TYPES, RELEVANCE & CLASSICAL LOGIC
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The world is as it ought to be, ALL THINGS considered

Abstract. Types were introduced into Logic as a defence mechanism. Without some segregation of formal entities into types, Russell's and other paradoxes would strike, it was feared, rendering every formula a theorem. More recently, types have come to play a similar role in computer science, to keep the bugs away. There are, moreover, famous interactions between systems of propositional logic and theories of types. These have been biased towards Heyting's intuitionist logic J, in view of the Curry-Howard correspondence. It is rather argued here, as in the author's work with Dezani, Motohama and Bono, that there is a better correspondence with the basic relevant logic B+. More than that, this paper develops the author's 1995 work showing that B+ may be conservatively extended to accommodate an outright Boolean negation. The resulting system, here called CB, will be a central focus of this paper.

1. Introduction

Types were introduced into Logic as a *defence* mechanism. Without some segregation of formal entities into types, Russell's and other paradoxes would strike, it was feared, rendering every formula a theorem. More recently, types have come to play a similar role in computer science, to keep the bugs away.

There are, moreover, famous interactions between systems of propositional logic and theories of types. On the whole, these have been biased towards Heyting's *intuitionist* logic **J**, in view of the Curry-Howard correspondence. It is rather argued here, as in Dezani *et al.* (2002), that there is a *better* correspondence with the basic *relevant* logic **B+**.

More than that, Meyer (1995) noted that **B+** may be *conservatively extended* to accommodate an outright Boolean negation ¬. The resulting system, here called **CB**, is studied further in Meyer *et al.* (2006). It will be an important focus of this paper.

2. Intersection Types and Basic Relevant Logic

B+ was introduced in Routley-Meyer (1972) as *the basic positive relevant logic*. This meant that, on the semantical analysis of Routley-Meyer (1973), only such postulates were imposed as went with the method. Thus the idea was that **B+** would stand to the strong positive relevant logics **E+** of entailment and **R+** of relevant implication roughly as the minimal normal modal logic **K** stands to the strong normal modal logics **S4** and **S5**, on the semantical analysis of Kripke (1963).

Unbeknownst to its authors, **B+** had (or would presently acquire) another life. For Coppo, Dezani and their European colleagues were independently developing, most notably in Barendregt *et al.* (1983), a theory **BCD** of *intersection types*. Types had been, since Russell (1908), a popular Way Out of the set-theoretic and semantic paradoxes. Types were introduced for Combinatory Logic (henceforth **CL**) and Lambda Calculus (henceforth **λ**) respectively in Curry (1934) and Church (1940). See also Curry-Feys 1958.

The **BCD** intersection type theory has important advantages over the Church and Curry schemes. First, *all combinators* can be typed in **BCD**. This contrasts with Curry-Feys 1958, where (for example) the combinator **WI** receives no type. Second, the intersection type theory actually provides *models* of **CL** and **λ.** Third, to reiterate observations from Dezani *et al.* (2002), the ternary relational semantics of relevant logics *applies to* **BCD**.

A. Syntactic Preliminaries

We presuppose a sentential language *L*, whose formulae are built up from a countable supply of atoms (sentential variables). We use ‘A’, etc., for the formulae and ‘p’, etc., for the atoms. We always suppose, here, that among the logical particles of *L* are the binary connectives (classical) conjunction $\wedge$ and (relevant) implication $\rightarrow$. (For the alternative story in which the formulae are taken as *types* and the particles are taken as *operations* on types, see Dezani *et al*. (2002).) We will have in mind some additional particles, such as classical negation $\neg$, disjunction $\vee$ and material implication $\supset$. Also interesting is a top truth **T** (the ω of Barendregt *et al.* (1983), taken there as the whole space of types). When the language *L* is so extended, we make the extra particles explicit thus: *L*[**T**] is the language in which **T** is an additional constant; *L*[$\neg$] that in which classical negation is primitive, etc. For ease in reading formulae we rank binary connectives $\wedge$, $\vee$, , $\supset$, $\rightarrow$ in order of *increasing* scope, otherwise associating equal particles to the *right*.

B. Ternary Relational Semantics

A 3-frame (formerly +ms) shall here be a triple

$$(1) \qquad \mathbf{K} = \langle 0, \mathrm{K}, \mathrm{R} \rangle,$$

where K is a set, $0 \in \mathrm{K}$, and $\mathrm{R} \subseteq \mathrm{K}^3$, subject to the following definition and postulates, for all a, b, c, a’, b’, c’ in K:

d$\subseteq$. $\quad a \subseteq b \ =\!\mathrm{df}\ \mathrm{R}0ab$

p1. $\quad a \subseteq a$

p2. $\quad (a' \subseteq a) \wedge (b' \subseteq b) \wedge (c \subseteq c') \supset \mathrm{R}abc \supset \mathrm{R}a'b'c'$

Metaphysically inclined readers may think of K as a set of worlds, 0 as a preferred logical world in K and R as a ternary *accessibility relation* on K. Demythologized, elements of K are just *logically closed* theories (or filters, if you are an algebraist). As our notation suggests, $\subseteq$ may be thought of as the *sub-theory* relation, which is by p1 reflexive and which by p2 is monotone decreasing in the first two arguments of the ternary relation R and which is monotone increasing in the final place. (By the time we are done going classical, dear Reader, p1 and p2 will have become *trivial.*)

C. Behind the Ternary Relation

The most conspicuous element in the semantics just presented is the *ternary relation* R. We would do well, accordingly, to demythologize R a little further. Behind R stands the *modus ponens product,* or *fusion,* operation o defined on *arbitrary* sets S and T of formulae. The underlying thought here is that a formula tells you *on its face,* so to speak, *why* it should be a *theorem* of Logic. (Or *not,* since there is disagreement about which formulae should be taken as *logically true*.) More than that, since Logic is above all an *applied science,* telling us how we should get on deductively in worldly theories and in ordinary life, the *shape* of a formula is also its central clue to the *prescriptions* it offers for everyday inferences from premises to conclusions. So here is the Powers (1976) definition of *modus ponens product*, for S, T $\subseteq$ *L*.

$$\text{Do. } S\text{o}T = \{C: \exists A[(A \rightarrow C \in S) \wedge (A \in T)]\}$$

That is, the *fusion* SoT of two theories S and T consists of all the formulae C obtained by performing →E on major premises A→C from S and minor premises A from T.

Why, you may wonder, have we chosen a binary operation o on theories to motivate a ternary relation R? The reason is that the operation has priority. For it is o that tracks *modus ponens,* and (an appropriate) respect for *modus ponens* is what the semantic analysis of → is all about.

Let us have another look at what → formulae are trying to tell us. We had better look at such a formula A→C as a *tree,*

$$\begin{array}{c} A \rightarrow C \\ / \quad \backslash \\ A \qquad C \end{array}$$

whose import is to tell us that, when we've got A, we can also get C.[1] Put otherwise, we clearly have, for any set of formulae S and formulae A and C,

$$(2) \qquad A \rightarrow C \in S \text{ iff } C \in S\text{o}\{A\}$$

We recall below how (2) yields an appropriate truth-condition T→ in models based on the 3-frame semantics.

Meanwhile, an open question: *Which* sets of formulae S should be taken with semantic *seriousness?* Philosophers will be tempted to answer, "Those S that might be taken to describe a *possible world*." We shall eventually see this answer as on the right track. (After all, we did suggest above that the members of our 3-frames K might be called "worlds".) But we do not want to arrive too quickly at such a conclusion. Rather what we aim for, in the wonderful phrase of Anderson, Belnap and Dunn (1992, p. 122), are theories that are *truth-like*.

Specifically, when *L* contains particles that are *intended classically,* like ∧, ∨, ¬, we expect a truth-like theory T to treat them classically, satisfying conditions like

C∧. $A \wedge B \in T$ iff $A \in T \wedge B \in T$

C∨. $A \vee B \in T$ iff $A \in T \vee B \in T$

[1] When yet a wee lad, the author got a little mixed up. He was warned by his mother that playing on the nearby railroad tracks would get him hit by a train, when he would feel an awful pain. He parsed this as "Excruciating tummy ache" → "I've been hit by a train". He had the antecedent, and tried *modus ponens*. But Mother rejected the consequent!

C¬. $\neg A \in T$ iff $A \notin T$

So let us get a little more deeply into our *intended* models. As the *preferred* syntactic counterparts of the "worlds", Routley-Meyer (1973) and its successors chose truth-like theories,[2] satisfying in particular the conditions C∧, C∨. In addition, that Logic should receive its due, theories were required to be closed under *provable* logical entailment. So when a logic **L** is in focus, we impose, for each $S \subseteq L$,

(3) S is **L**-closed iff, for $\forall A, B \in L$, **L** $\vdash A \to B \supset A \in S \supset B \in S$

(4) S is ∧-closed iff, for $\forall A, B \in L$, $A \in S \wedge B \in S \supset A \wedge B \in S$

(5) S is an **L**-theory iff S is **L**-closed and S is ∧-closed

(6) S is ∨-prime iff, for $\forall A, B \in L$, $A \vee B \in S \supset A \in S \vee B \in S$

(7) S is a prime **L**-theory iff S is a ∨-prime **L**-theory

It follows quickly, using the distributive lattice axioms of relevant logics **L**, that S is a prime **L**-theory iff S is **L**-closed and satisfies C∧ and C∨.

We return to the rationale behind the ternary relation. It would have been useful to base relevant semantics on the fusion operation o. But while it is easy to show that the fusion SoT of two **L**-theories is again an **L**-theory, it is simply *false* that the fusion of two *prime* **L**-theories is again ∨-prime. (Cf. Dezani *et al.* 2002 for counterexamples.) This presents us with an immediate quandary. We may weaken our attachment to truth-like theories by dropping C∨ above. Or we may save C∨ and go *relational,* trading in the fusion *operation* o for a ternary *relation* R. When one gets to the nitty-gritty of semantical *completeness* proofs, what this relation amounts to canonically, for prime **L**-theories x', y', z' is

(8) Rx'y'z' iff $x'oy' \subseteq z'$

The contrasting policies on which we have dwelt have actually been instantiated in independent semantic developments. They are, roughly speaking, different ways of *packaging* what is at root *the same* mathematics. While Routley-Meyer (1973) can claim priority for the first semantical completeness proof for a full relevant logic, they were quickly followed by the relational-operational semantics of Fine (1974). The distinction is without much of a difference. The operation Fine called *fusion* was already inside the ternary *relation*. And the Routley-Meyer smooth truth-condition on ∨ is hidden inside Fine's more intricate one. Finally, priority in the *area* belongs to Urquhart (1972), which developed the *first* operational semantics for relevant →.[3]

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[2] Due to the vagaries of publication schedules, some of those successors appeared in 1972.

[3] Urquhart developed his ideas on analysis of the Fitch style natural deduction schemes for **R** and its kin of Anderson-Belnap (1975). This is *not* the same mathematics as the "calculus of theories" approach of Routley, Meyer and Fine. Sadly, Urquhart's hope to extend his successful analysis of $\mathbf{R}_{\to\wedge}$ to all of $\mathbf{R}_{+}$ ran into a counterexample. Still, he was *first*.

D. Truth Conditions

It is the job of a formal semantics to say under what conditions formulae are true and false; and, building on that, to say what logically entails what. Given a 3-frame **K** = <0, K, R>, we begin with the notion of a *possible interpretation* I of *L* in **K**. Let **2** = {0, 1} be the set {*false, true*} of truth-values. Then I : $L \times K \rightarrow \mathbf{2}$ is a possible interpretation. I. e., *any function* I which assigns exactly one of the truth-values to each formula A at each point in K counts as a possible interpretation.

A possible interpretation I being given, we assume in context some notation that links the semantics with a corresponding first-order language. We will write

(9)	[A]a	for	I(A, a) = 1
(10)	¬[A]a	for	I(A, a) = 0

and we also *use* henceforth ¬, ∧, ∨, ⊃, ≡, ∀, ∃ in the obvious senses in our classical metalogic (quantifiers having been taken, since K is fixed, to range over K).

Let **I** = <0, K, R, I> be a possible interpretation of the language *L* in the 3-frame **K**. **I** is moreover an *interpretation,* or *model*, iff the following conditions hold, for all formulae A and B in *L* and all c, d in K, with quantifiers ranging over K.

(11) Truth-conditions

T∧.	[A∧B]c iff [A]c ∧ [B]c
T→.	[A→B]c iff ∀a∀b(Rcab ⊃ [A]a ⊃ [B]b)

In the presence of additional logical particles (of which more later), we impose also

T**T**.	[**T**]c
T¬.	[¬A]c iff ¬[A]c
T**F**.	¬[**F**]c
T∨.	[A∨B]c iff [A]c ∨ [B]c
To.	[AoB]c iff ∃a∃b(Rabc ∧ [A]a ∧ [B]b)

E. Heredity conditions

For all formulae A in *L* and all c, d in K, we impose moreover

H. c ⊆ d ⊃ [A]c ⊃ [A]d

The heredity condition H is reminiscent of a similar condition in Kripke (1965) for the semantical analysis of *intuitionist* logic. It rests on the thought that the ⊆ of d⊆ really does mean *sub-theory*. In the presence of other postulates and truth-conditions, it can very often (as in Kripke (1965), Routley-Meyer (1973, 1972)) be reduced to the condition

Hp. c ⊆ d ⊃ [p]c ⊃ [p]d

where p is a sentential variable.

F. Models

Again let **I** = <0, K, R, I> be a possible interpretation of the language *L* in the 3-frame **K**. We call **I** moreover an *interpretation,* or a *model,* of *L* in **K** provided that the heredity condition H and the truth conditions T→, T∧, etc., hold for **I**.[4]

We say of a formula A

T0. A is *verified* on **I** iff [A]0

There is an intimate connection on the ternary relational semantics between *verification* of implication statements B→C and a binary relation ≤ of propositional *entailment*. We signal this in the model **I** by

d≤. $B \leq C =df \ \forall a([B]a \supset [C]a)$

We recall next from Routley-Meyer 1972 the important *Semantic Entailment Lemma*.

SemEnt. In every model **I** we have B ≤ C iff [B→C]0.

That is, a relevant implication B→C is true on I at the central point 0 in K iff, for every point a in K either ¬[B]a or [C]a. Put otherwise, B→C is verified in our model **I** iff I is truth-preserving at every point in the model. We conclude this sub-section with

V0. A is *valid* in **K** iff A is verified in all models **I** = <**K**, I>

B0. A is *basically valid* iff A is valid in all 3-frames **K**

3. B∧T and the Combinators

We pause to recapitulate the system **B∧T** (pronounced *bat*) of Dezani *et al.* (2002) and to recall the accompanying model of **λ** and **CL** in its theories.

A. B[→, ∧, T]

We formulate **B∧T** in *L*[**T**],[5] with the following axioms and rules:

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[4] *L* may vary. Specifically we require that each *primitive* particle c of *L* satisfy its truth-condition Tc on I. We frame definitions so that *defined* particles c naturally also satisfy their truth-conditions Tc for a model **I**.

AxI.	$A \le A$
Ax∧E.	$A \wedge B \le A$
	$A \wedge B \le B$
Ax→∧I.	$(A \to B) \wedge (A \to C) \le A \to B \wedge C$
Ax**T**.	$A \le \mathbf{T}$
Ax**T→**.	$\mathbf{T} \le \mathbf{T} \to \mathbf{T}$
Rul→E.	$A \le C \supset (\vdash A \supset \vdash C)$
Rul∧I.	$(\vdash A \wedge \vdash B) \supset \vdash (A \wedge B)$
Rul**B**.	$(B \le C) \supset (A \to B \le A \to C)$
Rul**B'**.	$(A \le B) \supset (B \to C \le A \to C)$

Characteristic of the minimal relevant environment of **B+** is that many principles which appear as *axioms* in stronger familiar systems have been weakened to *rules*. Note also our appeal to the *material* vocabulary in stating rules. What, for example, the *prefixing* Rul**B** says is that if $B \to C$ is a *theorem* then also $(A \to B) \to (A \to C)$ is a theorem of **B∧T**. But prefixing *formulae* $(B \to C) \to ((A \to B) \to (A \to C))$ are *not,* in general, basic theorems. As for $(B \to C) \supset ((A \to B) \to (A \to C))$, it is by no means a theorem, since $\supset$ is neither primitive nor definable in the language *L*[**T**].[6]

The role of the top truth **T** is also interesting. This Church constant stands in for the ω of the **BCD** system equivalent to **B∧T**. Of special interest is the (at first sight weird) Ax**T→**. This assures that $A \to \mathbf{T}$ will always be **B∧T** equivalent to **T,** mirroring the material principle that conditionals with true consequents are true. (*Other* material principles, relevantists will be relieved to hear, are *not* so mirrored.) Incidentally, **T** is in the vocabulary here (as ω was in **BCD**) to look after the (rather irrelevant) combinator **K**. Its semantical correlate in completeness proofs is that only *non-empty* theories (all of which contain **T**) will count. If we have no need of **K** and its fellow *cancellators,* as in the **λI** systems preferred in Church (1941), then **T** and its special treatment can also be chopped.

B. Combinator Theories

A *theory,* to reiterate, is here any subset $S \subseteq L(\mathbf{T})$ such that, for any formulae A, B, we have

Entailment closure: $(A \le B) \supset ((A \in S) \supset (B \in S))$

Conjunction closure: $(A \in S) \wedge (B \in S) \supset (A \wedge B \in S)$

T closure: $\mathbf{T} \in S$

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[5] So primitive particles are the constant **T** and just the binary connectives ∧ and →. We may write $A \le C$ in place of $\vdash A \to C$, to spare parentheses and suggest algebraic links.

[6] There is a nice clue here as to what will happen when we go classical and admit Boolean ¬ (and its mate Boolean ⊃) to the vocabulary. Then indeed the formula just displayed will *become* a theorem, while the rules may be shrunk to only one: *modus ponens* for ⊃.

We can put this more succinctly if we say that S is a theory iff it is closed under arbitrary conjunctions of its finite subsets and provable **B∧T** entailment. So put, this counts (as lattice theory might suggest) the top element **T** as the conjunction of all the members of the *empty* subset ∅ of S, forcing the theory S itself to be non-empty.

There are many **B∧T** theories, but some are more equal than others. Of particular interest are the theories generated by Curry-style *Combinators*. We indicate these briefly with the following notational device. Where A is a formula, let #[A] be the smallest theory that contains all substitution instances of A. We then define

I : #[p→p]
B: #[(q→r) → (p→q)→(p→r)]
C: #[(p→(q→r)) → (q→(p→r))]
W: #[(p→(p→q)) → (p→q)]
K: #[p → (q→p)]
S: #[(p→(q→r)) → (p→q)→(p→r)]

The formulae displayed are those that Curry-Feys (1958) **CL** fans will be expecting. But you may be a bit surprised at the *interpretation* here placed on them. For, unless you are also an *intersection type theory* **BCD** fan, you may not notice that our Interpreted Combinators (henceforth, IC's) are not simply formulae or types but whole *theories*. We begin with the splendid

I fact. A ∈ **I** iff A is a theorem of **B∧T**.

That is, as was also noted for **B+** in Dezani-Frisch *et al*. (2002), the interpreted combinator **I** consists *exactly* of the theorems of the underlying logic.

Proof of the **I** fact is by a simple deductive induction, left to the reader.

What shall we make of the other IC's? We may reasonably look at them as further axiom *candidates,* in substructural logics strictly stronger than **B∧T**. For example the corresponding fragment of **R** (should we call it **R∧T**?) is the theory determined by *all* of the Interpreted Combinators **B**, **C**, **I**, **W**, containing in particular all unions of fusions of the IC's. Not only that, but when an IC makes it into a Logic **L**, as a sub-theory of **L**, the combinator becomes available as a basis for *structural rules* in a corresponding Gentzen consecution calculus.[7]

It is also interesting when we *combine* IC's under *modus ponens product o*. Recall that ∧ and **T** are in the language. This means, in the first instance, that *every combinator* has a corresponding IC, including ones that, in the original Curry scheme, are untypeable. This is immediately clear, just from the fact that *all fusions* of theories contain **T**. (So, in the worst instance, any combinator built by application from **S** and **K** and their mates has at least the minimal IC whose members are **T** and its **B∧T** equivalents.)

C. Non-Curry IC's

We mused in the last sub-section how *all* of the usual Combinators of Curry's **CL** have interpretants among the **B∧T**-theories. But there is no need to stop there. *Any*

[7] Curry (1963) and the Display Logic of Belnap (1982) make much of this fact.

B∧**T**-theory may, in our present perspective, be counted among the IC's. Should not a Classicist, for example, view with some enthusiasm an IC

P: #[((p→q)→p) → p],

which adds to the formula schemes above one that will deliver *all* of Classical Logic? (After all, **S** and **K** alone, with *modus ponens,* suffice for the pure intuitionist implicational logic **J→**, to which it is well known that the addition of Peirce's Law **P** will produce the pure classical 2-valued implicational calculus **2→**.)

Or we might try other directions completely. Of the same shape as **P** but different content is the Axiom of Relativity, which Meyer-Slaney (1979, 1989) picked to formulate their Abelian logic.

Rel: #[((p→q)→q) → p]

To be sure, **Rel** is *not* a 2-valued tautology. Worse, it is incompatible with the **T** axioms. Still, it serves (with **B** and the ∧ axioms above) to axiomatize (in the →, ∧ vocabulary) the logic **A** of lattice-ordered Abelian groups.

4. Let's Go Classical

Despite our musings re Peirce's Law **P** in the last section, we have an entirely different route in mind for making the basic relevant logic a subsystem of classical 2-valued logic **2**. Here goes.

A. The system CB

The language of **CB**, to begin with, is the language *L*[¬], with Boolean ¬ primitive, along with →, ∧. A *Boolean tautology,* in this vocabulary, will be any substitution instance of a classical tautology in the Boolean particles ∧ and ¬ (and in Boolean particles immediately to be defined, as follows).

DEFINITION	DEFINIENDUM	DEFINIENS	CONDITION
D∨	A ∨ B	¬(¬A ∧ ¬B)	
D**T**	**T**	p ∨ ¬p	p is the *first* atom
D**F**	**F**	¬**T**	
D⊃	A ⊃ B	¬A ∨ B	
D≡	A ≡ B	(A ⊃ B)∧(B ⊃ A)	

When we go Boolean, we get the top truth **T** for free (our choice of the first sentential variable in the *definiens* is indifferent, all Boolean tautologies being equivalent). We also get free the terrible falsehood **F**, whose fate it is to entail *everything*. We can less interestingly define as well some properly *relevant* particles, which *do not count* as Boolean.

DEFINITION	DEFINIENDUM	DEFINIENS
D□	□A	**T** → A
D◊	◊A	¬□¬A
D©	A © B	¬(A → ¬B)
D↔	A ↔ B	(A→B) ∧ (B→A)

Of the particles just defined, ↔ is a familiar relevant equivalence. The new binary particle © is a relevant *consistency* operation (*not* to be confused with fusion o, which is not yet introduced). Finally □ and ◊ are unary modal operators, reminiscent of these (weak) operations in the minimal normal modal logic **K**.

It is time for some axioms and rules. Under the same conventions as in section **3A**, we choose the following:

AxBool.	A ≤ C, when A ⊃ C is a Boolean tautology
Ax→⊃.	(A→B) ⊃ (A ⊃ B)
AxAntilog.	(A∧B → C) ⊃ (A∧¬C → ¬B)
Ax**B**⊃.	(B → C) ⊃ ((A→B) → (A→C))
Ax**B'**⊃.	(A → B) ⊃ ((B→C) → (A→C))
Ax→∧.	(A→C) ∧ (B→D) ≤ A∧B → C∧D
Ax→∨.	(A→C) ∧ (B→D) ≤ A∨B → C∨D
Ax**T→**.	**T ≤ T→T**
Rul⊃E.	⊢ (A ⊃ C) ⊃ (⊢A ⊃ ⊢C)

Note that AxBool has rendered our previous AxI, Ax∧E and Ax**T** redundant.

5. The B∧T Model in the Classical Framework

We take the opportunity to review the **BCD** "filter model" of λ-calculus, with the aim of adapting it to the classical **CB** situation. This is for us a Theory **M**odel **M** in non-empty **B∧T**-theories. Given the definability of **λ** in **CL**, we treat **M** as a function whose *arguments* are **CL**-terms and whose *values* are **B∧T**-theories.

A. Terms and Environments

The **CL**-terms shall consist of countably many individual variables, for which we use 'x', etc.; primitive constants for combinators, among which shall be **S**, **K**, **I**, and perhaps others; terms shall then be built up as usual under the binary application operation ·, always eliminated for simple juxtaposition, with association to the *left*. We use 't', etc., for terms.

We already have a suggested interpretation for application · in **B∧T**-theories—namely, the *modus ponens product* (or *fusion*) operation o defined by Do in section 2C. It is next necessary to say what values the model **M** will take on primitive combinators like **I**. But that is easy. We exposed our policy in 3B, which readers may consult to see which theories are assigned to which primitive combinators.

But *what,* you may well wonder, should **M** assign to a *variable* x? The only reasonable answer is, "A non-empty theory S," in the sense of 3B. To bring this under the tent, we introduce the notion of an *environment*—which, here, shall be any function **m** such that

(11) **m**(x) is a *theory*, for each variable x
(12) **m**(t) = **M**(t), for each primitive combinator t
(13) **m**(tu) = **m**(t) o **m**(u), for all terms t and u

There are uncountably many environments **m**. But they all lead to **M**.

B. The Calculus CBT of B∧T-theories and its Relational Structure BRT

Let us now back up from the last section. We already know that, in the presence of additional Boolean connectives like ∨ and ¬, we can no longer count on the resulting theories to be *truth-like*. Still, we will concentrate on the Calculus **CBT** of **B∧T**-theories, and pass quickly to its relational structure.

i. **The Calculus CBT = <CBT, o, ⊆, I>.** Let CBT be the class of non-empty theories, and let o be defined by Do, where ⊆ is sub-theory and **I** is the theorems of **B∧T**. The environments **m** and the model **M** are realizations of **CL** in **CBT**.

ii. **The Relational Structure BRT = <I, CBT, R>.** We now look at CBT no longer as an *algebra* but instead as a 3-*frame* <0, K, R>, with a relevant ternary accessibility relation R defined on K. 0 is of course the "combinator" **I**, which by the **I** fact of section 3B is the set of **B∧T**-theorems. K remains the class CBT of *all* **B∧T**-theories. And R is defined as in (8) above, for any theories x, y, z, by

iii.

(dR) Rxyz =df xoy ⊆ z

C. Conservatively Extending B∧T to CB

It is almost but not quite obvious that whatever can be done with **B∧T** can equally well be done by its extension **CB**, in which Boolean ¬ becomes *explicit,* and ∨ and **T** accordingly become *definable*. So we adapt here the policy, already invoked in the related context of Meyer-Routley (1973), which enables us to tweak a countermodel in the smaller vocabulary so that it remains one after enrichment with ¬.

Let **I** = <**K**, I> be an interpretation, where **K** = <0, K, R> is a 3-frame. Let G be a new element (G ∉ K). We define the *Boolean enrichment* $\mathbf{K_G}$ = <G, K_G, R_G> thus:

(14) $K_G = K \cup \{G\}$
(15) For all a, b, c ∈ K, Rabc iff $R_G$abc

(16) For all a, b $\in K_G$, R_GGab iff a = b
(17) Except as constrained just above, $\neg R_G$abc

KG is the Boolean enrichment of the 3-frame **K**. We now define the Boolean enrichment I_G of the interpretation I, thus:

(18) For all a $\in$ K and formulae A, I_G(A, a) = I(A, a), where the latter is defined
(19) For all formulae A, I_G(A, G) = I(A, 0), where the latter is defined
(20) Otherwise define I_G(A) using whichever of T$\rightarrow$, T$\wedge$, T$\neg$ is appropriate

We leave it to the reader to verify that $\mathbf{I_G}$ **=** **<$\mathbf{K_G}$,** I_G> is indeed an interpretation of the full Boolean language *L[¬]* in the 3-frame $\mathbf{K_G}$. But worthy of attention is the

Boolean Enrichment Theorem. The following conditions are equivalent:

(i) The formula A of *L*[**T**] is a theorem of **B∧T**
(ii) The formula A of *L[¬]* is a theorem of **CB**

Proof. We assume that A is translated into *L[¬]* using the definition D**T**, if required. That (i) ➔ (ii) is a straightforward deductive induction. We conclude the proof by contraposition, assuming the denial of (i). By the semantical completeness of **B∧T**, there is then a countermodel **I = <K,** I> for A. I. e., A is false on I at 0. Consider now the Boolean enrichment $\mathbf{I_G}$ **=** **<$\mathbf{K_G}$,** I_G> of **I**. Note that, in K_G, we have switched the "logical world" from 0 to the new element G. Nonetheless, we may complete the proof by structural induction on A, noting the following for all subformulas B of A:

(a) For all a $\in$ K, I_G(B, a) = I(B, a)
(b) I_G(B, G) = I_G(B, 0)

It occurs to me, at this point, that I might have left a little *too much* to the reader in the remarks preceding the present theorem. For (a) and (b) merely restate (18) and (19) above. The *real* key to the theorem is that IG, as defined thereby, truly is an interpretation, satisfying T$\rightarrow$ on all of K_G. This could most conspicuously fail if, for some subformula B$\rightarrow$C of A, we had [B$\rightarrow$C]0 on I but, in violation of T$\rightarrow$, R_GGab and [B]a and ¬[C]b on I_G, for some a, b $\in K_G$. Readers who have done the homework will already have noted that this possibility *cannot arise,* in virtue of the Semantic Entailment observation SemEnt above. So every subformula of A, including A itself, is verified on I_G iff it is verified on I. But we picked I because it makes A false at 0; accordingly I_G makes A false at G. So, given the semantic completeness of **CB**, A remains a non-theorem of that system, ending the proof of the theorem.

6. The Key to the Universe?

The author has long held, and is in print with several co-authors to the effect that, the combinatory character of relevant postulates is the key to the semantical universe. Nonetheless, he has been pleasantly surprised by how good this connection has hitherto proved to be, especially in the work on **B∧T**-theories detailed above.

But I must say something more, in conclusion, about the degree to which we can make a *Boolean connection* in advancing the project further. Recall that the genesis of the present investigations lay in seeking understanding of highly *non-classical* logics. (So Boole certainly was *not* on the initial agenda, except as an *opponent*.) Still, certain particles always had a Boolean flavor; conjunction $\wedge$ led that charge, with the distributive lattice $\vee$ and the DeMorgan negation ~ not far behind.

We now have a look at some *clues,* garnered from semantical completeness proofs, which may prove useful in yielding full-blooded Boolean extensions of non-Boolean type theories.

Clue #1. Attend to *prime* theories

A theory S is *prime* just in case it satisfies

($\vee$EI) $\quad$ $A \vee B \in S$ iff $(A \in S) \vee (B \in S)$

Associated with prime theories are the following useful facts, which depend on the distributive lattice properties of $\wedge$ and $\vee$:

Intersection Fact A. A theory S is the *intersection* of its prime extensions
Extension Fact B. Suppose that $A \notin S$, where S is a theory. Then there is a prime theory S' such that $S \subseteq S'$ and $A \notin S'$
Squeezing Fact C. Let P and Q be theories, and let S' be a prime theory. Suppose $PoQ \subseteq S'$. Then there are prime theories P' and Q' such that $P \subseteq P'$, $Q \subseteq Q'$ and $P'oQ \subseteq S'$ and $PoQ' \subseteq S'$ and $P'oQ' \subseteq S'$.

Clue #2. Attend to *ultra* theories

A theory S is *ultra* (or *maximal*) just in case it satisfies

($\neg$EI) $\quad$ $\neg A \in S$ iff $A \notin S$

Ultratheories are what prime theories come to in the presence of Boolean $\neg$. Accordingly the intersection, extension, and squeezing facts just above continue to hold for them.

Clue #3. Watch the combinators

We listed above some **B$\wedge$T**-theories that correspond to axiom candidates for well-known substructural logics. These theories play a dual role. First, they *enforce* certain first-order postulates in the 3-frame semantics for logics that count the candidates among their theorems. For example, corresponding to axioms **B**, **C** and **W** above, get

Postulate **B**. $Rabcd \supset Ra(bc)d$
Postulate **C**. $Rabcd \supset Racbd$
Postulate **W**. $Rabc \supset Rabbc$

Second, the combinator theories already play their appointed role, under the Curry isomorphism, even in logics of which they are *non-theorems*. This is clearest in the calculus of theories. For using juxtaposition for fusion o and otherwise invoking **CL** notation, our **B**∧**T**-theories **B, C, W** induce the facts, for arbitrary theories a, b, c,

B fact. **B**abc = a(bc)
C fact. **C**abc = acb
W fact. **W**ab = abb

Clue #4. Attend to substitution-closed theories

The combinator theories have another pleasant property. Their members contain, together with their instances, all substitution instances of their instances. This is another item on which we should focus when we expand the vocabulary—to include, for example, Boolean ¬. The theory **I** (which is now **CB** itself) has got *bigger*. It contains, for example, ¬p → ¬p, which was not previously in the vocabulary. And just so have our other combinator theories got bigger.

References:

References for Venice6.doc

The following abbreviations are used:

JPL	**Journal of Philosophical Logic**
JSL	**The Journal of Symbolic Logic**
NDJFL	**Notre Dame Journal of Formal Logic**
SL	**Studia Logica**
ZML	**Zeitschrift für mathematische Logik und Grundlagen der Mathematik**

Anderson, Alan R. and Nuel D. Belnap, Jr., 1975. *Entailment, vol. 1,* Princeton University Press, Princeton.

Anderson, Alan R., Nuel D. Belnap, Jr., and J. Michael Dunn, 1992. *Entailment, vol. 2,* Princeton University Press, Princeton.

Barendregt, Henk, Mario Coppo and Mariangiola Dezani-Ciancaglini, 1983. *A filter lambda model and the completeness of type assignment,* **JSL 48:** 931-940.

Belnap, Nuel D., 1982. *Display Logic,* **JPL 11:** 357-417. Incorporated in Anderson *et al.* 1992.

Belnap, Nuel D., and J. Michael Dunn. 1981, *Entailment and the disjunctive syllogism.* In **Philosophy of language/philosophical logic,** ed. F. Floistad and G. H. von Wright, Martinus Nijhoff, The Hague, pp. 337-366. Reprinted as section 80 in Anderson, Belnap and Dunn (1992), pp. 488-506.

Church, Alonzo. 1940. *A formulation of the simple theory of types,* **JSL 5:** 56-68.

Church, Alonzo. 1941. *The calculi of lambda conversion,* Princeton.

Curry, Haskell Brooks, 1934. *Functionality in combinatory logic, Proceedings of the National Academy of Sciences U. S. A. 20:* 584-590.

Curry, Haskell B., 1963. *Foundations of Mathematical Logic,* McGraw-Hill, N. Y.

Curry, Haskell B., and R. Feys, 1958. *Combinatory Logic, vol. 1,* N. Holland, Amsterdam.

Dezani-Ciancaglini, M., R. K. Meyer and Y. Motohama, 2002. *The semantics of entailment omega,* **NDJFL 43:** 129-145.

Dezani-Ciancaglini, M., A. Frisch, E. Giovannetti and Y. Motohama, 2002. *The Relevance of Semantic Subtyping, Electronic Notes in Theoretical Computer Science 70 No. 1, 15pp.*

Kripke, Saul A. 1963, *Semantical analysis of modal logic I: normal modal propositional calculi,* **ZML 9**, pp. 67-96.

Kripke, Saul A. 1965, *Semantical analysis of intuitionist logic I.* In *Formal systems and recursive functions***,** ed. J. N. Crossley and M. Dummett, N. Holland, Amsterdam, pp. 92-129.

Meyer, Robert K., 1995. *Types and the Boolean system B+*

Meyer, Robert K., Yoko Motohama and Viviana Bono. 2006, *Truth Translations of Basic Relevant Logics,* t. a. in Bryson Brown and Francois Lepage, eds., *Truth and Probability: Essays in Honour of Hugues Leblanc,* presented to Society for Exact Philosophy, Leblanc Memorial Conference, 2001, http://www.collegepublications.co.uk

Meyer, Robert K., and Richard Routley. 1973, *Classical relevant logics. I,* **SL 32:** 51-66.

Meyer, Robert K., and Richard Routley, 1974. *Classical relevant logics II,* **SL 33:** 183-194.

Meyer, R. K., and J. K. Slaney, 1979. *Abelian Logic (From A to Z),* Research Paper No. 7, Logic Group, Dept. of Philosophy, RSSS, Australian National University.

Meyer, R. K., and J. K. Slaney, 1989. *Abelian Logic from A to Z,* in G. Priest, R. Sylvan and J. Norman, eds., *Paraconsistent Logic: Essays on the Inconsistent,* Philosophica Verlag, Munich: 245-288.

Powers, Larry, 1976. *On P-W,* **Relevance Logic Newsletter 1:** 131-142.

Routley, Richard, and Robert K. Meyer. 1973, *The semantics of entailment (I),* in H. Leblanc, ed., *Truth, syntax and modality,* N. Holland, Amsterdam, pp. 194-243.

Routley, Richard, and Robert K. Meyer. 1972, *The semantics of entailment III,* **JPL 1**: 192-208.

Russell, Bertrand. 1908, *Mathematical logic as based on the theory of types,* **American journal of mathematics 30:** 222-262.