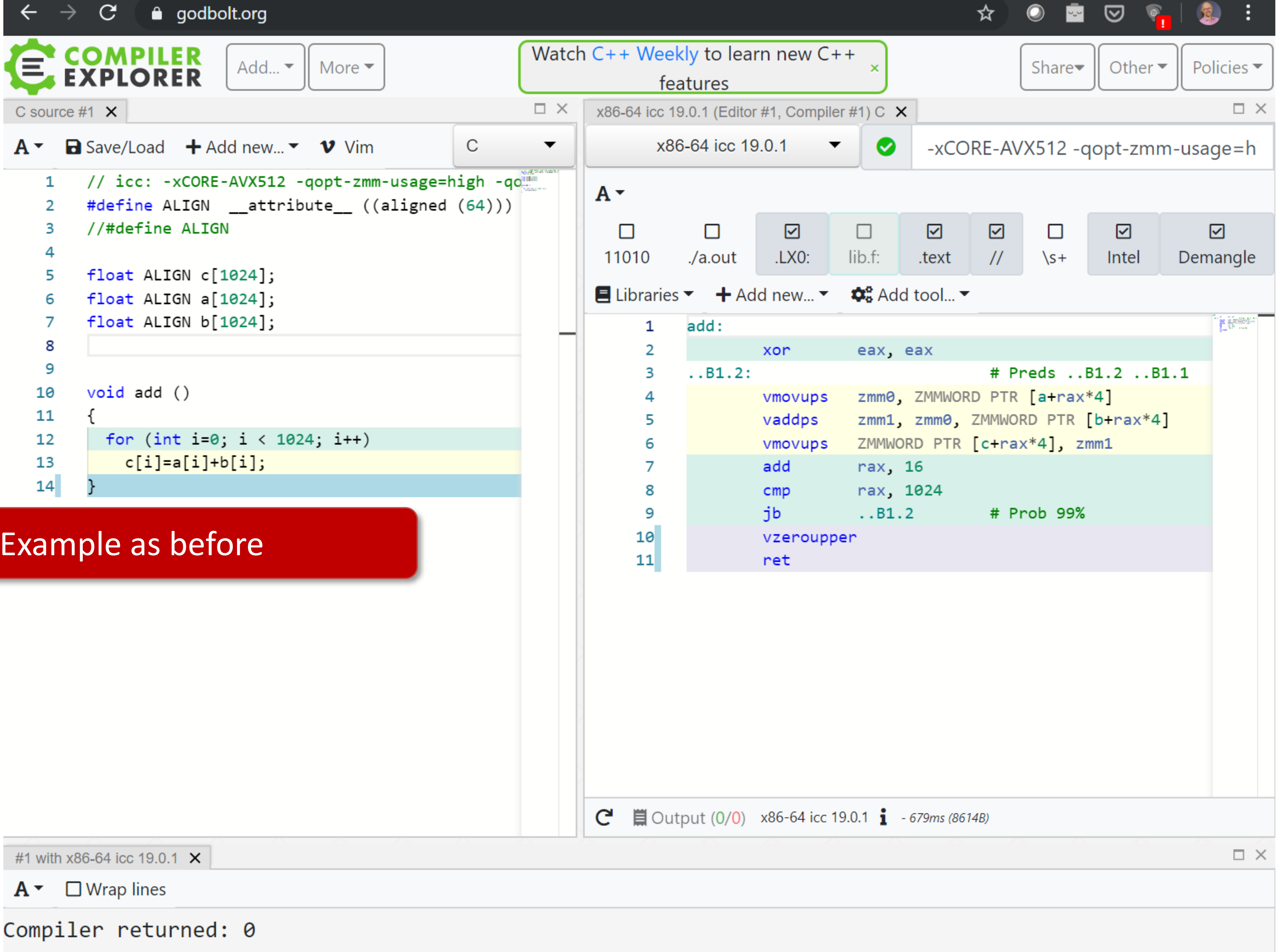
If the pointers might be aliases:

gcc reports:
test.c:6:3: note: loop vectorized
test.c:6:3: note: loop versioned for vectorization because of possible aliasing
test.c:6:3: note: loop peeled for vectorization to enhance alignment
test.c:6:3: note: loop turned into non-loop; it never loops.
test.c:6:3: note: loop with 3 iterations completely unrolled
test.c:1:6: note: loop turned into non-loop; it never loops.
test.c:1:6: note: loop with 3 iterations completely unrolled

What do we see?

- Actually exploiting vectorisation is a bit tricky even when the dependence analysis is easy

- In the following slides we start with an easily-vectorizable example
- And look at some of the things that make it complicated



Example as before

```c
// icc: -xCORE-AVX512 -qopt-zmm-usage=high -qo
#define ALIGN  __attribute__ ((aligned (64)))
//#define ALIGN

float ALIGN c[1024];
float ALIGN a[1024];
float ALIGN b[1024];
int ALIGN ind[1024];

void add ()
{
  for (int i=0; i < 1024; i++)
    c[i]=a[i]+b[ind[i]];
}
```

x86-64 icc 19.0.1 (Editor #1, Compiler #1) C — -xCORE-AVX512 -qopt-zmm-usage=h

```asm
add:
        xor       eax, eax
..B1.2:                         # Preds ..B1.2 ..B1.1
        vmovups   zmm0, ZMMWORD PTR [ind+rax*4]
        vpcmpeqb  k1, xmm0, xmm0
        vpxord    zmm1, zmm1, zmm1
        vgatherdps zmm1{k1}, DWORD PTR [b+zmm0*4]
        vaddps    zmm2, zmm1, ZMMWORD PTR [a+rax*4]
        vmovups   ZMMWORD PTR [c+rax*4], zmm2
        add       rax, 16
        cmp       rax, 1024
        jb        ..B1.2        # Prob 99%
        vzeroupper
        ret
```

Output (0/0) x86-64 icc 19.0.1 - 946ms (9359B)

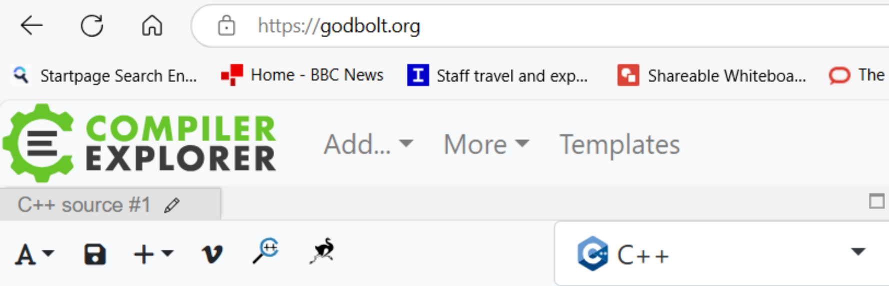
Indirection: b[ind[]]

We have a register containing a vector of pointers

We need a "gather" instruction:

- A vector load
- That loads from a different address in each lane

(how can this be implemented efficiently??)




```
// icx -Ofast -march=znver4 -qopt-report
#define SIZE 10240
#define ALIGN __attribute__ ((aligned(64)))

int ALIGN A[SIZE];
int ALIGN ind[SIZE];
int ALIGN C[SIZE];
int ALIGN D[SIZE];

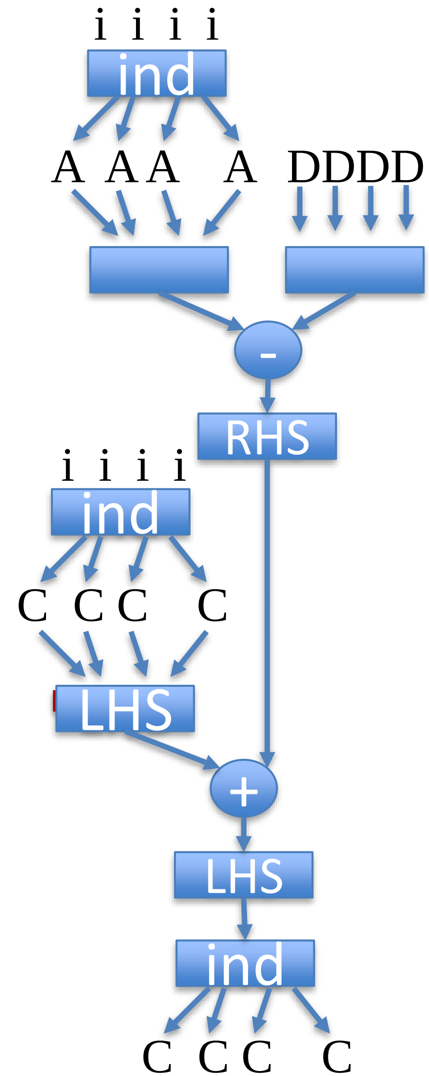
#define IB 32
#define JB 32

void P()
{
  int i, j;

  for (i=0; i<SIZE; i++) {
    C[ind[i]] += A[ind[i]] - D[i];
  }
}
```

Incrementing through indirection: ind[i]

1. Load a vector ind[i:i+16]
2. Gather a vector A[ind[i:i+16]]
3. Subtract the D[i] values:
4. RHS[0:16]=A[ind[i:i+16]] – D[i:i+16]
5. Gather the LHS[0:16] = C[ind[i:i+16]]
6. Add (+=): LHS[0:16] += RHS[0:16]
7. Scatter: C[ind[i:i+16]] = LHS[0:16]



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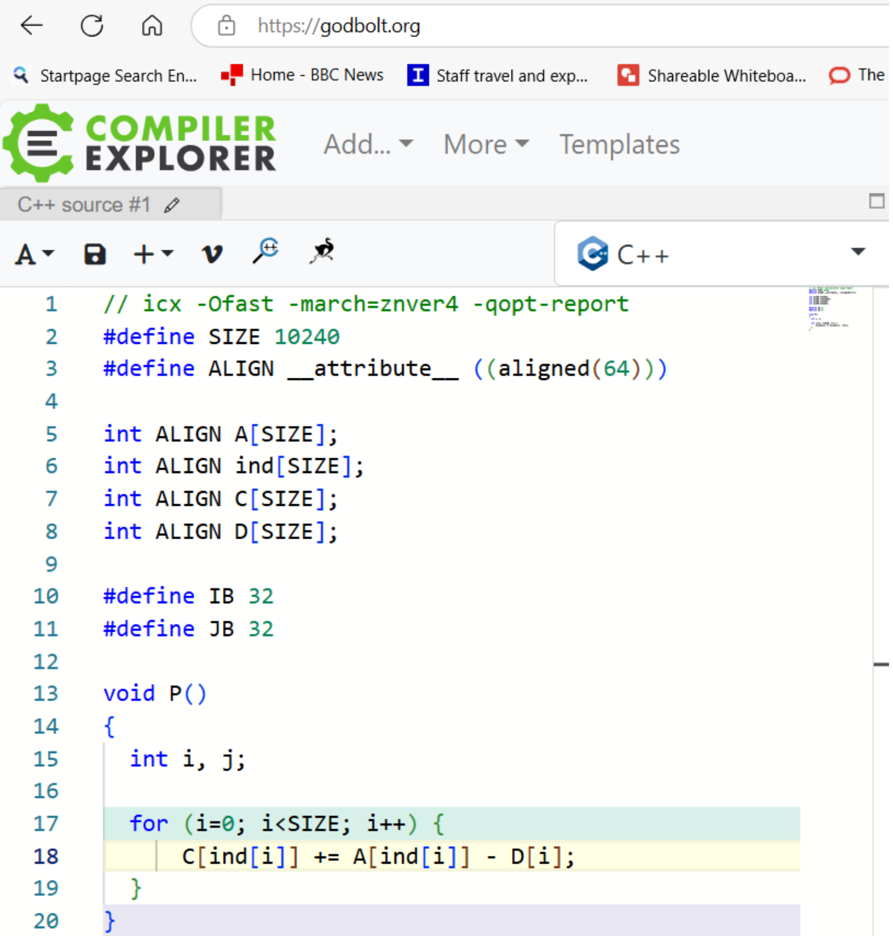


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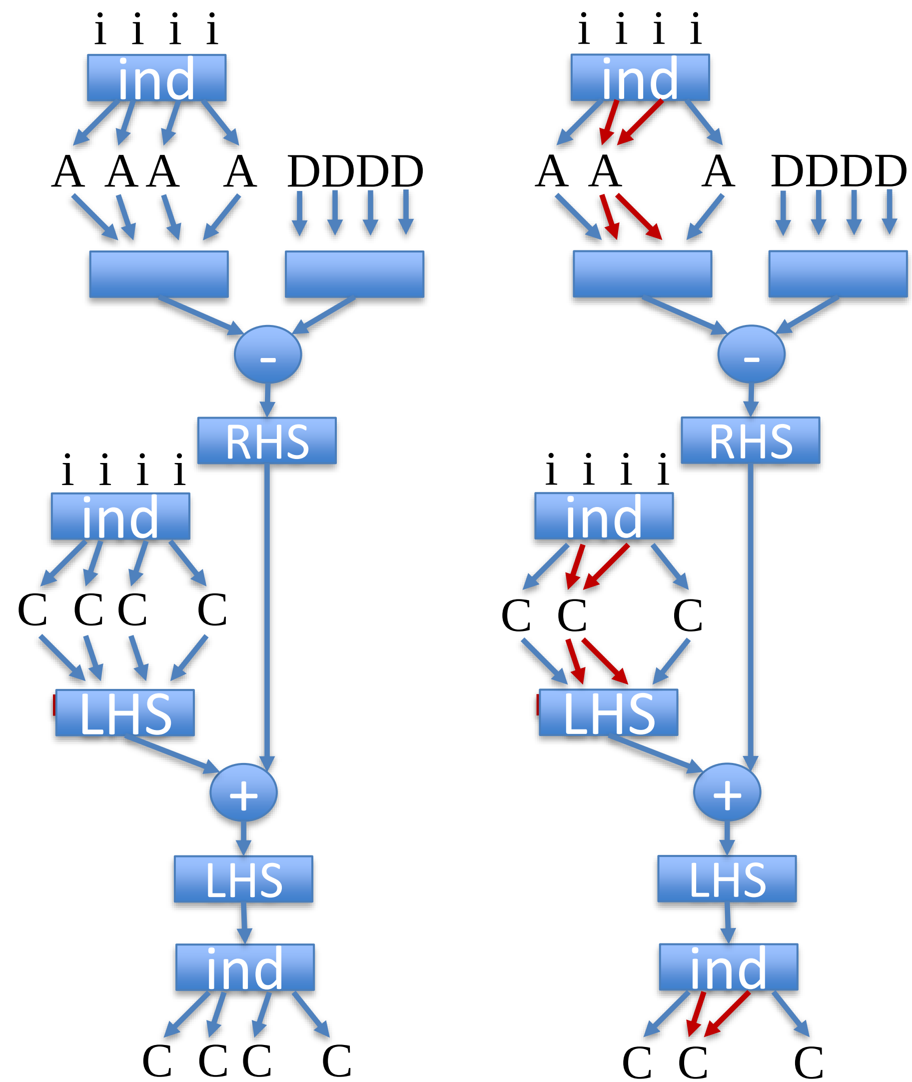
```

What would happen if there were duplicate indices in ind?



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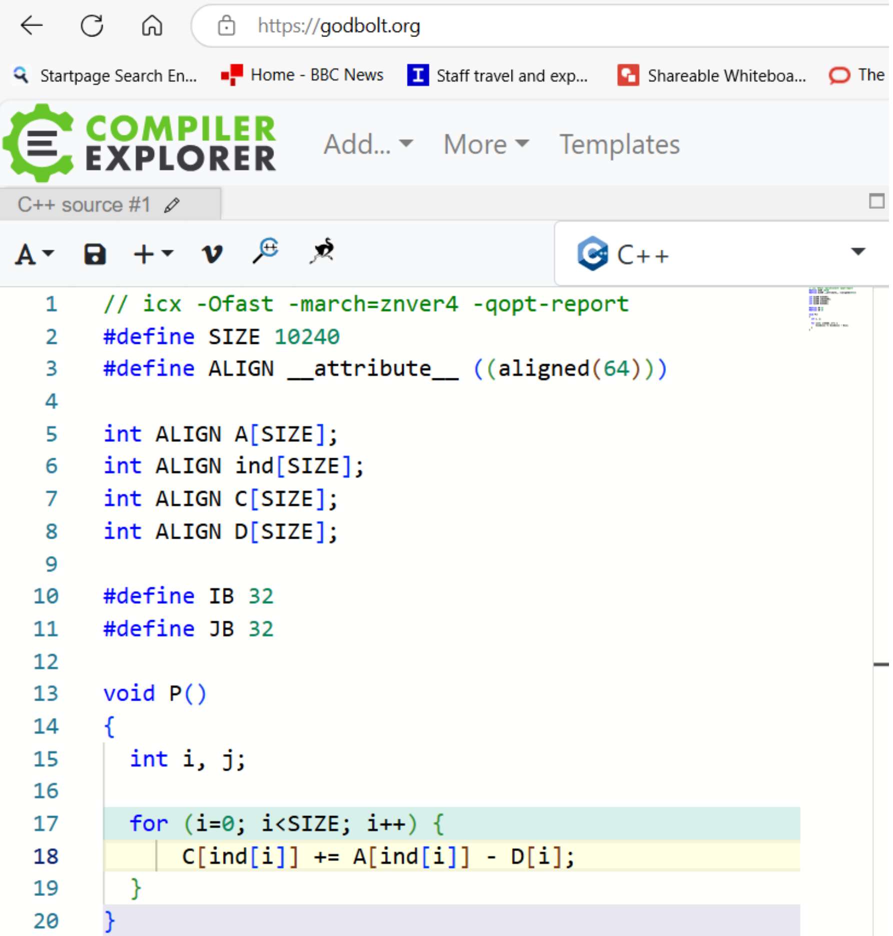

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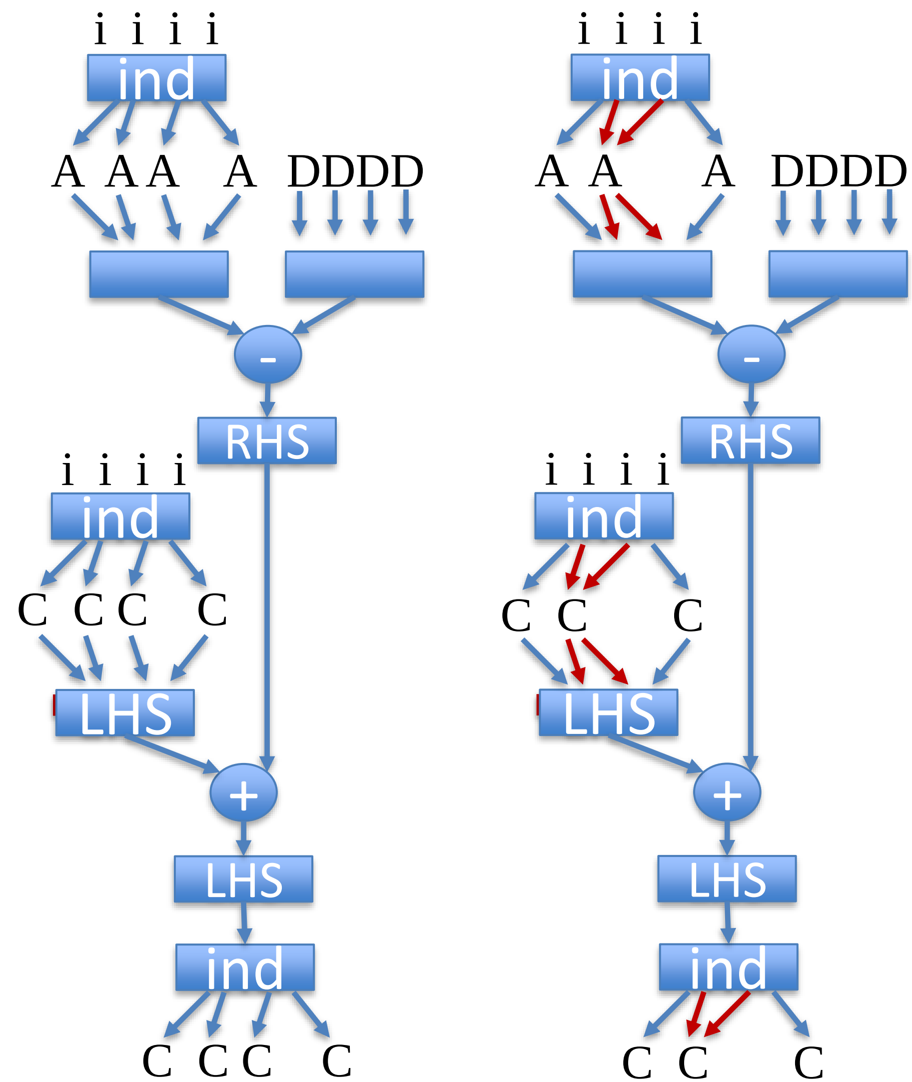

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It's not parallel! We have to sum two (or more) different values into the same C element

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https://godbolt.org

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C++ source #1   C++

```cpp
// icx -Ofast -march=znver4 -qopt-report
#define SIZE 10240
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int ALIGN A[SIZE];
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void P()
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  int i, j;

  for (i=0; i<SIZE; i++) {
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5. Gather the LHS[0:16] = C[ind[i:i+16]]
6. Add (+=): LHS[0:16] += RHS[0:16]
7. Scatter: C[ind[i:i+16]] = LHS[0:16]

```asm
.LCPI0_1:
        .long   63
P():
        vpbroadcastd    ymm0, dword ptr [rip + .LCPI0_0]
        vpbroadcastd    ymm1, dword ptr [rip + .LCPI0_1]
        vpcmpeqd        ymm2, ymm2, ymm2
        xor     eax, eax
        jmp     .LBB0_1
.LBB0_7:
        vpcmpeqb        k1, xmm0, xmm0
        vpxor   xmm6, xmm6, xmm6
        vpmovqd ymm3, zmm3
        lea     rcx, [rax + 8]
        add     rax, 16
        vpgatherdd      ymm6 {k1}, ymmword ptr [4*ymm4 + C]
        vpcmpeqb        k1, xmm0, xmm0
        vpaddd  ymm4, ymm6, ymm5
        vpscatterdd     ymmword ptr [4*ymm3 + C] {k1}, ymm4
        cmp     rcx, 10232
        jae     .LBB0_8
.LBB0_1:
        vmovdqu ymm4, ymmword ptr [4*rax + ind]
        vpcmpeqb        k1, xmm0, xmm0
        vpxor   xmm5, xmm5, xmm5
        vpmovsxdq       zmm3, ymm4
        vpgatherdd      ymm5 {k1}, ymmword ptr [4*ymm4 + A]
        vpsubd  ymm5, ymm5, ymmword ptr [4*rax + D]
        vpconflictq     zmm6, zmm3
        vplzcntq        zmm6, zmm6
        vpmovqd ymm6, zmm6
        vpxor   ymm7, ymm6, ymm0
        vptest  ymm7, ymm7
        je      .LBB0_4
        vpcmpneqd       k1, ymm6, ymm0
        vpsubd  ymm6, ymm1, ymm6
.LBB0_3:
        vpermd  ymm7, ymm6, ymm5
        vpermd  ymm6 {k1}, ymm6, ymm6
        vpaddd  ymm5 {k1}, ymm5, ymm7
        vpcmpneqd       k1, ymm6, ymm2
        vptest  ymm6, ymm2
        jae     .LBB0_3
.LBB0_4:
        vpcmpeqb        k1, xmm0, xmm0
        vpxor   xmm6, xmm6, xmm6
        vpmovqd ymm3, zmm3
        vpgatherdd      ymm6 {k1}, ymmword ptr [4*ymm4 + C]
        vpcmpeqb        k1, xmm0, xmm0
        vpaddd  ymm4, ymm6, ymm5
        vpxor   xmm5, xmm5, xmm5
        vpscatterdd     ymmword ptr [4*ymm3 + C] {k1}, ymm4
        vpcmpeqb        k1, xmm0, xmm0
        vmovdqu ymm4, ymmword ptr [4*rax + ind+32]
        vpmovsxdq       zmm3, ymm4
        vpgatherdd      ymm5 {k1}, ymmword ptr [4*ymm4 + A]
        vpsubd  ymm5, ymm5, ymmword ptr [4*rax + D+32]
        vpconflictq     zmm6, zmm3
        vplzcntq        zmm6, zmm6
        vpmovqd ymm6, zmm6
        vpxor   ymm7, ymm6, ymm0
        vptest  ymm7, ymm7
        je      .LBB0_7
        vpcmpneqd       k1, ymm6, ymm0
        vpsubd  ymm6, ymm1, ymm6
.LBB0_6:
        vpermd  ymm7, ymm6, ymm5
        vpermd  ymm6 {k1}, ymm6, ymm6
        vpaddd  ymm5 {k1}, ymm5, ymm7
        vpcmpneqd       k1, ymm6, ymm2
        vptest  ymm6, ymm2
        jae     .LBB0_6
        jmp     .LBB0_7
.LBB0_8:
        vzeroupper
        ret
```

Skipped

RHS

If no conflicts

Add conflicting lanes' value sequentially

Scatter back

Unrolled copy

vpconflictq instruction checks for duplicate values in ind[i:i+16]

If found, we branch to a loop over each distinct value

Roughly…

Not examinable

This is addressed by AVX512 "conflict detect" instructions which enable us to catch duplicates and serialise where needed

Health warning

- Automatic discovery of parallelism has a bad reputation
  - Deservedly! It looks great on simple examples
  - But real code has complexity that means it often just doesn't happen
- But in some application domains it can really work
- And some programming languages make it easier, maybe!
  - Functional languages lack anti- and output-dependences (but tend to add higher-order functions and lazy evaluation)
  - Some languages control pointer ownership and aliasing
  - Some programming models discourage explicit loops and explicit elementwise subscripting

So: we need a compiler algorithm to determine whether a loop is parallel…

Dependence

How?

- Define:
  - IN(S): set of memory locns which might be read by some execn of statement S
  - OUT(S): set of memory locns which might be written by some execn of statement S
- Reordering is constrained by dependences;
- There are four types:
  - Data ("true") dependence: S1 δ S2
    - OUT(S1) ∩ IN(S2)
  - Anti dependence: S1 $\bar{\delta}$ S2
    - IN(S1) ∩ OUT(S2)
  - Output dependence: S1 δ^o S2
    - OUT(S1) ∩ OUT(S2)
  - Control dependence: S1 δ^c S2

("S1 must write something before S2 can read it")

("S1 must read something before S2 overwrites it")

("If S1 and S2 might both write to a location, S2 must write after S1")

("S1 determines whether S2 should execute")

- These are static analogues of the dynamic RAW, WAR, WAW and control hazards which have to be considered in processor architecture

Loop-carried dependences

- **Recall:**

```
S1 :  A[0] := 0
      for I = 1 to 8
S2 :   A[I] := A[I-1] + B[I]
```

- **What does this loop do?**

B:		**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
A:	**0**								

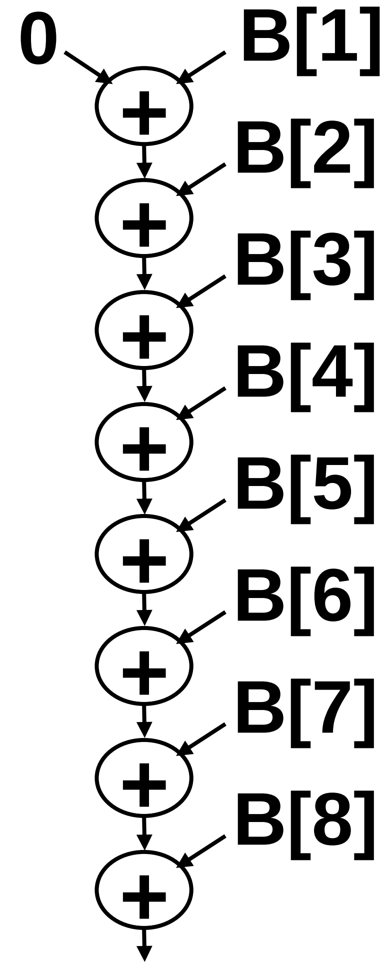
Loop-carried dependences

- **Recall:**

S1 : A[0] := 0

for I = 1 to 8

S2 : A[I] := A[I-1] + B[I]



- **What does this loop do?**

B:		**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
A:	**0**	**1**	**2**						

- **In this case, there is a data dependence**
  - **This is a loop-carried dependence - the dependence spans a loop iteration**
  - **This loop is inherently sequential**

Loop-carried dependences

- **Recall:**

S1 : A[0] := 0

for I = 1 to 8

S2 : A[I] := A[I-1] + B[I]

- **Loop carried:**

S2[1] : A[1] := A[0] + B[1]

S2[2] : A[2] := A[1] + B[2]

S2[3] : A[3] := A[2] + B[3]

S2[4] : A[4] := A[3] + B[4]

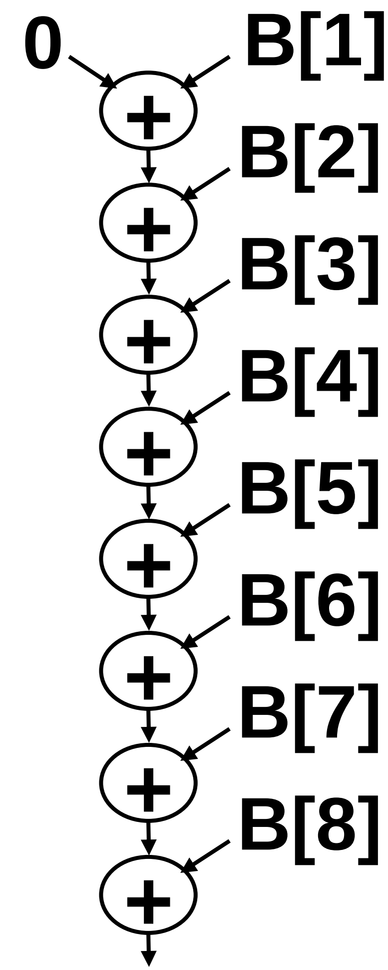
S2[5] : A[5] := A[4] + B[5]

S2[6] : A[6] := A[5] + B[6]

S2[7] : A[7] := A[6] + B[7]

S2[8] : A[8] := A[7] + B[8]

Dependences cross, from one iteration to next



What is a loop-carried dependence?

- Consider two iterations I^1 and I^2
- A dependence occurs between two statements S_p and S_q (not necessarily distinct), when an assignment in S_p^{I1} refers to the same location as a use in S_q^{I2}

- In the example,

```
S1 : A[0] := 0
     for I = 1 to 5
S2 :   A[I] := A[I-1] + B[I]
```

- The assignment is "$A[I^1]$:= ..."
- The use is "... := $A[I^2-1]$ ..."
- These refer to the same location when $I^1 = I^2-1$
- Thus $I^1 < I^2$, ie the assignment is in an earlier iteration

Notation: $S_2\ \delta_<\ S_2$

Definition: The dependence equation

- A dependence occurs
  - between two statements S_p and S_q (not necessarily distinct),
  - when there exists a pair of loop iterations I^1 and I^2,
  - such that a memory reference in S_p in I^1 may refer to the same location as a memory reference in S_q in I^2.

- This might occur if S_p and S_q refer to some common array A
- Suppose S_p refers to $A[\varphi_p(I)]$
- Suppose S_q refers to $A[\varphi_q(I)]$

($\varphi_p(I)$ is some subscript expression involving I)

- A dependence of some kind occurs between S_p and S_q if there exists a solution to the equation

$$\varphi_p(I^1) = \varphi_q(I^2)$$

- for integer values of I^1 and I^2 lying within the loop bounds

Types of dependence

- If a solution to the dependence equation exists, a dependence of some kind occurs

- The dependence type depends on what solutions exist

- The solutions consist of a set of pairs (I^1, I^2)

- We would appear to have a *data* dependence if

  $A[\phi_p(I)] \in OUT(S_p)$

  and

  $A[\phi_q(I)] \in IN(S_q)$

- But we only really have a data dependence if the assignments *precede* the uses, ie

  - $S_p \; \delta_< \; S_q$
  - if, for each solution pair (I^1, I^2), $I^1 < I^2$

Dependence versus anti-dependence

- If the *uses* precede the *assignments*, we actually have an *anti*-dependence, ie

$$S_p \ \overline{\delta}_< \ S_q$$

if, for each solution pair $(\mathbf{I}^1, \mathbf{I}^2)$, $\mathbf{I}^1 > \mathbf{I}^2$

- In this case we do have a constraint on execution order
- Because we (may) have to read a value before it (may) be overwritten
- And this anti-dependence is loop-carried
- Anti-dependences prevent re-ordering, and multi-thread parallelism

Dependence versus anti-dependence

- If there are some solution pairs (I^1,I^2) with $I^1 < I^2$ and some with $I^1 > I^2$, we write

$$S_p \;\delta_* \;S_q$$

  This represents that we know we must respect execution ordering, even though the compiler is unable to classify the dependence fully

- If, for all solution pairs (I^1,I^2), $I^1 = I^2$, there are dependences *within* an iteration of the loop, but there are no loop-carried dependences:

$$S_p \;\delta_= \;S_q$$

Dependence distance

In many common examples, the set of solution pairs is characterised easily:

- **Definition:** dependence distance
  - **If, for all solution pairs (I^1, I^2),**

    $I^1 = I^2 - k$

    then the dependence distance is k
- **For example in the loop we considered earlier,**

$$
\begin{array}{ll}
S_1: & \texttt{A[0] := 0} \\
 & \texttt{for I = 1 to 5} \\
S_2: & \quad \texttt{A[I] := A[I-1] + B[I]}
\end{array}
$$

We find that $S_2 \; \delta_{<} \; S_2$ with dependence distance 1.

- ***((of course there are many cases where the difference is not constant and so the dependence cannot be summarised this way)).***

Reuse distance

- **When optimising for cache performance, it is sometimes useful to consider the re-use relationship,**
  - $IN(S_1) \cap IN(S_2)$
- **Here there is no dependence - it doesn't matter which read occurs first**
- **Nonetheless, cache performance can be improved by minimising the *reuse distance***
- **The reuse distance is calculated essentially the same way**
- **Eg**

```
    for I = 5 to 100
S1:  B[I] := A[I] * 2
S2:  C[I] := A[I-5] * 10
```

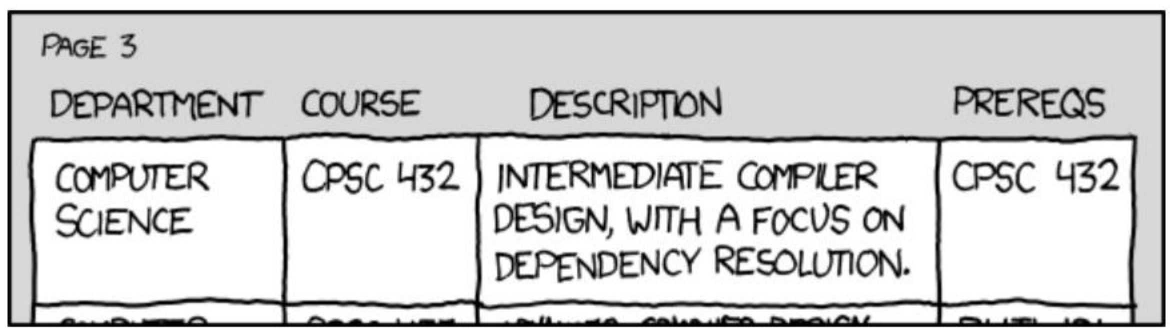
- **Here we have a loop-carried reuse with distance 5**

Compilers - Chapter 8:

Loop scheduling optimisations

Part 3: Dependence analysis in nested loops

- Lecturer:
  - Paul Kelly (p.kelly@imperial.ac.uk)



(It turns out that this cartoon is good for almost any compilers topic, definitely this one) https://xkcd.com/754/

Nested loops

- Up to now we have looked at single loops
- Now let's generalise to loop "nests"
- We begin by considering a very common dependence pattern, called the "wavefront":

$$\texttt{for } I_1 \texttt{ = 0 to 3 do}$$
$$\quad\texttt{for } I_2 \texttt{ = 0 to 3 do}$$
$$S: \quad \texttt{A[}I_1\texttt{,}I_2\texttt{] := A[}I_1 - 1\texttt{,}I_2\texttt{] + A[}I_1\texttt{,}I_2 - 1\texttt{]}$$

- Dependence structure?

Nested loops

- Up to now we have looked at single loops
- Now let's generalise to loop "nests"
- We begin by considering a very common dependence pattern, called the "wavefront":

```
for I = 0 to 3 do
  for J = 0 to 3 do
 S: A[I,J] = A[I-1,J] + A[I,J-1]
```

- Dependence structure?

$$\begin{bmatrix} I \text{ is } I_1 \\ J \text{ is } I_2 \end{bmatrix}$$

System of dependence equations

- Consider the dependence equations for this loop nest:

```
for I1 = 0 to 3 do
  for I2 = 0 to 3 do
S :   A[I1,I2] := A[I1 - 1,I2] + A[I1,I2 - 1]
```

- There are two potential dependences arising from the three references to A, so two systems of dependence equations to solve:

1. Between $A[I_1^1, I_2^1]$ and $A[I_1^2 - 1, I_2^2]$:

$$\begin{cases} I_1^1 = I_1^2 - 1 \\ I_2^1 = I_2^2 \end{cases}$$

2. Between $A[I_1^1, I_2^1]$ and $A[I_1^2, I_2^2 - 1]$:

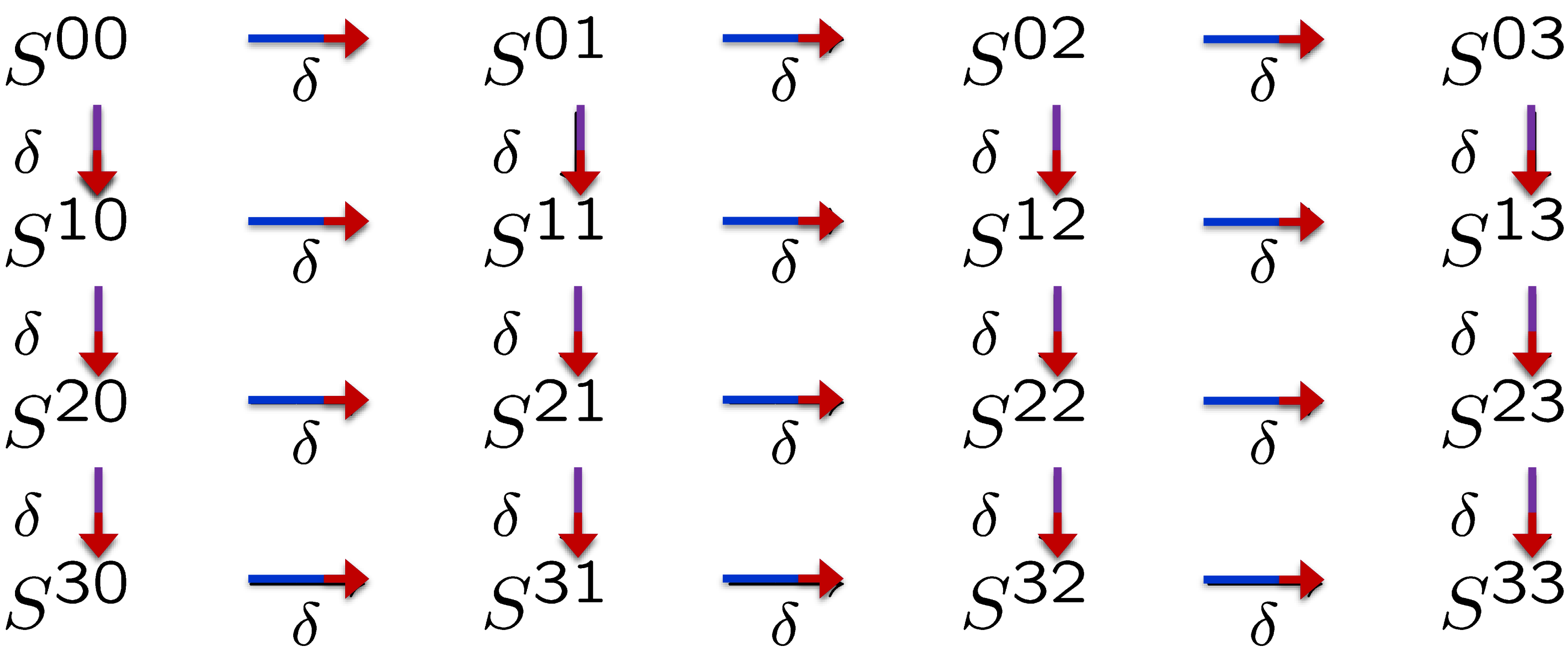
$$\begin{cases} I_1^1 = I_1^2 \\ I_2^1 = I_2^2 - 1 \end{cases}$$

Iteration space graph

- **The same loop:**

$$\texttt{for } I_1 = \texttt{0 to 3 do}$$
$$\texttt{for } I_2 = \texttt{0 to 3 do}$$
$$S: \quad \texttt{A}[I_1, I_2] := \texttt{A}[I_1 - 1, I_2] + \texttt{A}[I_1, I_2 - 1]$$

- **For humans the easy way to understand this loop nest is to draw the *iteration space graph* showing the iteration-to-iteration dependences:**



- **The diagram shows an arrow for each solution of each dependence equation.**

Iteration space graph

- **The same loop:**

$$\texttt{for } I_1 = \texttt{0 to 3 do}$$
$$\quad \texttt{for } I_2 = \texttt{0 to 3 do}$$
$$S: \quad \texttt{A}[I_1, I_2] := \texttt{A}[I_1 - 1, I_2] + \texttt{A}[I_1, I_2 - 1]$$

- **For humans the easy way to understand this loop nest is to draw the *iteration space graph* showing the iteration-to-iteration dependences:**

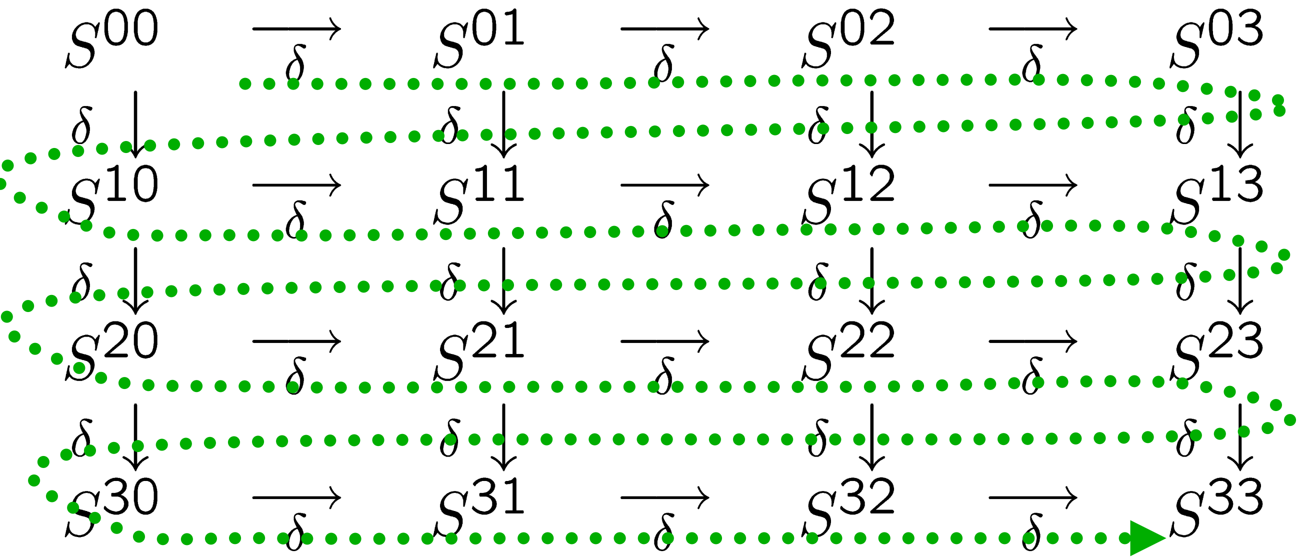
$$\begin{array}{ccccccc}
S^{00} & \xrightarrow[\delta]{} & S^{01} & \xrightarrow[\delta]{} & S^{02} & \xrightarrow[\delta]{} & S^{03} \\
\delta \downarrow & & \delta \downarrow & & \delta \downarrow & & \delta \downarrow \\
S^{10} & \xrightarrow[\delta]{} & S^{11} & \xrightarrow[\delta]{} & S^{12} & \xrightarrow[\delta]{} & S^{13} \\
\delta \downarrow & & \delta \downarrow & & \delta \downarrow & & \delta \downarrow \\
S^{20} & \xrightarrow[\delta]{} & S^{21} & \xrightarrow[\delta]{} & S^{22} & \xrightarrow[\delta]{} & S^{23} \\
\delta \downarrow & & \delta \downarrow & & \delta \downarrow & & \delta \downarrow \\
S^{30} & \xrightarrow[\delta]{} & S^{31} & \xrightarrow[\delta]{} & S^{32} & \xrightarrow[\delta]{} & S^{33}
\end{array}$$

- **The diagram shows an arrow for each solution of each dependence equation. Is there any parallelism?**

$$
\begin{array}{ccccccc}
S^{00} & \xrightarrow[\delta]{} & S^{01} & \xrightarrow[\delta]{} & S^{02} & \xrightarrow[\delta]{} & S^{03} \\
\delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow \\
S^{10} & \xrightarrow[\delta]{} & S^{11} & \xrightarrow[\delta]{} & S^{12} & \xrightarrow[\delta]{} & S^{13} \\
\delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow \\
S^{20} & \xrightarrow[\delta]{} & S^{21} & \xrightarrow[\delta]{} & S^{22} & \xrightarrow[\delta]{} & S^{23} \\
\delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow \\
S^{30} & \xrightarrow[\delta]{} & S^{31} & \xrightarrow[\delta]{} & S^{32} & \xrightarrow[\delta]{} & S^{33}
\end{array}
$$

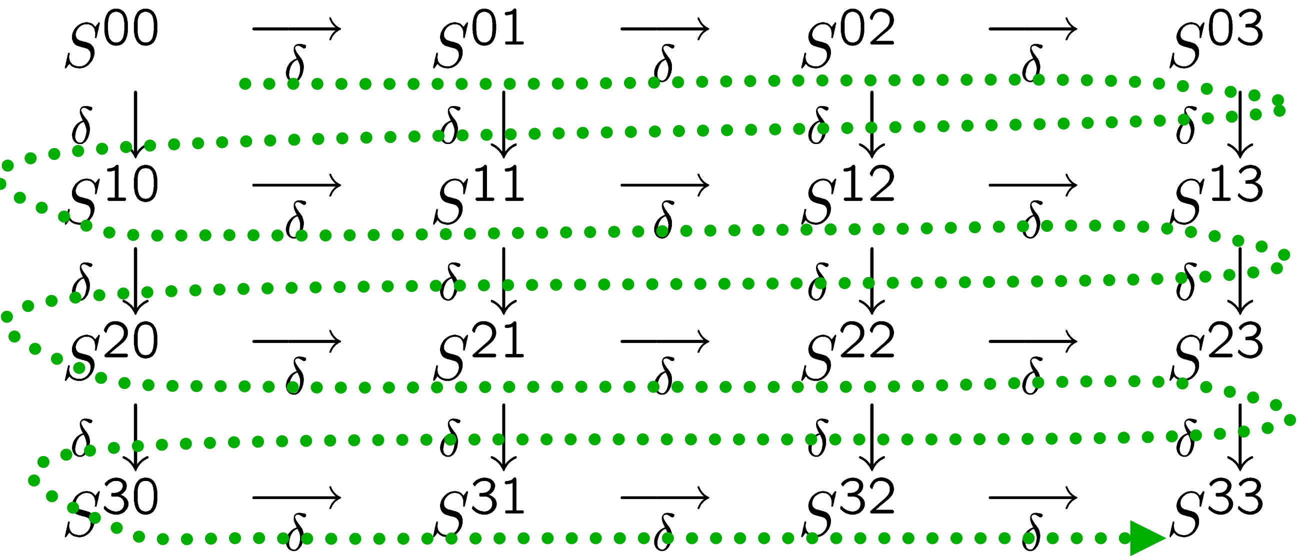
```
for I1 = 0 to 3 do
  for I2 = 0 to 3 do
S :   A[I1,I2] := A[I1-1,I2] + A[I1,I2-1]
```

- **The inner loop is not vectorisable since there is a dependence chain linking successive iterations.**
  - **(to use a vector instruction, need to be able to operate on each element of the vector in parallel)**
- **Similarly, the outer loop is not parallel**




```
for I1 = 0 to 3 do
  for I2 = 0 to 3 do
S :   A[I1,I2] := A[I1-1,I2] + A[I1,I2-1]
```

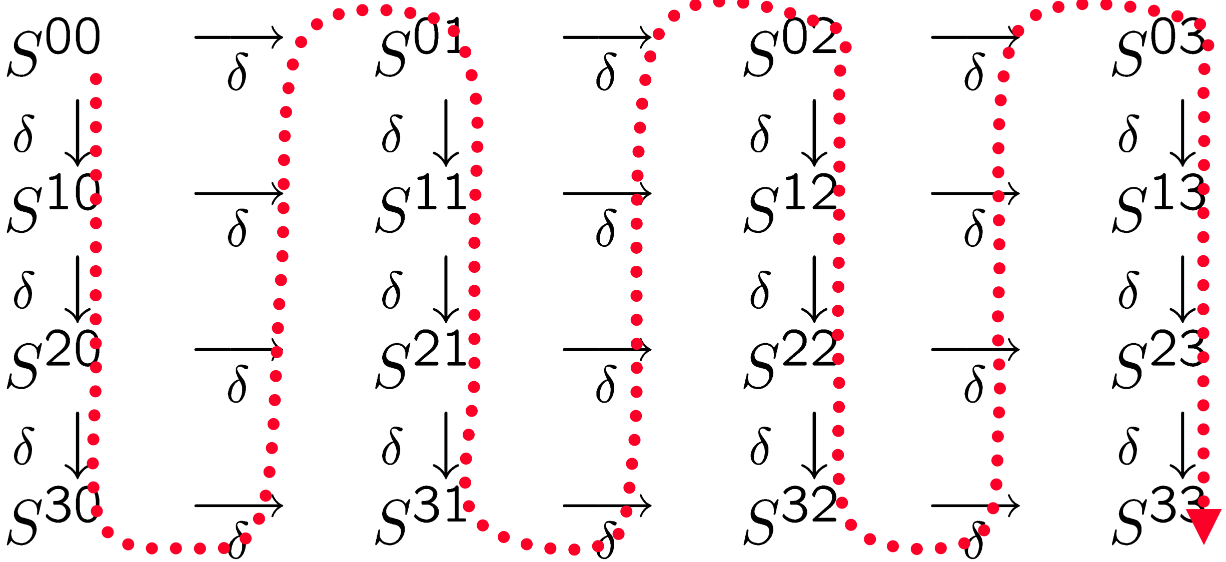
- **The inner loop is not vectorisable since there is a dependence chain linking successive iterations.**
  - **(to use a vector instruction, need to be able to operate on each element of the vector in parallel)**
- **Similarly, the outer loop is not parallel**




```
for I1 = 0 to 3 do
  for I2 = 0 to 3 do
S :   A[I1,I2] := A[I1 − 1,I2] + A[I1,I2 − 1]
```

- **The inner loop is not vectorisable since there is a dependence chain linking successive iterations.**
  - **(to use a vector instruction, need to be able to operate on each element of the vector in parallel)**
- **Similarly, the outer loop is not parallel**

- **This loop nest has two dependence distance vectors:**
  - **(1,0) carried by the outer loop** ↓ **Direction vector: (<,=)**
  - **(0,1) carried by the inner loop** → **Direction vector: (=,>)**




```
for I1 = 0 to 3 do
  for I2 = 0 to 3 do
S:    A[I1,I2] := A[I1-1,I2] + A[I1,I2-1]
```

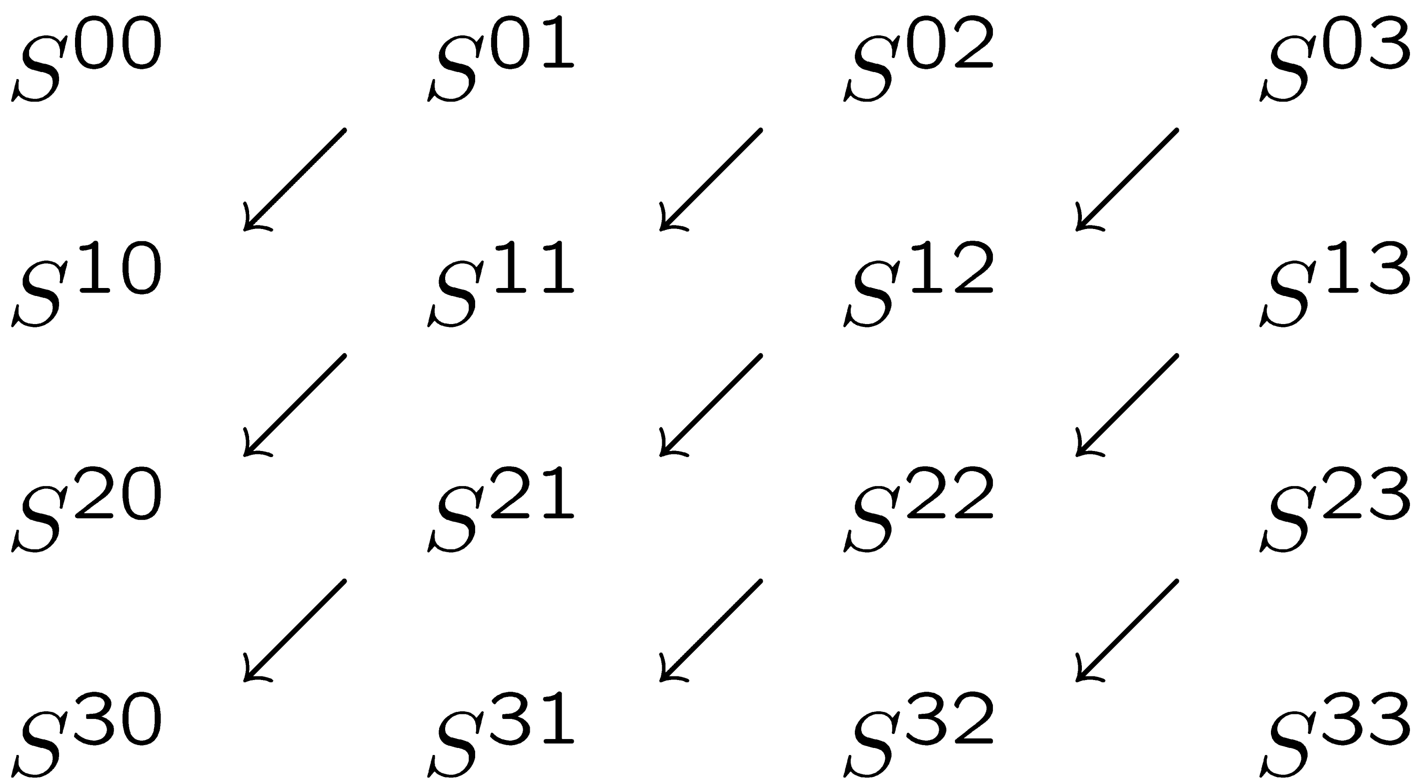
- **The inner loop is not vectorisable since there is a dependence chain linking successive iterations.**
  - **(to use a vector instruction, need to be able to operate on each element of the vector in parallel)**
- **Similarly, the outer loop is not parallel**
- **This loop is *interchangeable*: the top-to-bottom, left-to-right execution order is also valid since all dependence constraints (as shown by the arrows) are still satisfied.**
- **Interchanging the loop does not improve vectorisability or parallelisability**

Interchange: counter-example

```
for I₁ = 0 to 3 do
  for I₂ = 0 to 3 do
    A[I₁,I₂] := A[I₁ - 1, I₂ + 1] + B[I₁, I₂]
```

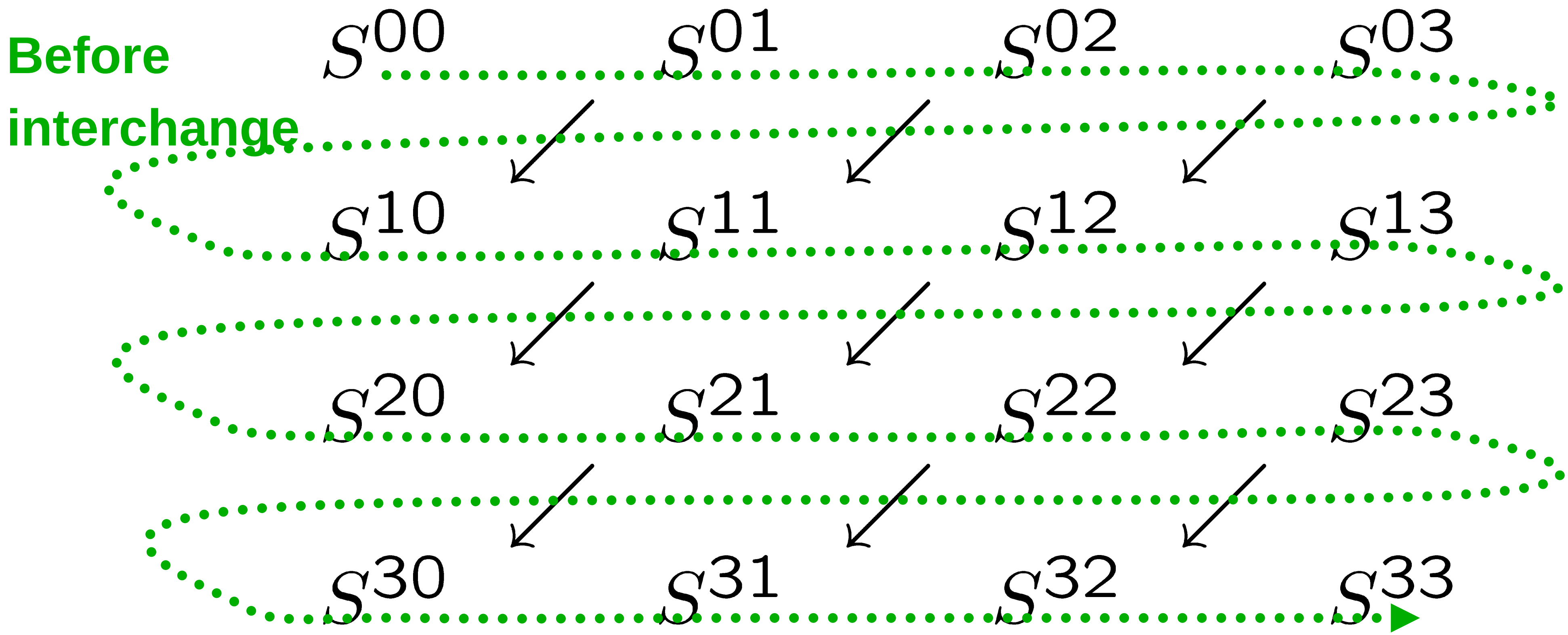
Interchange: counter-example

```
for I1 = 0 to 3 do
   for I2 = 0 to 3 do
      A[I1,I2] := A[I1 - 1, I2 + 1] + B[I1, I2]
```



Interchange: counter-example

```
for I1 = 0 to 3 do
   for I2 = 0 to 3 do
      A[I1,I2] := A[I1 - 1, I2 + 1] + B[I1, I2]
```

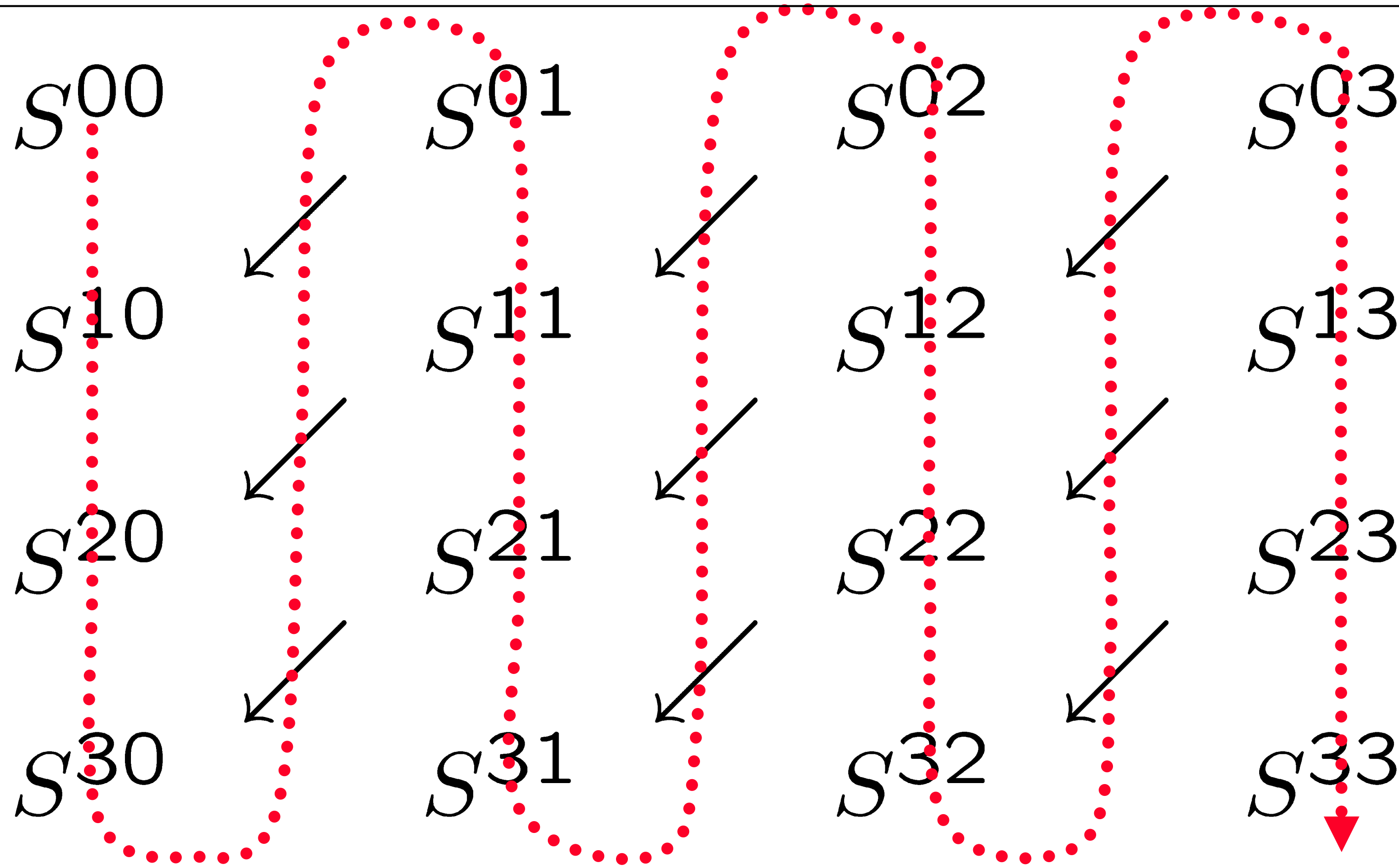


Interchange: counter-example

```
for I1 = 0 to 3 do
   for I2 = 0 to 3 do
      A[I1,I2] := A[I1 - 1, I2 + 1] + B[I1, I2]
```

After interchange:

New traversal order crosses dependence arrows backwards



Interchange: condition

- **A loop is *interchangeable* if all dependence constraints (as shown by the arrows) are still satisfied by the top-to-bottom, left-to-right execution order**

- **How can you tell whether a loop can be interchanged?**

- **Look at its dependence direction vectors:**
  - **Is there a dependence direction vector with the form (<,>) ?**
    - **ie there is a dependence distance vector (k_1,k_2) with k_1>0 and k_2<0 ?**
    - **If so, interchange would be** *invalid*
  - **Because the arrows would be traversed backwards**
  - **All other dependence directions are OK.**

Skewing

- Consider this variation on the wavefront loop:

```
for k1 := 0 to 3 do
   for k2 := k1 to k1+3 do
S : A[k1,k2-k1] := A[k1-1,k2-k1]+A[k1,k2-k1-1]
```

- The inner loop's control variable runs from k_1 to k_1+3.
- The iteration space of this loop has 4^2 iterations just like the original loop.
- If we draw the iteration space with each iteration S^{K_1,K_2} at coordinate position (K_1,K_2), it is skewed to form a lozenge shape:

S^{00}	S^{01}	S^{02}	S^{03}			
	S^{11}	S^{12}	S^{13}	S^{14}		
		S^{22}	S^{23}	S^{24}	S^{25}	
			S^{33}	S^{34}	S^{35}	S^{36}

This loop performs the same computation as the original.

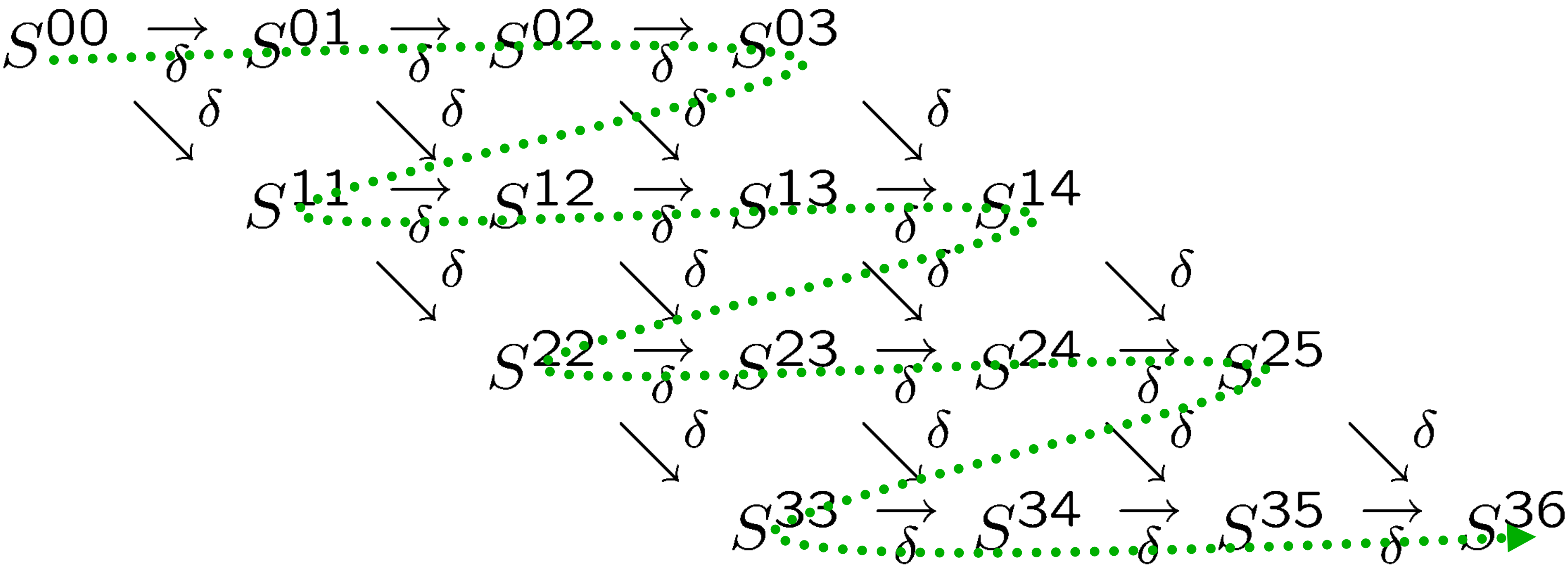
- Thus the dependence structure of the skewed loop is shown by marking the iteration space with all the dependences:

$$
\begin{array}{ccccccccccccc}
S^{00} & \xrightarrow[\delta]{} & S^{01} & \xrightarrow[\delta]{} & S^{02} & \xrightarrow[\delta]{} & S^{03} & & & & & & \\
& \searrow^{\delta} & & \searrow^{\delta} & & \searrow^{\delta} & & \searrow^{\delta} & & & & & \\
& & S^{11} & \xrightarrow[\delta]{} & S^{12} & \xrightarrow[\delta]{} & S^{13} & \xrightarrow[\delta]{} & S^{14} & & & & \\
& & & \searrow^{\delta} & & \searrow^{\delta} & & \searrow^{\delta} & & \searrow^{\delta} & & & \\
& & & & S^{22} & \xrightarrow[\delta]{} & S^{23} & \xrightarrow[\delta]{} & S^{24} & \xrightarrow[\delta]{} & S^{25} & & \\
& & & & & \searrow^{\delta} & & \searrow^{\delta} & & \searrow^{\delta} & & \searrow^{\delta} & \\
& & & & & & S^{33} & \xrightarrow[\delta]{} & S^{34} & \xrightarrow[\delta]{} & S^{35} & \xrightarrow[\delta]{} & S^{36}
\end{array}
$$

- Can this loop nest be vectorised?
- Can this loop nest be interchanged?

Skewing changes effect of interchange

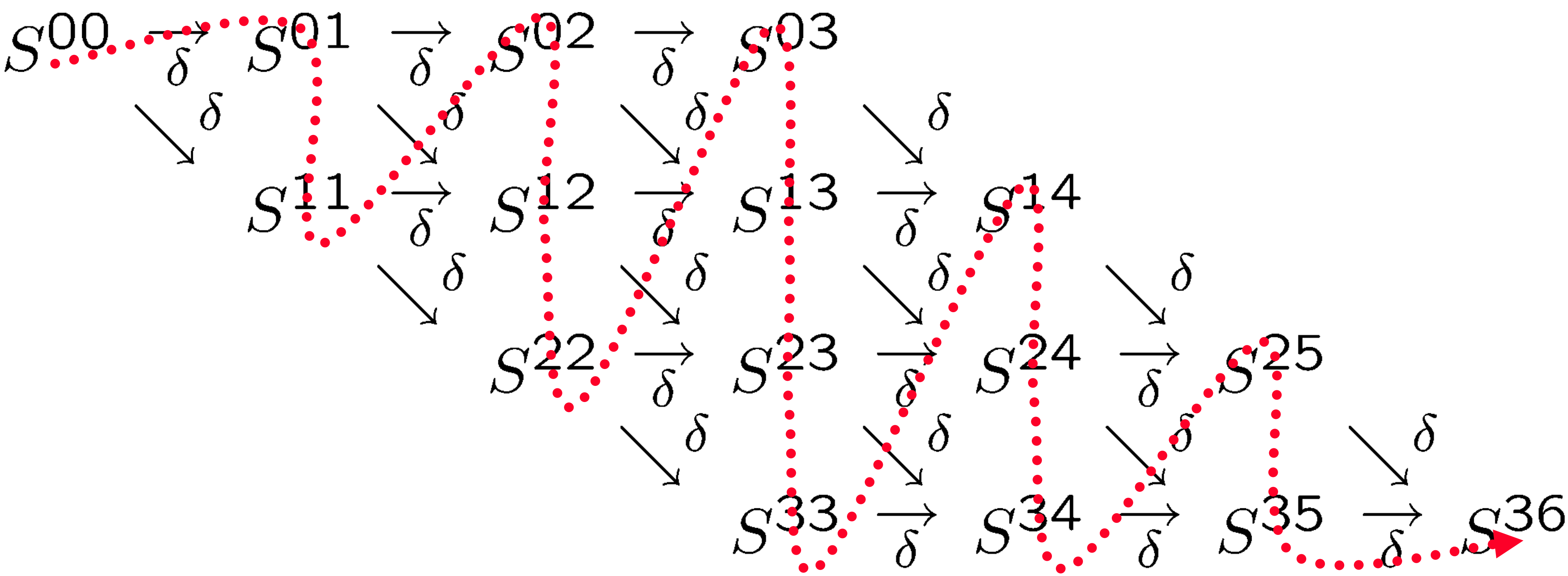
- Thus the dependence structure of the skewed loop is shown by marking the iteration space with all the dependences:



- Original execution order

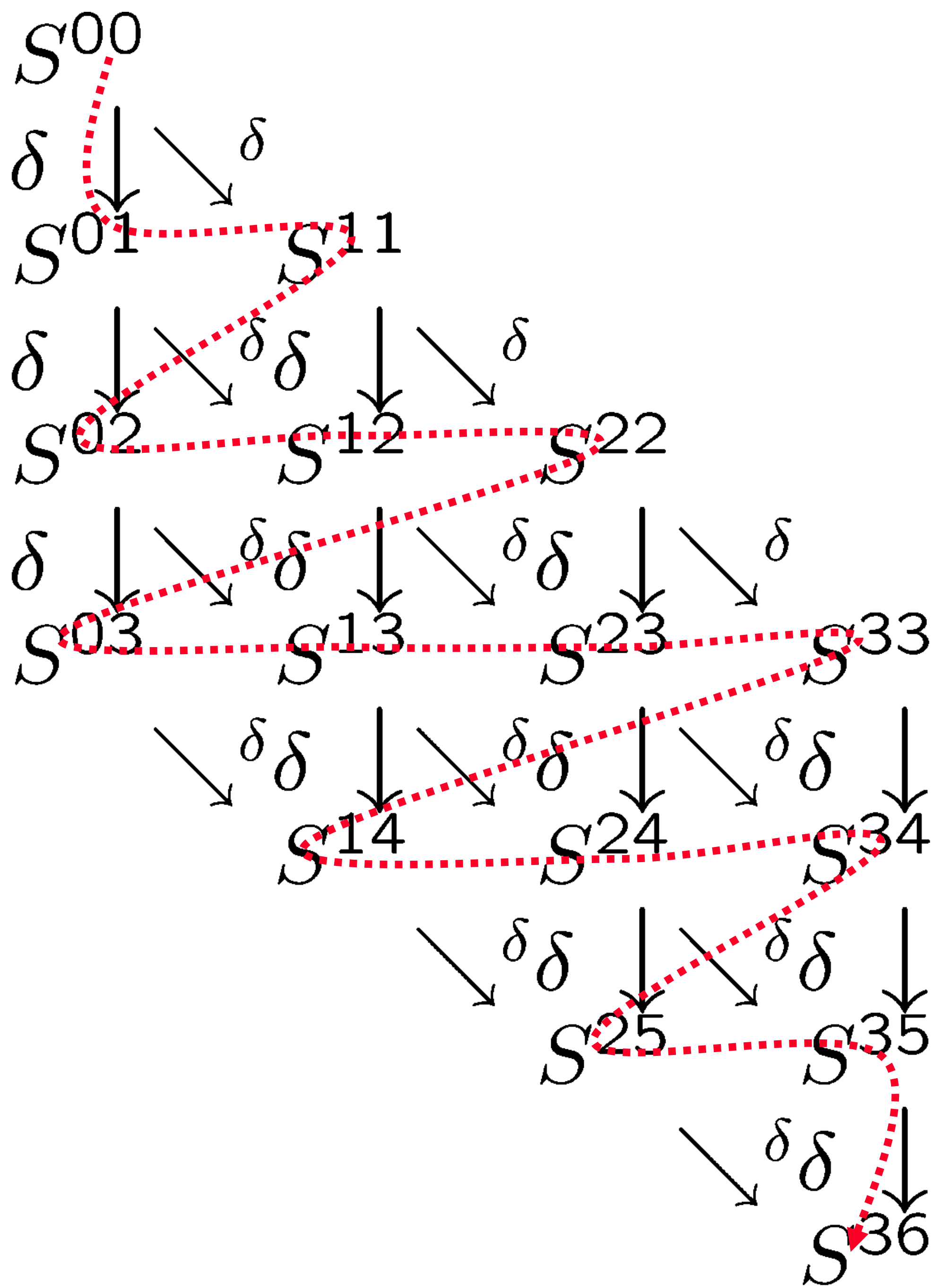
Interchange after skewing

- Thus the dependence structure of the skewed loop is shown by marking the iteration space with all the dependences:



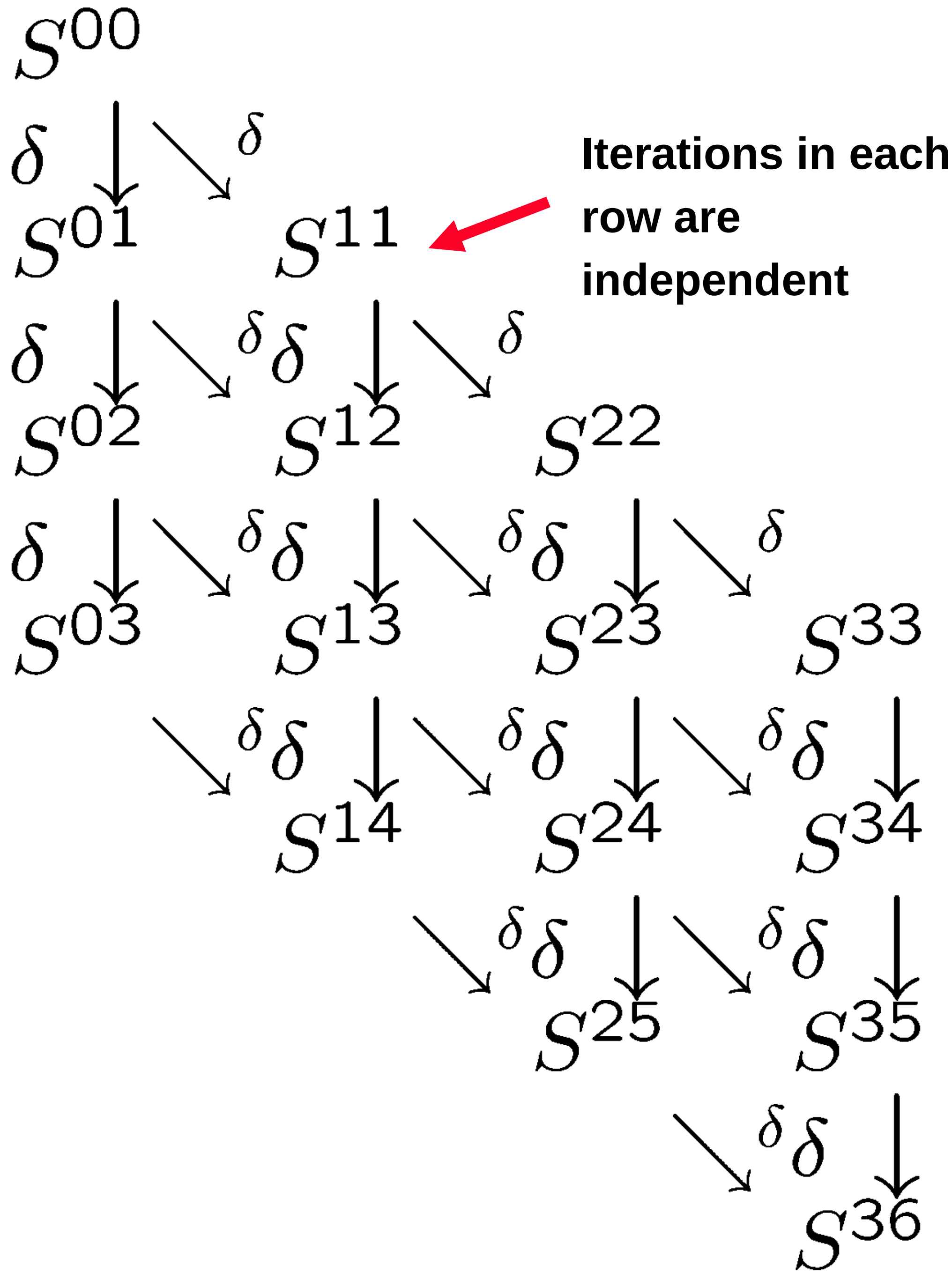
- Transposed execution order

- You can think of loop interchange as changing the way the iteration space is traversed
- Alternatively, you can think of it as a change to the way the runtime code instances are mapped onto the iteration space
- Traversal is always lexicographic – ie left-to-right, top-down

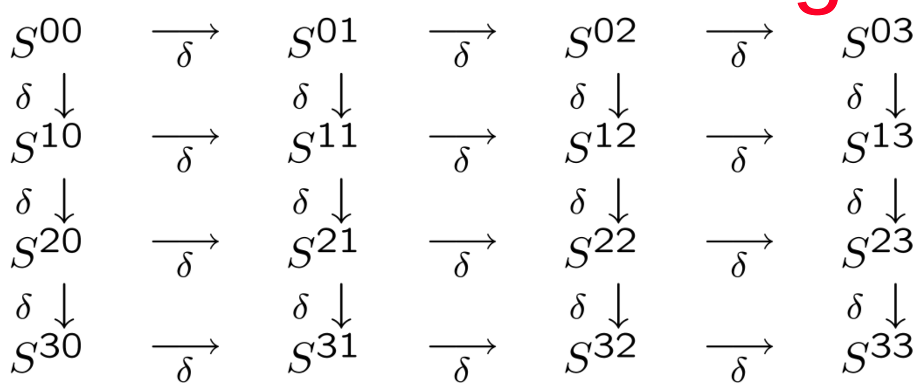


- The inner loop is now vectorisable, since it has no loop-carried dependence

- The skewed iteration space has N rows and 2N-1 columns, but still only N^2 actual statement instances.

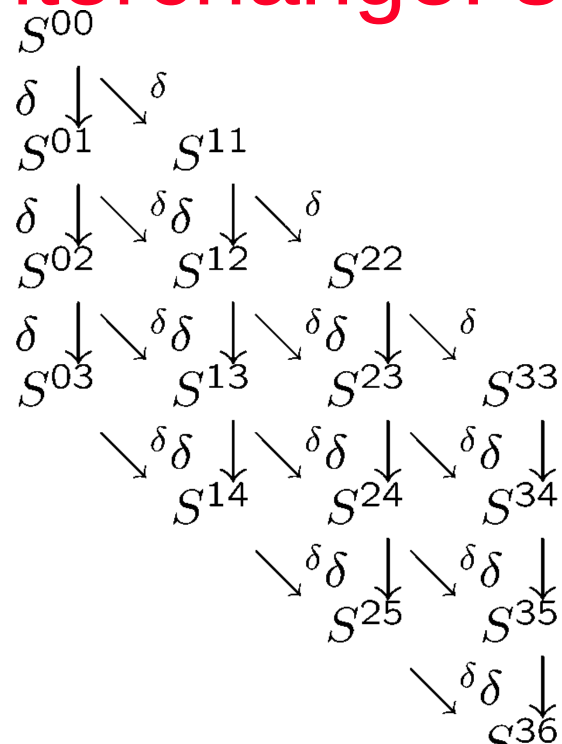


Skewing and interchange: summary



```
for I1 = 0 to 3 do
  for I2 = 0 to 3 do
S :   A[I1,I2] := A[I1-1,I2] + A[I1,I2-1]
```

- **Original loop interchangeable but not vectorisable.**
- **We skewed inner loop by outer loop by factor 1.**
- **Still not vectorisable, but interchangeable.**
- **Interchanged, skewed loop *is* vectorisable.**
- **Bounds of new loop not simple!**



```
for k2 := 0 to 2N2 - 2 do
  for k1 := max(0, K2 - N2 + 2) to min(K2, N1) do
S : A[k1,k2-k1] := A[k1-1,k2-k1]+A[k1,k2-k1-1]
```

- **Is skewing ever invalid?**
- **Does skewing affect interchangeability?**
- **Does skewing affect dependence distances?**
- **Can you predict value of skewing?**

Summary: dependence

- Dependence equation for single loop:
  - **Suppose S_p refers to $A[\varphi_p(I)]$**
  - **Suppose S_q refers to $A[\varphi_q(I)]$**
  - **A dependence of some kind occurs between S_p and S_q if there exists a solution to the equation**

    $$\varphi_p(I^1) = \varphi_q(I^2)$$

  - **for integer values of I^1 and I^2 lying within the loop bounds**
- For multidimensional arrays, and nested for-loops, we generalise this to a system of simultaneous dependence equations for two iterations, (I_1^1, I_2^1) and (I_1^2, I_2^2)
- Iteration space graph**, lexicographic schedule of execution**
- **Arrows in graph show solutions to dependence equation**
- Dependence distance vectors **characterise families of congruent arrows**

Summary: transformations

- A loop can be executed in parallel **if it has no loop-carried dependence**
- A loop nest can be interchanged **if the transposed dependence distance vectors are lexicographically forward**
- Strip-mining **is always valid**
- Tiling **= strip-mining + interchange**

Not explained yet

- **Skewing is always valid**
- Skewing can expose parallelism **by aligning parallel iterations with one of the loops**
- **Skewing can make interchange (and therefore tiling) valid**

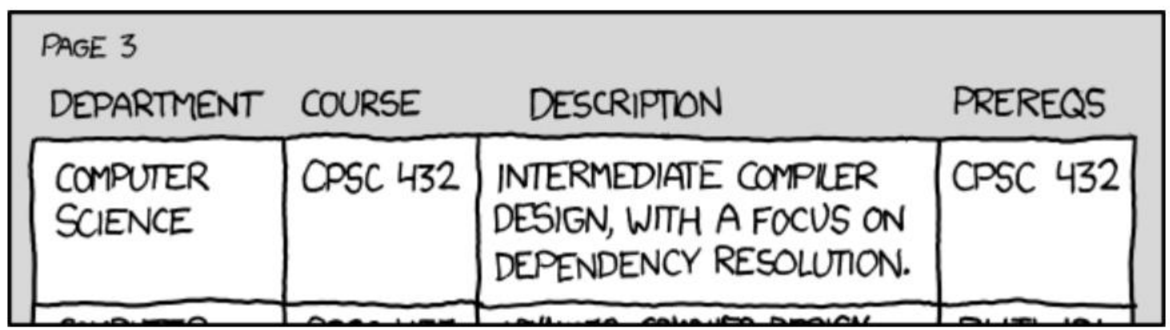
Compilers - Chapter 8:

Loop scheduling optimisations

Part 4: Representing loop transformations as matrix multiplications

- Lecturer:
  - Paul Kelly (p.kelly@imperial.ac.uk)

This section is not examinable



(It turns out that this cartoon is good for almost any compilers topic, definitely this one) https://xkcd.com/754/

Matrix representation of loop transformations

- **To skew the inner loop by the outer loop by factor 1 we adjust the loop bounds, and replace I_1 by K_1, and I_2 by K_2-K_1. That is,**

$$(K_1,K_2) = (I_1,I_2) \,.\, U$$

- **where U is a 2 x 2 matrix**

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- **That is,**

$$(K_1,K_2) = (I_1,I_2) \,.\, U = (I_1,I_2+I_1)$$

- **The inverse gets us back again:**

$$(I_1,I_2) = (K_1,K_2) \,.\, U^{-1} = (K_1,K_2-K_1)$$

The matrix representation is not examinable

- Matrix U maps each statement instance $S^{I_1 I_2}$ to its position in the new iteration space, $S^{K_1 K_2}$:

- Original iteration space:

I_1 \ I_2:	0	1	2	3
0	S^{00}	S^{01}	S^{02}	S^{03}
1	S^{10}	S^{11}	S^{12}	S^{13}
2	S^{20}	S^{21}	S^{22}	S^{23}
3	S^{30}	S^{31}	S^{32}	S^{33}

```
for I1 = 0 to 3 do
  for I2 = 0 to 3 do
S :   A[I1,I2] := A[I1 - 1,I2] + A[I1,I2 - 1]
```

```
for k1 := 0 to 3 do
  for k2 := k1 to k1+3 do
S : A[k1,k2-k1] := A[k1-1,k2-k1]+A[k1,k2-k1-1]
```

The subscripts are mapped back using U^{-1}

K_1 \ K_2:	0	1	2	3	4	5	6
0	S^{00}	S^{01}	S^{02}	S^{03}			
1		S^{11}	S^{12}	S^{13}	S^{14}		
2			S^{22}	S^{23}	S^{24}	S^{25}	
3				S^{33}	S^{34}	S^{35}	S^{36}

The dependences are subject to the same transformation.

The matrix representation is not examinable

$(K_1,K_2) = (I_1,I_2) \cdot U = (I_1,I_2+I_1)$

$(I_1,I_2) = (K_1,K_2) \cdot U^{-1} = (K_1,K_2-K_1)$

Using matrices to reason about dependence

Recall that:

- There is a dependence between two iterations (I_1^1, I_2^1) and (I_1^2, I_2^2) if there is a memory location which is assigned to in iteration (I_1^1, I_2^1), and read in iteration (I_1^2, I_2^2).

  ((unless there is an intervening assignment))

- If (I_1^1, I_2^1) precedes (I_1^2, I_2^2) it is a *data*-dependence.
- If (I_1^2, I_2^2) precedes (I_1^1, I_2^1) it is a *anti*-dependence.
- If the location is assigned to in both iterations, it is an *output*-dependence.

- The dependence distance vector (D_1, D_2) is $(I_1^1 - I_1^2, I_2^1 - I_2^2)$.

The matrix representation is not examinable

Transforming dependence vectors

- **If there is a dependence between two iterations (I_1^1, I_2^1) and (I_1^2, I_2^2)**
- **Then iterations $(I_1^1, I_2^1) . U$ and $(I_1^2, I_2^2) .U$ will also read and write the same location**

- **The transformation U is *valid* iff**

  $(I_1^1, I_2^1) . U$ precedes $(I_1^2, I_2^2) . U$

  whenever there is a dependence between

  (I_1^1, I_2^1) and (I_1^2, I_2^2).

- **In the transformed loop the dependence distance vector is also transformed, to**

  $(D_1, D_2) . U$

- **U is a valid transformation if all the program's dependence distance vectors are still "forward" when transformed by U**

The matrix representation is not examinable

Transforming dependence vectors

- **What do we mean by "precedes"?**

> **Definition: Lexicographic ordering:**
>
> **(I^1,J^1) precedes (I^2,J^2)**
>
> **if $I^1 < I^2$, or $I^1 = I^2$ and $J^1 < J^2$**

- **"Lexicographic" is dictionary order – both "baz" and "can" precede "cat"**
- **So (1,2) precedes (1,3)**
- **But (0,3) precedes (1,4)**
- **A dependence distance vector (D_1,D_2) is lexicographically "forward" if it precedes (0,0)**

The matrix representation is not examinable

Example: loop given earlier

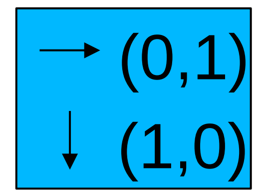
Before transformation we had two dependences:

1. Distance: (1,0), direction: (<,.)
2. Distance: (0,1), direction: (.,<)

```
for I1 = 0 to 3 do
  for I2 = 0 to 3 do
S :   A[I1,I2] := A[I1 − 1,I2] + A[I1,I2 − 1]
```

```
for k1 := 0 to 3 do
  for k2 := k1 to k1+3 do
S : A[k1,k2-k1] := A[k1-1,k2-k1]+A[k1,k2-k1-1]
```

I_1 \ I_2 :	0	1	2	3
0	S^{00}	S^{01}	S^{02}	S^{03}
1	S^{10}	S^{11}	S^{12}	S^{13}
2	S^{20}	S^{21}	S^{22}	S^{23}
3	S^{30}	S^{31}	S^{32}	S^{33}



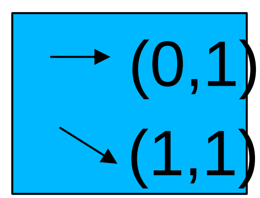
- After transformation by matrix

$$\mathbf{U} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- (i.e. skewing of inner loop by outer) we get:

1. Distance: (1,0).U = (1,1), direction: (<,<)
2. Distance: (0,1).U = (0,1), direction: (.,<)

K_1 \ K_2 :	0	1	2	3	4	5	6
0	S^{00}	S^{01}	S^{02}	S^{03}			
1		S^{11}	S^{12}	S^{13}	S^{14}		
2			S^{22}	S^{23}	S^{24}	S^{25}	
3				S^{33}	S^{34}	S^{35}	S^{36}



The matrix representation is not examinable

- We can also represent loop interchange by a matrix transformation.
- After transforming the skewed loop by matrix

$$\mathbf{V} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- (i.e. loop interchange) we get:

1. Distance: (1,0).U.V = (1,1).V = (1,1), direction: (<,<)
2. Distance: (0,1).U.V = (0,1).V = (1,0), direction: (<,.)

- The transformed iteration space is the transpose of the skewed iteration space:

$$\begin{array}{llll} S^{00} & & & \\ S^{10} & S^{11} & & \\ S^{20} & S^{21} & S^{22} & \\ S^{30} & S^{31} & S^{32} & S^{33} \\ & S^{41} & S^{42} & S^{43} \\ & & S^{52} & S^{53} \\ & & & S^{63} \end{array}$$

The matrix representation is not examinable

Summary

- (I_1,I_2) . U maps each statement instance (I_1,I_2) to its new position (K_1,K_2) in the transformed loop's execution sequence
- (D_1,D_2) . U gives new dependence distance vector, giving test for validity
- Captures skewing, interchange and reversal
- Compose transformations by matrix multiplication

  $U_1 . U_2$

- Resulting loop's bounds may be a little tricky
  - Efficient algorithms exist [Banerjee90] to maximise parallelism by skewing and loop interchanging
  - Efficient algorithms exist to optimise cache performance by finding the combination of blocking, block size, interchange and skewing which leads to the best reuse [Wolf91]

The matrix representation is not examinable

Restructuring compilers - conclusions:

- Restructuring compilers can find parallelism
- And enhance locality
- **For a very restricted class of programs**
  - For-loops over arrays with array subscripts that are simple (“affine”) expressions involving loop control variables

- But for this restricted class there is a rather elegant theory (the “polyhedral” or “polytope” model, http://en.wikipedia.org/wiki/Polytope_model)
- Extending beyond this is a big research problem

- Current compilers (GCC, Clang/LLVM, Intel, Microsoft etc) can do some of this, in theory – but are often defeated by program complexity

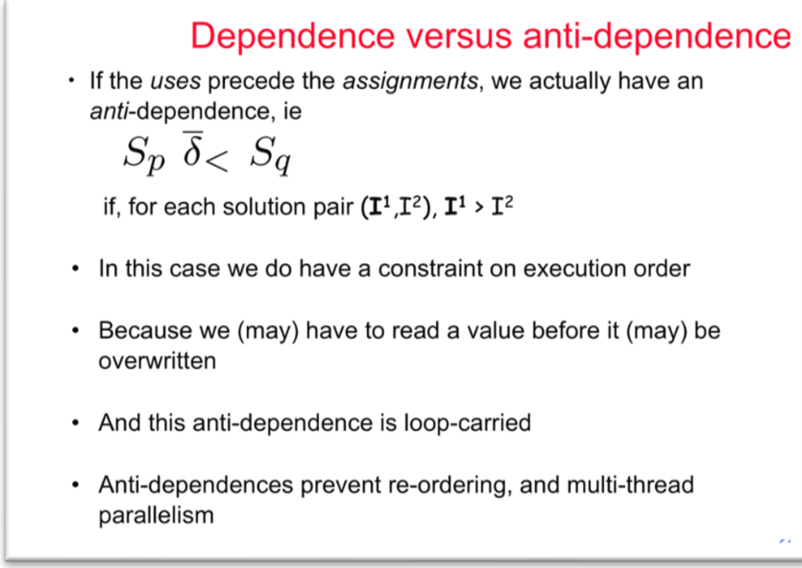
References

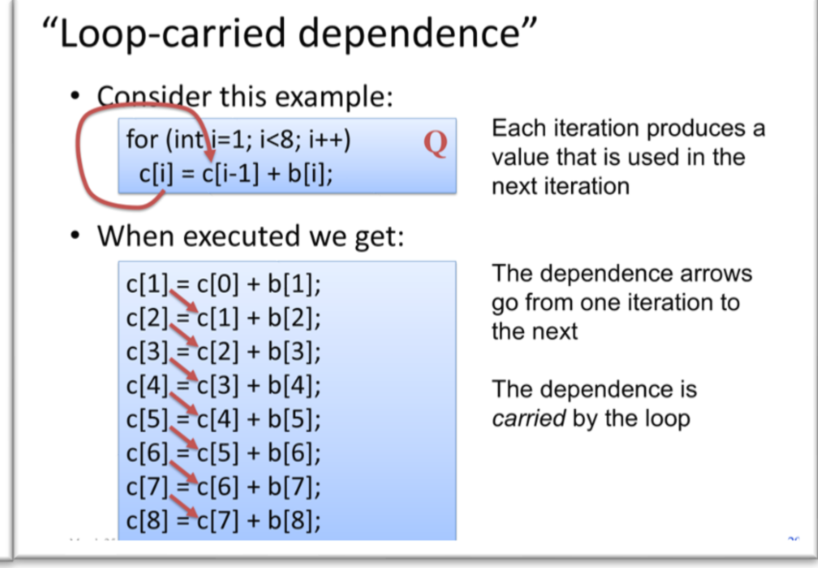
- **Textbooks covering restructuring compilers**
  - **Michael Wolfe. High Performance Compilers for Parallel Computing. Addison Wesley, 1996.**
  - **Steven Muchnick, Advanced Compiler Design and Implementation. Morgan Kaufmann, 1997.**
  - **Ken Kennedy and Randy Allen, Optimizing Compilers for Modern Architectures. Morgan Kaufmann, 2001.**
- **Research papers:**

  - **D. F. Bacon and S. L. Graham and O. J. Sharp, "Compiler Transformations for High-Performance Computing". ACM Computing Surveys V26 N4 Dec 1994 http://doi.acm.org/10.1145/197405.197406**
  - **U. Banerjee. Unimodular transformations of double loops. In Proceedings of the Third Workshop on Programming Languages and Compilers for Parallel Computing, Irvine, CA. Pitman/MIT Press, 1990.**
  - **M.E. Wolf and M.S. Lam. A data locality optimizing algorithm. In Proceedings of the ACM SIGPLAN '91 Conference on Programming Language Design and Implementation, volume 26, pages 30-44, Toronto, Ontario, Canada, June 1991.**

Student question:

"why is antidependence a dependence?"





Loop-carried true dependence:

```
for i
  A[i] = A[i-1] + B[i]
```

Loop-carried anti-dependence:

```
for i
  A[i] = A[i+1] + B[i]
```

“Loop-carried anti-dependence”

- Consider this example:

```
for (int i=0; i<7; i++)
  c[i] = c[i+1] + b[i];
```

Z

- When executed we get:

```
c[0] = c[1] + b[1];
c[1] = c[2] + b[2];
c[2] = c[3] + b[3];
c[3] = c[4] + b[4];
c[4] = c[5] + b[5];
c[5] = c[6] + b[6];
c[6] = c[7] + b[7];
```

Each iteration uses a value which is overwritten in the next iteration

We need the use to happen before the overwrite

So we have a *precedence* requirement due to an anti-dependence

The anti-dependence arrows go from one iteration to the next

The anti-dependence is *carried* by the loop

Implementing shared-memory parallel loop

```
for (i=0; i<N; i++) {
  A[i]  = A[i] + B[i];
}
```

```
if (myThreadId() == 0)
  i = 0;
barrier();
// on each thread
while (true) {
  local_i = FetchAndAdd(&i);
  if (local_i >= N) break;
  A[local_i] = A[local_i] + B[local_i];
}
barrier();
```

Barrier(): block until all threads reach this point

- "self-scheduling" loop
- FetchAndAdd() is atomic operation to get next un-executed loop iteration:

```
Int FetchAndAdd(int *i) {
  lock(i);
  r = *i;
  *i = *i+1;
  unlock(i);
  return(r);
}
```

Optimisations:
- Work in chunks
- Avoid unnecessary barriers
- Exploit "cache affinity" from loop to loop

There are smarter ways to implement FetchAndAdd….

```
for (i=0; i<N; i++) {
  A[i]  = A[i] + B[i];
}
```

Thread #0

```
if (myThreadId() == 0)
  i = 0;
barrier();
while (true) {
  local_i = FetchAndAdd(&i);
  if (local_i >= N) break;
  A[local_i] = A[local_i] + B[local_i];
}
barrier();
```

Thread #1

```
if (myThreadId() == 0)
  i = 0;
barrier();
while (true) {
  local_i = FetchAndAdd(&i);
  if (local_i >= N) break;
  A[local_i] = A[local_i] + B[local_i];
}
barrier();
```

Thread #0 gets some sequence of iterations to do

Thread #1 gets some sequence of iterations to do

What could possibly go wrong?

```
for (i=0; i<N; i++) {
  A[i]  = A[i+1] + B[i];
}
```

Thread #0

```
if (myThreadId() == 0)
  i = 0;
barrier();
while (true) {
  local_i = FetchAndAdd(&i);
  if (local_i >= N) break;
  A[local_i] = A[local_i+1] + B[local_i];
}
barrier();
```

Thread #1

```
if (myThreadId() == 0)
  i = 0;
barrier();
while (true) {
  local_i = FetchAndAdd(&i);
  if (local_i >= N) break;
  A[local_i] = A[local_i+1] + B[local_i];
}
barrier();
```

Thread #0 gets some sequence of iterations to do

Thread #1 gets some sequence of iterations to do

What could possibly go wrong?

```
for (i=0; i<N; i++) {
  A[i]  = A[i+1] + B[i];
}
```

This example has a loop-carried anti-dependence.
We must read from A before overwriting A

Thread #0

```
if (myThreadId() == 0)
  i = 0;
barrier();
while (true) {
  local_i = FetchAndAdd(&i);
  if (local_i >= N) break;
  A[local_i] = A[local_i+1] + B[local_i];
}
barrier();
```

Thread #1

```
if (myThreadId() == 0)
  i = 0;
barrier();
while (true) {
  local_i = FetchAndAdd(&i);
  if (local_i >= N) break;
  A[local_i] = A[local_i+1] + B[local_i];
}
barrier();
```

Thread #0 gets some sequence of iterations to do, eg: **0, 2, 4, 6…**

Thread #1 gets some sequence of iterations to do, eg: **1, 3, 5, 7…**

Feeding curiosity: solving the dependence equation (not examinable)

```
from z3 import *
N=100
i1 = Int("i1")
i2 = Int("i2")
# consider a loop like this:
#  for i = 1 to N
#   a[phi1(i)] = a[phi2(i)] + b[i]
# So the dependence equation is
#   exists i1, i2: 1<i<n s.t. phi1(i1) == phi2(i2)
def DependenceTest(bounds, dependence_equation):
   s = Solver()
   s.add( bounds, dependence_equation )
   if s.check() == unsat:
      print ("No dependence is present")
   else:
      print("Dependence is found, for example when:")
      m = s.model()
      print ("i1 = %s (LHS)" % m[i1])
      print ("i2 = %s (RHS)" % m[i2])
```

Example 1:

```
print("for i = 1 to N")
print("  a[i] = a[i-1] + b[i]")
DependenceTest( And(i1>=1, i1<N, i2>=1, i2<N),
                i1 == i2-1 )
```

```
for i = 1 to N
  a[i] = a[i-1] + b[i]
Dependence is found, for example when:
i1 = 1 (LHS)
i2 = 2 (RHS)
```

Just add the constraints and call the solver

Feeding curiosity: solving the dependence equation

```
def DependenceTest(bounds, dependence_equation):
   s = Solver()
   s.add( bounds, dependence_equation )
   if s.check() == unsat:
      print ("No dependence is present")
   else:
      print("Dependence is found, for example when:")
      m = s.model()
      print ("i1 = %s (LHS)" % m[i1])
      print ("i2 = %s (RHS)" % m[i2])
      # Is there a loop-carried true dependence?
      s2 = Solver()
      s2.add( bounds, dependence_equation, i1<i2 )
      if s2.check() == unsat:
         print ("No loop-carried true dependence is present")
      else:
         print("Loop-carried true dependence found, for example when:")
         m = s2.model()
         print ("i1 = %s" % m[i1])
         print ("i2 = %s" % m[i2])
       # Is there a loop-carried anti-dependence?
      s3 = Solver()
      s3.add( bounds, dependence_equation, i1>i2 )
      if s3.check() == unsat:
         print ("No loop-carried anti-dependence is present")
      else:
         print("Loop-carried anti-dependence found, for example when:")
         m = s3.model()
         print ("i1 = %s" % m[i1])
         print ("i2 = %s" % m[i2])
```

Example 1:

```
print("for i = 1 to N")
print("  a[i] = a[i-1] + b[i]")
DependenceTest( And(i1>=1, i1<N, i2>=1, i2<N),
                        i1 == i2-1 )
```

```
for i = 1 to N
  a[i] = a[i-1] + b[i]
Dependence is found, for example when:
i1 = 1 (LHS)
i2 = 2 (RHS)
Loop-carried true dependence found, for example when:
i1 = 1
i2 = 2
No loop-carried anti-dependence is present
```

Extend to distinguish loop-carried true and anti-dependencies

Feeding curiosity: solving the dependence equation

```
def DependenceTest(bounds, dependence_equation):
  s = Solver()
  s.add( bounds, dependence_equation )
  if s.check() == unsat:
    print ("No dependence is present")
  else:
    print("Dependence is found, for example when:")
    m = s.model()
    print ("i1 = %s (LHS)" % m[i1])
    print ("i2 = %s (RHS)" % m[i2])
    # Is there a loop-carried true dependence?
    s2 = Solver()
    s2.add( bounds, dependence_equation, i1<i2 )
    if s2.check() == unsat:
      print ("No loop-carried true dependence is present")
    else:
      print("Loop-carried true dependence found, for example when:")
      m = s2.model()
      print ("i1 = %s" % m[i1])
      print ("i2 = %s" % m[i2])
    # Is there a loop-carried anti-dependence?
    s3 = Solver()
    s3.add( bounds, dependence_equation, i1>i2 )
    if s3.check() == unsat:
      print ("No loop-carried anti-dependence is present")
    else:
      print("Loop-carried anti-dependence found, for example when:")
      m = s3.model()
      print ("i1 = %s" % m[i1])
      print ("i2 = %s" % m[i2])
```

Example 2:

```
print("for i = 1 to N")
print("  a[i] = a[i] + b[i]")
DependenceTest( And(i1>=1, i1<N, i2>=1, i2<N),
                i1 == i2 )
```

```
for i = 1 to N
  a[i] = a[i] + b[i]
Dependence is found, for example when:
i1 = 1 (LHS)
i2 = 1 (RHS)
No loop-carried true dependence is present
No loop-carried anti-dependence is present
```

In this case the dependence is present but not loop-carried

Feeding curiosity: solving the dependence equation

```
def DependenceTest(bounds, dependence_equation):
  s = Solver()
  s.add( bounds, dependence_equation )
  if s.check() == unsat:
    print ("No dependence is present")
  else:
    print("Dependence is found, for example when:")
    m = s.model()
    print ("i1 = %s (LHS)" % m[i1])
    print ("i2 = %s (RHS)" % m[i2])
    # Is there a loop-carried true dependence?
    s2 = Solver()
    s2.add( bounds, dependence_equation, i1<i2 )
    if s2.check() == unsat:
      print ("No loop-carried true dependence is present")
    else:
      print("Loop-carried true dependence found, for example when:")
      m = s2.model()
      print ("i1 = %s" % m[i1])
      print ("i2 = %s" % m[i2])
    # Is there a loop-carried anti-dependence?
    s3 = Solver()
    s3.add( bounds, dependence_equation, i1>i2 )
    if s3.check() == unsat:
      print ("No loop-carried anti-dependence is present")
    else:
      print("Loop-carried anti-dependence found, for example when:")
      m = s3.model()
      print ("i1 = %s" % m[i1])
      print ("i2 = %s" % m[i2])
```

Example 3:

```
print("for i = 1 to N")
print("  a[2*i] = a[2*i-1] + b[i]")
DependenceTest( And(i1>=1, i1<N, i2>=1, i2<N),
                2*i1 == 2*i2-1 )
```

```
for i = 1 to N
  a[2*i] = a[2*i-1] + b[2*i]
No dependence is present
```

Feeding curiosity: solving the dependence equation

```
def DependenceTest(bounds, dependence_equation):
  s = Solver()
  s.add( bounds, dependence_equation )
  if s.check() == unsat:
    print ("No dependence is present")
  else:
    print("Dependence is found, for example when:")
    m = s.model()
    print ("i1 = %s (LHS)" % m[i1])
    print ("i2 = %s (RHS)" % m[i2])
    # Is there a loop-carried true dependence?
    s2 = Solver()
    s2.add( bounds, dependence_equation, i1<i2 )
    if s2.check() == unsat:
      print ("No loop-carried true dependence is present")
    else:
      print("Loop-carried true dependence found, for example when:")
      m = s2.model()
      print ("i1 = %s" % m[i1])
      print ("i2 = %s" % m[i2])
    # Is there a loop-carried anti-dependence?
    s3 = Solver()
    s3.add( bounds, dependence_equation, i1>i2 )
    if s3.check() == unsat:
      print ("No loop-carried anti-dependence is present")
    else:
      print("Loop-carried anti-dependence found, for example when:")
      m = s3.model()
      print ("i1 = %s" % m[i1])
      print ("i2 = %s" % m[i2])
```

Example 4:

```
print("for i = 1 to N")
print("  a[3*i] = a[5*i-10] + b[i]")
DependenceTest( And(i1>=1, i1<N, i2>=1, i2<N),
                3*i1 == 5*i2-20 )
```

```
for i = 1 to N
  a[3*i] = a[5*1-20] + b[i]
Dependence is found, for example when:
i1 = 5 (LHS)
i2 = 7 (RHS)
Loop-carried true dependence found, for example when:
i1 = 5
i2 = 7
Loop-carried anti-dependence found, for example when:
i1 = 15
i2 = 13
```

In this case we have both true and anti-dependences: weird!

S1 : A[0] := 0

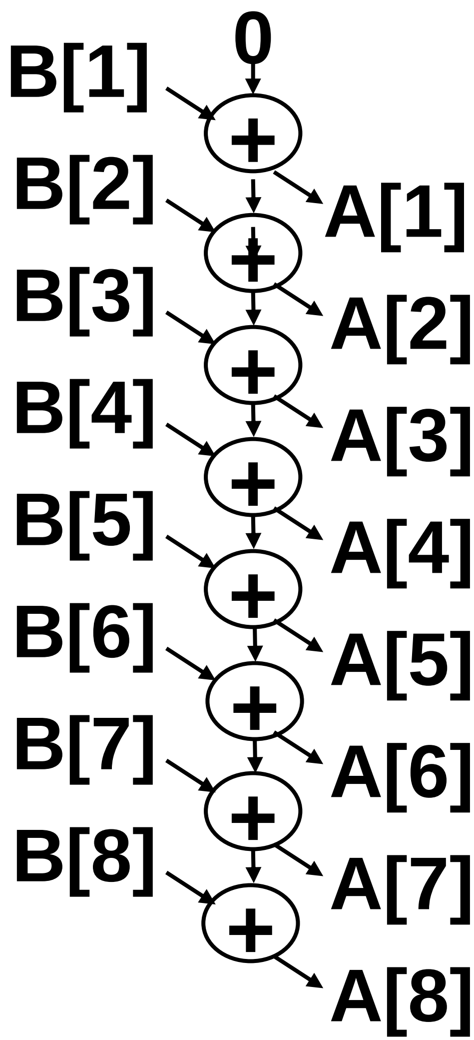
for i = 1 to 8

S2 : A[i] := A[i-1] + B[i]

Feeding curiosity (not examinable)

Loop-carried dependences can sometimes still be parallelised

- **Appears to be inherently sequential**



S1 : A[0] := 0

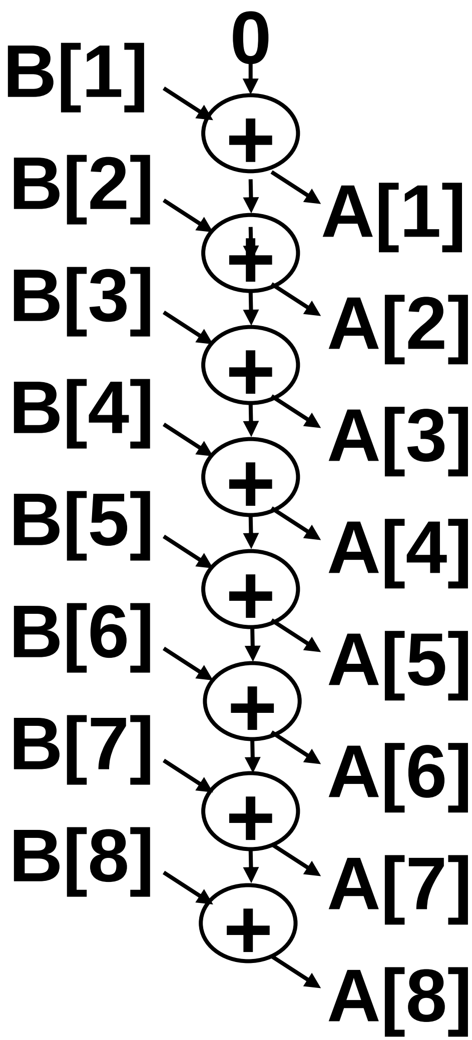
for i = 1 to 8

S2 : A[i] := A[i-1] + B[i]

Feeding curiosity (not examinable)

Loop-carried dependences can sometimes still be parallelised

- **Appears to be inherently sequential**



- **But parallel is possible:**

B:	1	1	1	1	1	1	1	1	
>>1:		1	1	1	1	1	1	1	+
A1:	1	2	2	2	2	2	2	2	
>>2:			1	2	2	2	2	2	+
A2:	1	2	3	4	4	4	4	4	
>>4:					1	2	3	4	+
A3:	1	2	3	4	5	6	7	8	

"Parallel scan" or "parallel prefix sum"

S1 : A[0] := 0

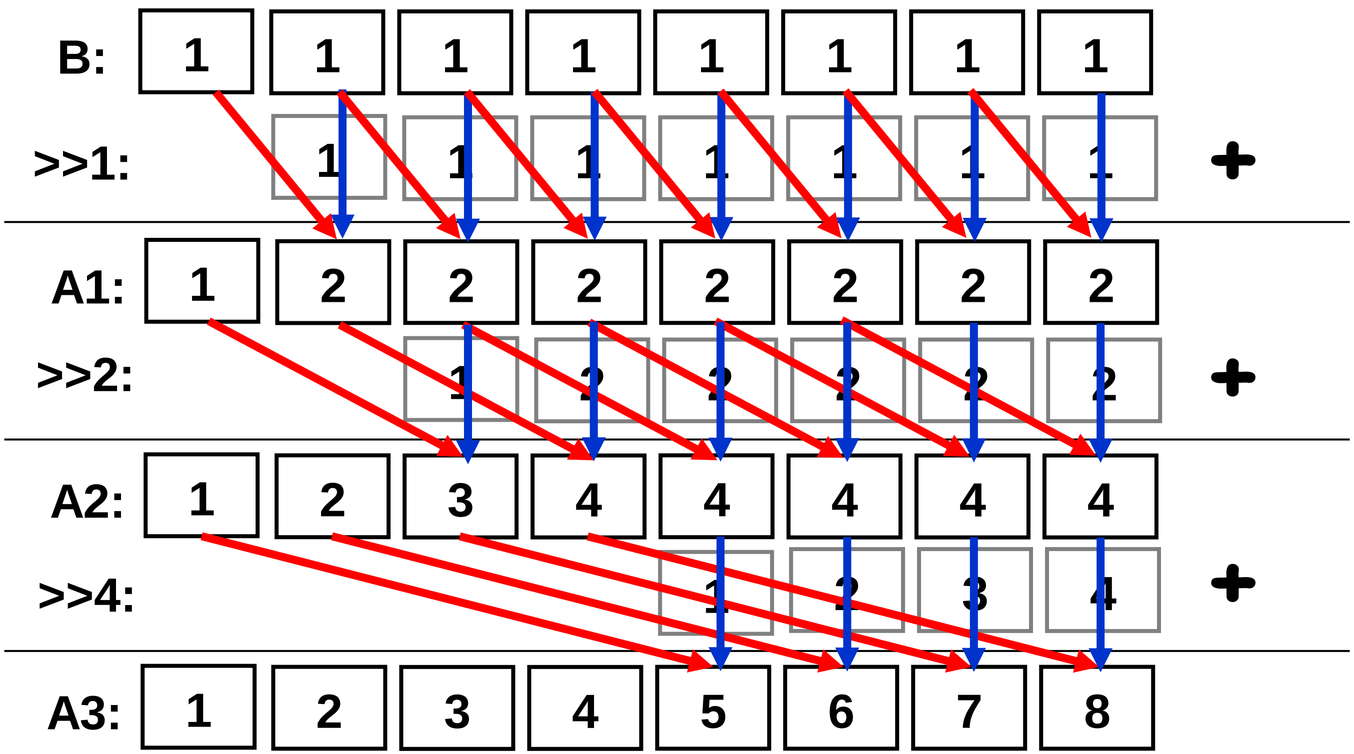
for i = 1 to 8

S2 : A[i] := A[i-1] + B[i]

Feeding curiosity (not examinable)

Loop-carried dependences can sometimes still be parallelised

- **Appears to be inherently sequential**
- **But parallel implementation is possible**



- **Each step is a vector-parallel operation**
- **Of decreasing size**
- **We have log(N) steps**

"Parallel scan" or "parallel prefix sum"

S1 : A[0] := 0

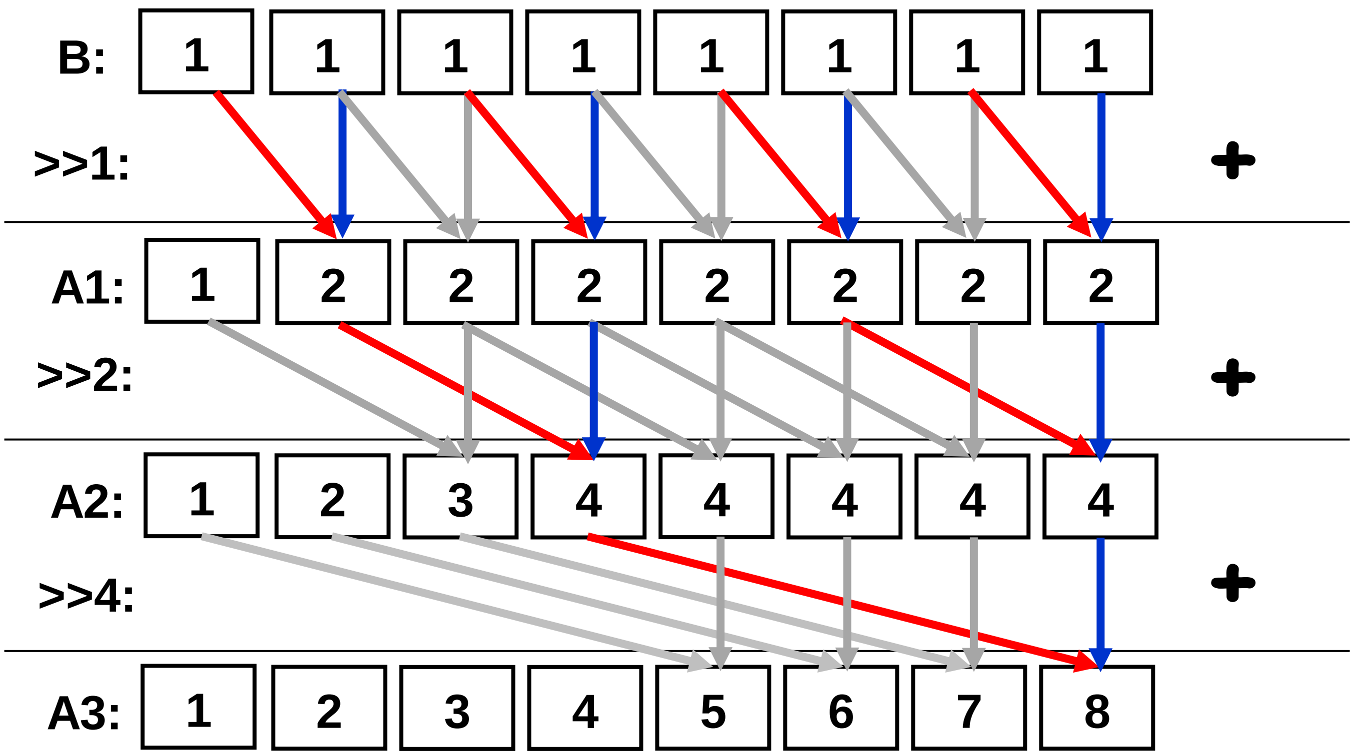
for i = 1 to 8

S2 : A[i] := A[i-1] + B[i]

Feeding curiosity (not examinable)

Loop-carried dependences can sometimes still be parallelised

- **Appears to be inherently sequential**
- **But parallel implementation is possible**



We can see that the last element is computed with a reduction tree

S1 : A[0] := 0

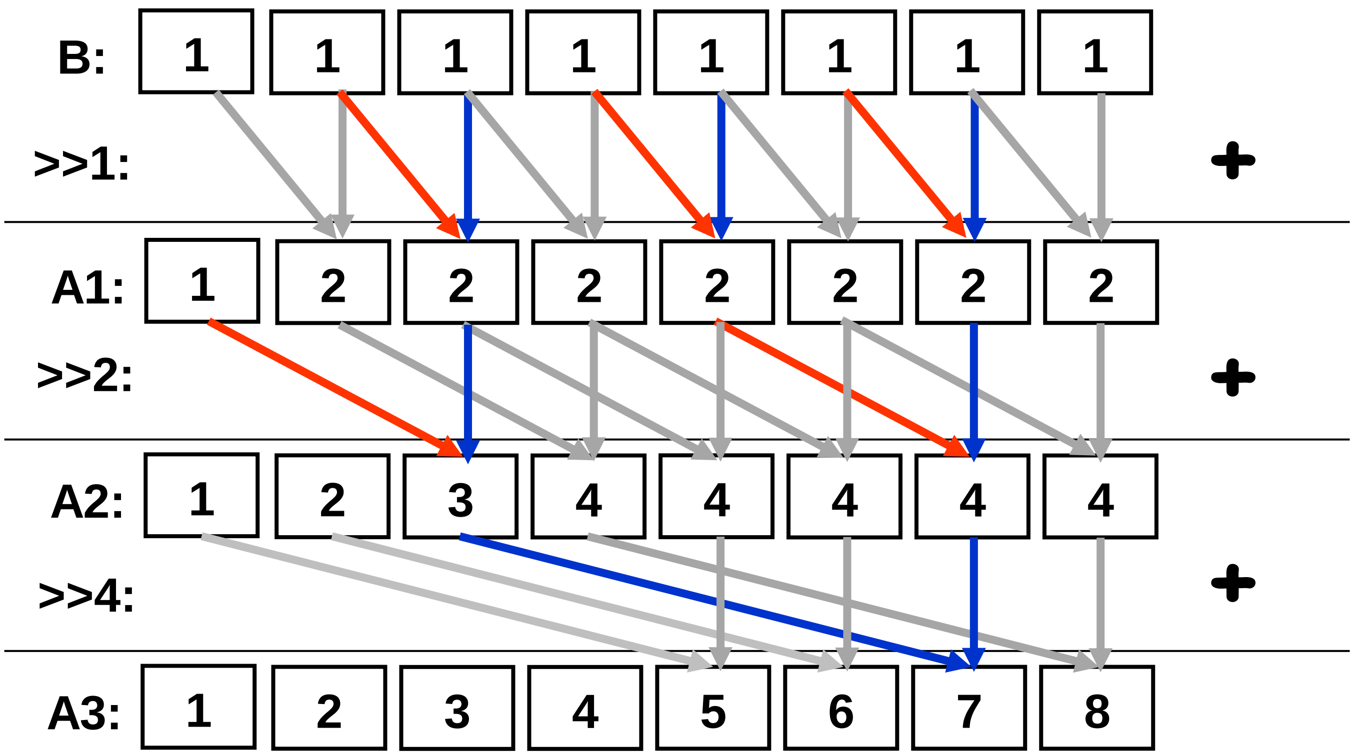
for i = 1 to 8

S2 : A[i] := A[i-1] + B[i]

Feeding curiosity (not examinable)

Loop-carried dependences can sometimes still be parallelised

- **Appears to be inherently sequential**
- **But parallel implementation is possible**



All the elements are computed by reduction trees of depth log(N) – for example element 7

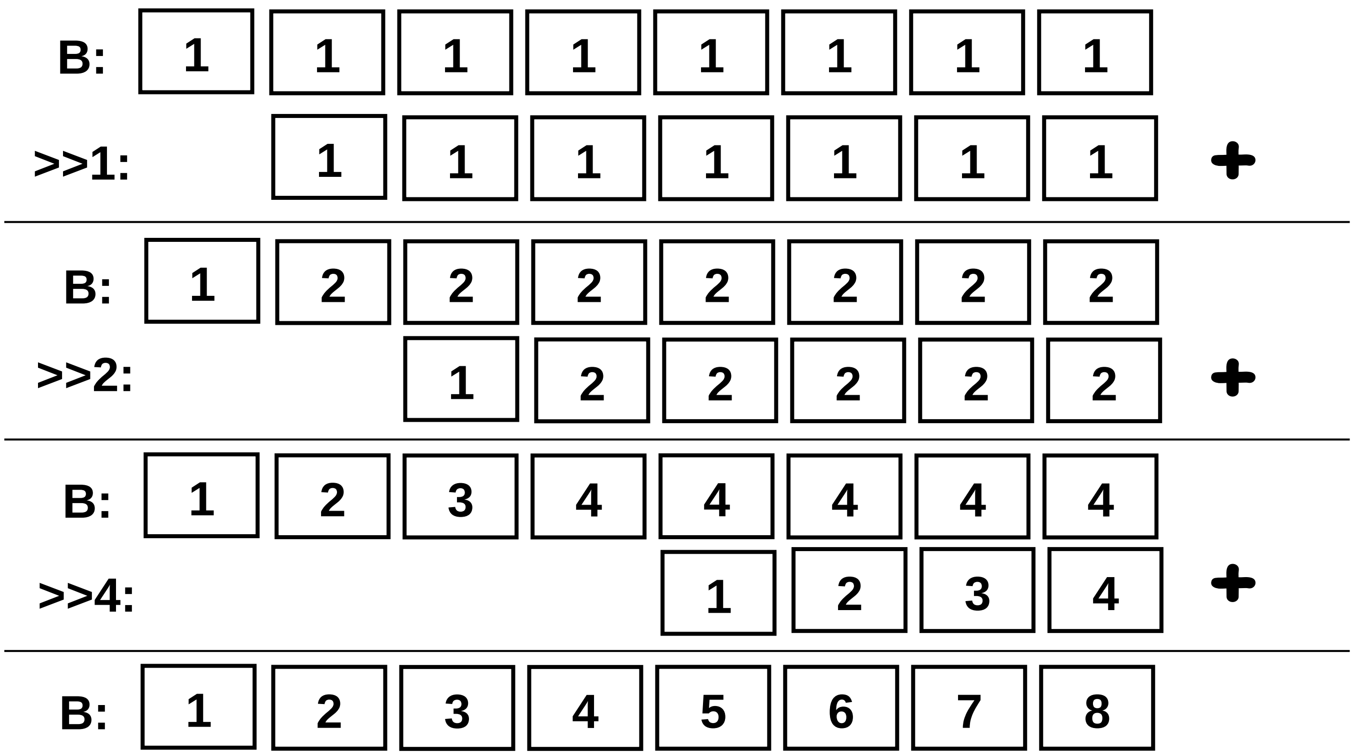
Feeding curiosity

S1 : A[0] := 0

for i = 1 to 8

S2 : A[i] := A[i-1] + B[i]

- **Appears to be inherently sequential**
- **But parallel implementation is possible**



“Parallel scan” or “parallel prefix sum”

- **This is the “naïve” parallel scan**
- **It does more work than the sequential scan – but it does use parallelism**
- **There are “work-efficient” parallel scans**
- **Eg see Mark Harris, GPU Gems Ch39**
  https://developer.nvidia.com/gpugems/gpugems3/part-vi-gpu-computing/chapter-39-parallel-prefix-sum-scan-cuda

Feeding curiosity

Matrix transpose

Try this link to the the Compiler Explorer:

https://godbolt.org/#g:!((g:!((g:!((h:codeEditor,i:(filename:'1',fontScale:14,fontUsePx:'0',j:1,lang:c%2B%2B,selection:(endColumn:2,endLineNumber:20,positionColumn:2,positionLi
neNumber:20,selectionStartColumn:2,selectionStartLineNumber:20,startColumn:2,startLineNumber:20),source:'%23define+SIZE+10240%0A//%23define+SIZE+20480%0A%23defin
e+TOTALBYTES+SIZE*SIZE*4%0A%0Aint+A%5BSIZE%5D%5BSIZE%5D%3B%0Aint+B%5BSIZE%5D%5BSIZE%5D%3B%0A%0A%23define+IB+32%0A%23define+JB+32%0A%0Avo
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march%3Dznver4',overrides:!(),selection:(endColumn:1,endLineNumber:1,positionColumn:1,positionLineNumber:1,selectionStartColumn:1,selectionStartLineNumber:1,startColu
mn:1,startLineNumber:1),source:1),l:'5',n:'0',o:'+x86-64+clang+(trunk)+(Editor+%231)',t:'0')),k:50,l:'4',n:'0',o:'',s:0,t:'0')),l:'2',n:'0',o:'',t:'0')),version:4

Collision detect

Try this link to the the Compiler Explorer:

https://godbolt.org/#g:!((g:!((g:!((h:codeEditor,i:(filename:'1',fontScale:14,fontUsePx:'0',j:1,lang:c%2B%2B,selection:(endColumn:13,endLineNumber:18,positionColumn:13,positio
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%0A%23define+JB+32%0A%0Avoid+P(int+N,+int+M)%0A%7B%0A++int+i,+j%3B%0A%0A++for+(i%3D0%3B+i%3CN%3B+i%2B%2B)+%7B%0A++++++C%5BB%5Bi%5D%5D+%2B%
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umn:1,startLineNumber:1),source:1),l:'5',n:'0',o:'+x86-64+icx+2025.0.0+(Editor+%231)',t:'0')),k:33.333333333333336,l:'4',n:'0',o:'',s:0,t:'0'),(g:!((h:output,i:(compilerName:'x86-
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64+icx+2025.0.0+(Compiler+%231)',t:'0')),k:33.33333333333333,l:'4',n:'0',o:'',s:0,t:'0')),l:'2',n:'0',o:'',t:'0')),version:4

Feeding curiosity

Ask me about….

- **Loop interchange for locality**
  - For i, j, k matrix multiply vs
  - For i, k, j matrix multiply
- **Tiling for locality**
  - For the transpose example shown in the last chapter
  - For matrix multiply
- **Stencils and convolutions**
  - skewed, split, diamond
- **Graphs and unstructured meshes**