Classical Logic of Bunched Implications

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Abstract. We consider a classical (propositional) version, CBI, of O'Hearn and Pym's logic of bunched implications (BI) from a model- and proof-theoretic perspective. We present a class of classical models of BI which extend the usual BI-models, based on partial commutative monoids, with an algebraic notion of "resource negation". This class of models gives rise to natural definitions of multiplicative falsity, negation and disjunction. We demonstrate that a sequent calculus proof system for CBI is sound with respect to our classical models by translating CBI sequent proofs into proofs in BI^+, a sound extension of sequent calculus for BI.

1 Introduction

The *logic of bunched implications* (BI), due to O'Hearn and Pym [6], is a substructural logic suitable for reasoning about various domains that incorporate a notion of *resource* [5]. Its best-known application to computer science is in *separation logic*, a logic for reasoning about imperative, pointer-manipulating programs, which essentially is obtained by considering a particular model of BI based on heaps [9]. Semantically, BI arises by considering cartesian doubly closed categories. This viewpoint gives rise to the following (propositional) connectives for BI:

Additive:	$\top$	$\bot$	$\wedge$	$\vee$	$\rightarrow$
Multiplicative:	$\top^*$		$*$		$-\!\!*$

From the aforementioned categorical perspective, this presentation of BI is necessarily intuitionistic. However, there is also an algebraic semantics of BI in which the multiplicatives are modelled by partial commutative monoids, and the additive connectives can be interpreted either classically or intuitionistically according to preference. When the additives are interpreted classically (e.g. using a boolean algebra) the resulting logic is often called *boolean* BI. In this paper, we consider the extension of boolean BI to *classical* BI, in which both the additives and the multiplicatives are treated classically. Specifically, classical BI includes the multiplicative analogues of additive falsity, negation and disjunction, which are "missing" in BI. We give an algebraic semantics for classical BI, and present a sequent calculus proof system that is sound with respect to this semantics.

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In BI, the presence of the two implications ($\rightarrow$ and $-\!\!*$), each with a suitably adjunctive conjunction ($\wedge$ and $*$ respectively), gives rise to two context-forming operations, ';' and ',':

$$\frac{\Gamma; F_1 \vdash F_2}{\Gamma \vdash F_1 \rightarrow F_2}\ (\rightarrow\text{R}) \qquad \frac{\Gamma, F_1 \vdash F_2}{\Gamma \vdash F_1 -\!\!* F_2}\ (-\!\!*\text{R})$$

Accordingly, contexts on the left-hand side of sequents are not sets or lists, as in standard sequent systems, but rather *bunches*: trees whose leaves are formulas and whose internal nodes are either ';' or ',', denoting additive and multiplicative combination respectively. The crucial difference between the two operations is that weakening and contraction are possible for ';' but not for ','. Since BI is an intuitionistic logic, bunches arise only on the left-hand side of sequents, with a single formula on the right.

From a proof-theoretic perspective, it is natural to consider a full two-sided sequent calculus for a classical extension of BI, i.e., a proof system employing sequents of the form $\Gamma \vdash \Delta$ where both Γ and Δ are bunches. We would expect the context-forming operations in Δ to logically correspond to additive and multiplicative disjunction (again in analogy to ordinary sequent calculus). Formulating a notion of multiplicative disjunction, however, entails formulating a multiplicative falsity $\bot^*$, which would act as the unit of ',' on the right of sequents, just as $\top^*$ acts as the unit of ',' on the left. Also, additive negation has natural rules in a two-sided calculus, swapping sides with respect to ';'. We would expect a corresponding notion of multiplicative negation, $\sim$, that swaps sides with respect to ',':

$$\frac{\Gamma \vdash F, \Delta}{\Gamma, \sim F \vdash \Delta}\ (\sim\text{L}) \qquad \frac{\Gamma, F \vdash \Delta}{\Gamma \vdash \sim F, \Delta}\ (\sim\text{R})$$

Indeed, just such rules are considered (briefly) by Pym in his monograph on BI although, as mentioned there, there are problems with cut-elimination [7]. An important concern is therefore the following one originally considered in the context of standard BI:

> "It is all very well to postulate proof rules in this way, but what meaning or significance, if any, does the resulting logic have?" [6]

It is with this question that we primarily concern ourselves in this paper. After briefly recalling the BI sequent calculus and its partial commutative monoid semantics in Section 2, we present in Section 3 a class of so-called "classical BI-models" in which multiplicative versions of falsity, negation, and disjunction all have natural interpretations. Our models extend partial commutative monoids with a natural notion of negation, –. In fact, they are almost Abelian groups, except that the result of combining an element x with its "inverse" $-x$ using the monoid operation is not necessarily the monoid identity, but rather some arbitrary element, denoted by ∞. In Section 4 we present a classical version of BI, called CBI, and define satisfaction of its formulas with respect to our classical BI-models. Interestingly, the natural interpretations of multiplicative falsity and negation are not ∞ and $-$ respectively, as one might expect, but rather $\neg\infty$ and $\neg-$. (Similar ideas are also employed by relevant logicians [8, 4].) We also give a two-sided sequent calculus for CBI (similar to the presentation in [7]). In Section 5 we present the main technical result of this paper, which is a proof of soundness of this sequent calculus with respect to our classical models. To do so, we first define an extension BI^+ of the BI sequent calculus with a formula representing ∞ and two axioms representing the fact that $-$ behaves as an involution in our models. BI^+ is easily proven sound with respect to our models, whence the required soundness theorem for CBI follows by showing admissibility of its proof rules under a validity-preserving translation from CBI-sequents to BI^+-sequents. Finally, in Section 6, we conclude and identify the main directions for future work.

Due to space limitations, some of the details of our proofs have been suppressed. For full proofs — and reports on substantial developments since the initial submission of the present paper — we point the interested reader to a recent paper by the authors, currently available as a technical report [3].

2 Propositional boolean BI

In this section we give a brief overview of (propositional) boolean BI, focussing on its monoid semantics and sequent calculus presentation (cf. [6, 7]).

Formulas of propositional BI are obtained by combining atomic formulas — $\top, \bot$ and $\top^*$, plus propositional variables drawn from a set $\mathcal{V}$ — using the binary connectives $\wedge, \vee, \rightarrow, *$ and $-\!\!*$. We write $\neg F$ to abbreviate the formula $F \rightarrow \bot$. A *model* M of BI is then partial commutative monoid $\langle R, \circ, e\rangle$. An *environment* for a BI-model M is a function $\rho : \mathcal{V} \rightarrow R$ interpreting propositional variables as propositions, relative to elements of the monoid. Intuitively, the set R corresponds to a set of *resources* which can (possibly) be combined by $\circ$; the unit e of $\circ$ is then the empty or null resource. *Satisfaction* of a formula in a model $M = \langle R, \circ, e\rangle$ under an environment ρ for M is then defined relative to resources $r \in R$, with the key clauses being those for the propositional variables and multiplicative connectives:

$$\begin{array}{rcl} r \models P & \Leftrightarrow & r \in \rho(P) \\ r \models \top^* & \Leftrightarrow & r = e \\ r \models F_1 * F_2 & \Leftrightarrow & \exists r_1, r_2.\, r = r_1 \circ r_2 \text{ and } r_1 \models F_1 \text{ and } r_2 \models F_2 \\ r \models F_1 -\!\!* F_2 & \Leftrightarrow & \forall r'.\, r \circ r' \text{ defined and } r' \models F_1 \text{ implies } r \circ r' \models F_2 \end{array}$$

The clauses for the additive connectives are defined in the standard way, i.e. without performing any operation on the resource r. In particular, additive implication is interpreted classically, so that $r \models \neg F$ iff $r \not\models F$.

One can then give a sequent calculus for BI, writing sequents of the form $\Gamma \vdash F$, where F is a formula and Γ is a *bunch*, given by the following grammar:

$$\Gamma ::= F \mid \Gamma; \Gamma \mid \Gamma, \Gamma$$

where F ranges over formulas. Thus bunches are trees whose leaves are formulas and whose internal nodes are either ';' or ','. We write $\Gamma(\Delta)$ for a bunch of which Δ is a distinguished sub-bunch (i.e. subtree), and in such cases write $\Gamma(\Delta')$ for the bunch obtained by replacing Δ by the bunch Δ' in $\Gamma(\Delta)$. In analogy to the use of sets in ordinary sequent calculus, we consider bunches up to *coherent equivalence*:

Definition 2.1 (Coherent equivalence). $\equiv$ is the least relation on bunches satisfying commutative monoid equations for ';' and $\top$, and for ',' and $\top^*$, plus the rule of congruence: if $\Delta \equiv \Delta'$ then $\Gamma(\Delta) \equiv \Gamma(\Delta')$.

The sequent calculus for BI employs sequents of the form $\Gamma \vdash F$, where Γ is a bunch and F is a formula. We give the sequent calculus rules for BI in Figure 1.

Structural rules:

$$\frac{}{F \vdash F}\,(\text{Id}) \qquad \frac{\Gamma(\Delta) \vdash F}{\Gamma(\Delta;\Delta') \vdash F}\,(\text{Weak}) \qquad \frac{\Gamma(\Delta;\Delta) \vdash F}{\Gamma(\Delta) \vdash F}\,(\text{Contr})$$

$$\frac{\Gamma' \vdash F}{\Gamma \vdash F}\ \Gamma \equiv \Gamma'\ (\text{Equiv}) \qquad \frac{\Delta \vdash G \quad \Gamma(G) \vdash F}{\Gamma(\Delta) \vdash F}\,(\text{Cut})$$

Propositional rules:

$$\frac{}{\Gamma(\bot) \vdash F}\,(\bot\text{L}) \qquad \frac{\Gamma(F_1) \vdash F \quad \Gamma(F_2) \vdash F}{\Gamma(F_1 \vee F_2) \vdash F}\,(\vee\text{L}) \qquad \frac{\Gamma(F_1;F_2) \vdash F}{\Gamma(F_1 \wedge F_2) \vdash F}\,(\wedge\text{L})$$

$$\frac{}{\Gamma \vdash \top}\,(\top\text{R}) \qquad \frac{\Gamma \vdash F_i}{\Gamma \vdash F_1 \vee F_2}\ i \in \{1,2\}\ (\vee R_i) \qquad \frac{\Gamma \vdash F_1 \quad \Gamma \vdash F_2}{\Gamma \vdash F_1 \wedge F_2}\,(\wedge\text{R})$$

$$\frac{\Delta \vdash F_1 \quad \Gamma(F_2) \vdash F}{\Gamma(\Delta, F_1 \mathrel{-\!*} F_2) \vdash F}\,(\mathrel{-\!*}\text{L}) \qquad \frac{\Delta \vdash F_1 \quad \Gamma(\Delta;F_2) \vdash F}{\Gamma(\Delta; F_1 \to F_2) \vdash F}\,(\to\text{L}) \qquad \frac{\Gamma(F_1,F_2) \vdash F}{\Gamma(F_1 * F_2) \vdash F}\,(*\text{L})$$

$$\frac{\Gamma, F_1 \vdash F_2}{\Gamma \vdash F_1 \mathrel{-\!*} F_2}\,(\mathrel{-\!*}\text{R}) \qquad \frac{\Gamma; F_1 \vdash F_2}{\Gamma \vdash F_1 \to F_2}\,(\to\text{R}) \qquad \frac{\Gamma \vdash F_1 \quad \Delta \vdash F_2}{\Gamma, \Delta \vdash F_1 * F_2}\,(*\text{R})$$

$$\frac{}{\neg\neg F \vdash F}\,(\text{DNE})$$

Fig. 1. Sequent calculus rules for propositional BI.

3 Classical models for BI

Definition 3.1 (Classical BI-model). A *classical* BI-*model* is given by $\langle R, \circ, e, -, \infty \rangle$, where $\langle R, \circ, e \rangle$ is a partial commutative monoid, $\infty \in R$ and $- : R \to R$ is a function such that for each $x \in R$, $-x$ is the unique element of R satisfying $x \circ -x = \infty$. We extend $-$ to subsets of R by $-X = \{-x \mid x \in X\}$.

Proposition 3.2. *If $\langle R, \circ, e, -, \infty\rangle$ is a classical* BI*-model then:*

1. $-e = \infty$*;*
2. $\forall x \in R. -(-x) = x$*;*
3. $\forall X \subseteq R. -(-X) = X$*;*
4. $\forall X \subseteq R. R \setminus (-X) = -(R \setminus X)$*;*
5. *if $e = \infty$ then $\circ$ is total and $\langle R, \circ, e, -\rangle$ is an Abelian group.*

Proof. 1. We have $-e = e \circ -e = \infty$ by definition of $-$ and e.
2. We have both $-(-x) \circ -x = \infty$ and $x \circ -x = \infty$ by definition of $-$, so by the uniqueness property of $-$ we must have $-(-x) = x$.
3. Follows immediately from 2.
4. ($\subseteq$) Suppose $x \in R \setminus (-X)$, i.e. $x \notin -X$. Since x satisfies $x \circ -x = \infty$ it must be the case that $-x \notin X$. Since $-(-x) = x$ we have $x \in \{-y \mid y \notin X\} = -(R \setminus X)$ as required.
   ($\supseteq$) Suppose $x \in -(R \setminus X)$, i.e. $x = -y$ where $y \notin X$. By the uniqueness property of $-$, x is not $-z$ for any $z \in X$. Thus $x \notin \{-z \mid z \in X\}$, i.e. $x \in R \setminus (-X)$ as required.
5. To see that $x \circ y$ is defined for any $x, y \in R$, observe that $-x \circ (x \circ y) = (-x \circ x) \circ y = e \circ y = y$. Since y is defined, so is $-x \circ (x \circ y)$, which can only be the case if $x \circ y$ is defined.
   To see that $\langle R, \circ, e, -\rangle$ is an Abelian group, observe that it is already a partial commutative monoid by definition, $\circ$ is a total operation by the above, and $-x$ is the unique inverse of x for any $x \in R$.

□

Example 3.3. For any $n \in \mathbb{N}$, the tuple $\langle \{0,1\}^n, \mathrm{XOR}, \{0\}^n, \mathrm{NOT}, \{1\}^n\rangle$ is a classical BI-model. In this model, the resources can be seen as n-bit binary numbers, which can be combined and "inverted" using the logical operations XOR and NOT respectively. Accordingly, the resources e and ∞ are respectively the n-bit representations of 0 and $2^n - 1$.

Example 3.4. Consider the monoid $(\mathbb{Z}_n, +, 0)$, where $\mathbb{Z}_n$ is the set of integers modulo n, and $+$ is addition modulo n. We can form a classical BI-model by choosing, for any $m \in \mathbb{Z}_n$,

$$\infty_m =_{\mathrm{def}} m \qquad -_m(k) =_{\mathrm{def}} m - k$$

This example shows that, even when the monoid structure is fixed, the choice of ∞ is not unique in general.

Example 3.5. Given an arbitrary monoid $\langle R, \circ, e\rangle$, we give a syntactic construction to generate a classical BI-model $\langle R', \circ', e', -', \infty'\rangle$. Consider the set T of terms

$$t \in T ::= r \in R \mid \infty \mid t \cdot t \mid -t$$

and let $\approx$ be the least congruence such that: $r_1 \cdot r_2 \approx r$ when $r_1 \circ r_2 = r$, $t_1 \cdot t_2 \approx t_2 \cdot t_1$, $t_1 \cdot (t_2 \cdot t_3) \approx (t_1 \cdot t_2) \cdot t_3$, $--t \approx t$, $t \cdot (-t) \approx \infty$, $t_1 \approx -t_2$ whenever $t_1 \circ t_2 \approx \infty$. Write $T/{\approx}$ for the quotient of T by the relation $\approx$, and $[t]$ for the equivalence class of t. The required classical BI-model is obtained by defining $R' =_{\mathrm{def}} T/{\approx}$, $\circ'([t_1],[t_2]) =_{\mathrm{def}} [t_1 \circ t_2]$, $e' =_{\mathrm{def}} [e]$, $-'(t) =_{\mathrm{def}} [-t]$, $\infty' =_{\mathrm{def}} [\infty]$.

Example 3.6. A natural question is whether BI models used in separation logic are also classical BI-models. Consider the partial commutative monoid $\langle H, \circ, e\rangle$, where $H =_{\mathrm{def}} \mathbb{Z}_{>0} \rightharpoonup \mathbb{Z}$ is the set of partial functions from positive integers to integers, $\circ$ is disjoint union of the graph of functions, and e is the function with empty domain. Unfortunately, no choice of ∞ gives rise to a classical BI-model. However, it is possible to embed the heap monoid into a more general structure $\langle H', \circ', e'\rangle$, where $H' =_{\mathrm{def}} \mathcal{P}(\mathbb{Z}_{>0} \times \mathbb{Z})$ is the set of relations instead of partial functions, $\circ$ is disjoint union, and e is the empty relation. A classical BI-model is then obtained by setting $\infty =_{\mathrm{def}} \mathbb{Z}_{>0} \times \mathbb{Z}$, and $-r =_{\mathrm{def}} (\mathbb{Z}_{>0} \times \mathbb{Z}) \setminus r$.

Example 3.7. As a final example, we consider a heap monoid with fractional permissions [2] $\langle H_p, \circ_p, e_p\rangle$, where $H_p =_{\mathrm{def}} \mathbb{Z}_{>0} \rightharpoonup \mathbb{Z} \times (0, 1]$ consists of functions which in addition return a permission in the real interval $(0, 1]$, and $\circ$ is defined on functions with overlapping domains using a partial composition function $\oplus$: $(\mathbb{Z} \times (0, 1]) \times (\mathbb{Z} \times (0, 1]) \rightharpoonup (\mathbb{Z} \times (0, 1])$ such that $\oplus((v_1, p_1), (v_2, p_2))$ is defined if and only if $v_1 = v_2$ and $p_1 + p_2 \leq 1$, and returns $(v_1, p_1 + p_2)$. The unit e_p is again the function with empty domain. In analogy with ordinary heaps, we define a more general structure $\langle H'_p, \circ'_p, e'_p\rangle$, where $H'_p =_{\mathrm{def}} \mathbb{Z}_{>0} \times \mathbb{Z} \to [0, 1]$ is the set of *total* functions, and $\circ'_p$ is defined point-wise using $+ : [0, 1] \times [0, 1] \rightharpoonup [0, 1]$, which is ordinary addition restricted to be defined only when the result is ≤ 1. The function e'_p maps everything to 0. A classical BI-model is then obtained by setting ∞ as mapping everything to 1, and $-r =_{\mathrm{def}} \{(l, v, 1 - p) \mid (l, v, p) \in r\}$. Observe that, in this case, the general model is in a way simpler, and that the $-$ operation returns the complement of the permissions.

4 Classical propositional BI (CBI)

Definition 4.1 (CBI**-formula).** *Formulas* of CBI are obtained by the following grammar:

$$\begin{aligned} F ::= \ & P \mid \top \mid \bot \mid \neg F \mid F \wedge F \mid F \vee F \mid F \to F \mid \\ & \top^* \mid \bot^* \mid {\sim} F \mid F * F \mid F \otimes F \mid F \mathrel{-\!\!*} F \end{aligned}$$

where P ranges over the propositional variables $\mathcal{V}$.

Definition 4.2 (Satisfaction). Let $M = \langle R, \circ, e, -, \infty\rangle$ be a classical BI-model and let ρ be an environment for M. We extend the definition of *satisfaction* of a formula F under a resource $r \in R$ in BI (cf. Section 2) to all formulas of CBI as follows:

$$\begin{aligned} r \models \bot^* \quad & \Leftrightarrow \quad r \neq \infty \\ r \models {\sim} F \quad & \Leftrightarrow \quad -r \not\models F \\ r \models F_1 \otimes F_2 \quad & \Leftrightarrow \quad \forall r_1, r_2.\ -r = r_1 \circ r_2 \text{ implies } -r_1 \models F_1 \text{ or } -r_2 \models F_2 \end{aligned}$$

We remark that these interpretations are justified by the resulting semantic equivalences between formulas. For example, $\sim F$ is equivalent to $F \twoheadrightarrow \bot^*$, and $F \twoheadrightarrow G$ is equivalent to $\sim F \otimes G$. As expected, $\otimes$ is interpreted as the de Morgan dual of $*$ with respect to $\sim$.

Definition 4.3 (Two-sided coherent equivalence). We define two relations $\equiv_L$ and $\equiv_R$ on bunches as follows.

- $\equiv_L$ is the relation $\equiv$ given in Definition 2.1;
- $\equiv_R$ is the least relation satisfying commutative monoid equations for ';' and $\bot$, and for ',' and $\bot^*$, plus congruence.

In Figure 2 we give sequent calculus proof rules for CBI, writing sequents of the form $\Gamma \vdash \Delta$, where Γ and Δ are both bunches. Our presentation is similar to Pym's formulation in [7], except that we use two cut rules in order to overcome a problem with soundness in the formulation of the cut rule there.

We remark that, as formulated, the sequent calculus proof rules of CBI suffer from the problem that they do not admit cut-elimination (cf. [7]). For example, as the rules for negation operate at the top level of bunches only, there is no cut-free proof of, e.g., the sequent $F, (G; \neg G) \vdash H$. However, since, e.g., $G; \neg G \vdash \bot$ is provable, we have a cut-proof of the sequent using (MCut) and ($\bot$L).

Validity of CBI-sequents with respect to our classical models is now defined in the obvious manner.

Definition 4.4 (Bunches as formulas). For any bunch Γ we define two formulas, Φ_Γ and Ψ_Γ by recursion on the structure of bunches as follows:

$$\begin{aligned} \Phi_F &= F & \Psi_F &= F \\ \Phi_{\Gamma_1;\Gamma_2} &= \Phi_{\Gamma_1} \wedge \Phi_{\Gamma_2} & \Psi_{\Gamma_1;\Gamma_2} &= \Psi_{\Gamma_1} \vee \Psi_{\Gamma_2} \\ \Phi_{\Gamma_1,\Gamma_2} &= \Phi_{\Gamma_1} * \Phi_{\Gamma_2} & \Psi_{\Gamma_1,\Gamma_2} &= \Psi_{\Gamma_1} \otimes \Psi_{\Gamma_2} \end{aligned}$$

Definition 4.5 (Truth / Validity). Let $M = \langle R, \circ, e, -, \infty \rangle$ be a classical BI-model. A sequent $\Gamma \vdash \Delta$ is said to be *true in* M if for any environment ρ for M and for all $r \in R$, $r \models \Phi_\Gamma$ implies $r \models \Psi_\Delta$, where $\Phi(-)$ and $\Psi(-)$ are the formulas defined in Defn. 4.4 above. A sequent is said to be *valid* if it is true in all classical BI-models.

5 Proof of soundness of CBI wrt. classical BI-models

Definition 5.1 (BI$^+$)**.** BI$^+$ is an extension of boolean BI (given in Section 2) defined as follows:

1. The formulas of BI$^+$ are the formulas of BI plus a new atomic propositional formula $\bowtie$. We extend the usual satisfaction relation for BI-formulas in a classical BI-model M under an environment ρ for M by:

$$r \models \bowtie \;\Leftrightarrow\; r = \infty$$

Structural rules:

$$\frac{}{F \vdash F}\,(\mathrm{Id}) \qquad \frac{\Gamma' \vdash \Delta'}{\Gamma \vdash \Delta}\;\begin{array}{l}\Gamma \equiv_L \Gamma' \\ \Delta \equiv_R \Delta'\end{array}\;(\mathrm{Equiv})$$

$$\frac{\Gamma(\Gamma') \vdash \Delta}{\Gamma(\Gamma';\Gamma'') \vdash \Delta}\,(\mathrm{WkL}) \qquad \frac{\Gamma(\Gamma';\Gamma') \vdash \Delta}{\Gamma(\Gamma') \vdash \Delta}\,(\mathrm{CtrL}) \qquad \frac{\Gamma' \vdash \Delta; F \qquad \Gamma; F \vdash \Delta'}{\Gamma; \Gamma' \vdash \Delta; \Delta'}\,(\mathrm{ACut})$$

$$\frac{\Gamma \vdash \Delta(\Delta')}{\Gamma \vdash \Delta(\Delta';\Delta'')}\,(\mathrm{WkR}) \qquad \frac{\Gamma \vdash \Delta(\Delta';\Delta')}{\Gamma \vdash \Delta(\Delta')}\,(\mathrm{CtrR}) \qquad \frac{\Gamma' \vdash \Delta, F \qquad \Gamma, F \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,(\mathrm{MCut})$$

Additive rules:

$$\frac{}{\Gamma(\bot) \vdash \Delta}\,(\bot\mathrm{L}) \qquad \frac{}{\Gamma \vdash \Delta(\top)}\,(\top\mathrm{R})$$

$$\frac{\Gamma \vdash F; \Delta}{\Gamma; \neg F \vdash \Delta}\,(\neg\mathrm{L}) \qquad \frac{\Gamma(F_1) \vdash \Delta \quad \Gamma(F_2) \vdash \Delta}{\Gamma(F_1 \vee F_2) \vdash \Delta}\,(\vee\mathrm{L}) \qquad \frac{\Gamma(F_1; F_2) \vdash \Delta}{\Gamma(F_1 \wedge F_2) \vdash \Delta}\,(\wedge\mathrm{L})$$

$$\frac{\Gamma; F \vdash \Delta}{\Gamma \vdash \neg F; \Delta}\,(\neg\mathrm{R}) \qquad \frac{\Gamma \vdash \Delta(F_1; F_2)}{\Gamma \vdash \Delta(F_1 \vee F_2)}\,(\vee\mathrm{R}) \qquad \frac{\Gamma \vdash \Delta(F_1) \quad \Gamma \vdash \Delta(F_2)}{\Gamma \vdash \Delta(F_1 \wedge F_2)}\,(\wedge\mathrm{R})$$

$$\frac{\Gamma' \vdash F_1 \quad \Gamma(\Gamma'; F_2) \vdash \Delta}{\Gamma(\Gamma'; F_1 \to F_2) \vdash \Delta}\,(\to\mathrm{L}) \qquad \frac{\Gamma; F_1 \vdash F_2; \Delta}{\Gamma \vdash F_1 \to F_2; \Delta}\,(\to\mathrm{R})$$

Multiplicative rules:

$$\frac{\Gamma \vdash F, \Delta}{\Gamma, \sim F \vdash \Delta}\,(\sim\mathrm{L}) \qquad \frac{\Gamma(F_1, F_2) \vdash \Delta}{\Gamma(F_1 * F_2) \vdash \Delta}\,(*\mathrm{L}) \qquad \frac{\Gamma, F_1 \vdash \Delta \quad \Gamma', F_2 \vdash \Delta'}{\Gamma, \Gamma', F_1 \otimes F_2 \vdash \Delta, \Delta'}\,(\otimes\mathrm{L})$$

$$\frac{\Gamma, F \vdash \Delta}{\Gamma \vdash \sim F, \Delta}\,(\sim\mathrm{R}) \qquad \frac{\Gamma \vdash \Delta, F_1 \qquad \Gamma' \vdash \Delta', F_2}{\Gamma, \Gamma' \vdash \Delta, \Delta', F_1 * F_2}\,(*\mathrm{R}) \qquad \frac{\Gamma \vdash \Delta(F_1, F_2)}{\Gamma \vdash \Delta(F_1 \otimes F_2)}\,(\otimes\mathrm{R})$$

$$\frac{\Gamma' \vdash F_1 \quad \Gamma(F_2) \vdash \Delta}{\Gamma(\Gamma', F_1 \mathrel{-\!\!*} F_2) \vdash \Delta}\,(\mathrel{-\!\!*}\mathrm{L}) \qquad \frac{\Gamma, F_1 \vdash F_2, \Delta}{\Gamma \vdash F_1 \mathrel{-\!\!*} F_2, \Delta}\,(\mathrel{-\!\!*}\mathrm{R})$$

Fig. 2. Sequent calculus rules for propositional CBI.

2. The proof rules of BI^+ are the proof rules of BI plus the following axioms:

$$\frac{}{--F \vdash F}\ (\mathrm{DIE}) \qquad \frac{}{F \vdash --F}\ (\mathrm{DII})$$

where $–F$ is an abbreviation for the formula $\neg(F \multimap (\neg \bowtie))$.

We remark that the definition of BI^+ is intended to capture as directly as possible the special features of our classical BI-models.

Lemma 5.2. *Let $M = \langle R, \circ, e, -, \infty\rangle$ be a classical* BI*-model and let ρ be an environment for M. For any $r \in R$ and formula F we have $r \models -F$ iff $-r \models F$.*

Proof. We have by the definitions of $-F$ and of satisfaction:

$$\begin{array}{rcl} r \models -F & \Leftrightarrow & r \models (F \multimap (\bowtie \to \bot)) \to \bot \\ & \Leftrightarrow & r \not\models F \multimap (\bowtie \to \bot) \\ & \Leftrightarrow & \exists r'.\ r \circ r' \text{ defined and } r' \models F \text{ but } r \circ r' \not\models \bowtie \to \bot \\ & \Leftrightarrow & \exists r'.\ r \circ r' \text{ defined and } r' \models F \text{ and } r \circ r' = \infty \\ & \Leftrightarrow & \exists r'.\ r \circ r' \text{ defined and } r' \models F \text{ and } r' = -r \\ & \Leftrightarrow & -r \models F \end{array}$$

Note that the penultimate equivalence above is justified by the fact that $-r$ is the unique element of R satisfying $r \circ -r = \infty$.

Proposition 5.3. BI^+ *is sound with respect to validity in classical* BI*-models.*

Proof. As usual, soundness follows from the fact that the proof rules of BI^+ preserve truth in classical BI-models. First, note that the rules of BI preserve truth in BI-models and thus in classical BI-models in particular. Thus it only remains to show that the axioms added to BI in BI^+ are true in every classical BI-model. We fix a classical BI-model $\langle R, \circ, e, -, \infty\rangle$ and environment ρ, and observe that Lemma 5.2 and part 2 of Proposition 3.2 give:

$$r \models --F \ \Leftrightarrow\ -r \models -F \ \Leftrightarrow\ --r \models F \ \Leftrightarrow\ r \models F$$

so that $r \models --F \leftrightarrow F$ as required. □

We write $F \dashv\vdash G$ to mean that both $F \vdash G$ and $G \vdash F$ are derivable in BI^+, and call $F \dashv\vdash G$ a *derivable equivalence*. The following lemma says that we can rewrite formulas in BI^+ sequents according to derivable equivalences without affecting BI^+-derivability.

Lemma 5.4. *Write $F(G)$ for a formula F of which G is a distinguished subformula, and when $F(G)$ is understood write $F(G')$ for the formula obtained by replacing G by G' in F. (This is analogous to the notation for bunches.)*

Now suppose that $A \dashv\vdash B$ is a derivable equivalence of BI^+ *(where A, B are formulas). Then the following two proof rules are derivable in* BI^+*:*

$$\frac{\Gamma(F(A)) \vdash C}{\Gamma(F(B)) \vdash C}\ (\dashv\vdash L) \qquad \frac{\Gamma \vdash F(A)}{\Gamma \vdash F(B)}\ (\dashv\vdash R)$$

Proof. By considering the following two instances of (Cut):

$$\frac{F(B) \vdash F(A) \quad \Gamma(F(A)) \vdash C}{\Gamma(F(B)) \vdash C} \text{(Cut)} \qquad \frac{\Gamma \vdash F(A) \quad F(A) \vdash F(B)}{\Gamma \vdash F(B)} \text{(Cut)}$$

it suffices to prove that $F(A) \vdash F(B)$ is derivable in BI^+, whence it follows by symmetry that $F(B) \vdash F(A)$ is also derivable. If $F(A) = A$ then this is immediate by assumption. Otherwise A is a (distinguished) strict subformula of F and we proceed by an easy structural induction on F. □

Lemma 5.5. *The following are all derivable in* BI^+*:*

1. $\neg - F \dashv\vdash F \mathrel{-\!\!*} \neg \bowtie$
2. $\neg - F \dashv\vdash - \neg F$
3. $\neg - \neg - F \dashv\vdash F$
4. $\neg - (F * \neg - G) \dashv\vdash F \mathrel{-\!\!*} G$
5. $F \mathrel{-\!\!*} G \dashv\vdash \neg - G \mathrel{-\!\!*} \neg - F$
6. $F \dashv\vdash \neg - (\neg - \neg \bowtie * \neg - F)$

Definition 5.6 (Embedding function). We define an *embedding function*, $\ulcorner - \urcorner$, from CBI-formulas to BI^+-formulas by recursion on the structure of CBI-formulas, as follows:

$$\begin{array}{rcll}
\ulcorner F \urcorner & = & F & \text{where } F \in \{P \mid P \in \mathcal{V}\} \cup \{\top, \bot, \top^*\} \\
\ulcorner F_1 \,?\, F_2 \urcorner & = & \ulcorner F_1 \urcorner \,?\, \ulcorner F_2 \urcorner & \text{where } ? \in \{\wedge, \vee, \rightarrow, *, \mathrel{-\!\!*}\} \\
\ulcorner \neg F \urcorner & = & \ulcorner F \urcorner \rightarrow \bot & \\
\ulcorner \bot^* \urcorner & = & \bowtie \rightarrow \bot & \\
\ulcorner \sim F \urcorner & = & \ulcorner F \urcorner \mathrel{-\!\!*} \ulcorner \bot^* \urcorner = - \ulcorner F \urcorner \rightarrow \bot & \\
\ulcorner F_1 \otimes F_2 \urcorner & = & \ulcorner \sim(\sim F_1 * \sim F_2) \urcorner &
\end{array}$$

We extend the domain of the function $\ulcorner - \urcorner$ to bunches in the natural manner:

$$\ulcorner \Gamma_1, \Gamma_2 \urcorner = \ulcorner \Gamma_1 \urcorner , \ulcorner \Gamma_2 \urcorner \text{ and } \ulcorner \Gamma_1 ; \Gamma_2 \urcorner = \ulcorner \Gamma_1 \urcorner ; \ulcorner \Gamma_2 \urcorner$$

Finally, we extend $\ulcorner - \urcorner$ to a function from CBI-sequents to BI^+-sequents by:

$$\ulcorner \Gamma \vdash \Delta \urcorner = \ulcorner \Gamma \urcorner \vdash \ulcorner \Psi_\Delta \urcorner$$

Lemma 5.7. *The function* $\ulcorner - \urcorner$ *preserves validity with respect to (i) formulas and (ii) sequents. That is,* $\Gamma \vdash \Delta$ *is valid iff* $\ulcorner \Gamma \vdash \Delta \urcorner$ *is.*

Proof. We fix a classical BI-model $M = \langle R, \circ, e, -, \infty \rangle$ and an environment ρ for M. For (i), we let $r \in R$ and prove by structural induction on F that $r \models F$ iff $r \models \ulcorner F \urcorner$:

Case $F \in \{P \mid P \in \mathcal{V}\} \cup \{\top, \bot, \top^*\}$. Trivial.

Case $F = F_1 \mathbin{?} F_2$, where $? \in \{\wedge, \vee, \rightarrow, *, \mathrel{-\!\!*}\}$. We are immediately done by the induction hypothesis.

Case $F = \neg G$. We need to show $r \models \neg G$ iff $r \models \ulcorner G \urcorner \rightarrow \bot$, i.e., $r \not\models G$ iff $r \not\models \ulcorner G \urcorner$, which follows by the induction hypothesis.

Case $F = \bot^*$. We need to show $r \models \bot^*$ iff $r \models \bowtie \rightarrow \bot$. We have both $r \models \bot^*$ iff $r \neq \infty$ and $r \models \bowtie \rightarrow \bot$ iff $r \neq \infty$ by definition, so are done.

Case $F = \sim G$. We require to show $r \models \sim G$ iff $r \models -\ulcorner G \urcorner \rightarrow \bot$, i.e., $-r \not\models G$ iff $r \not\models -\ulcorner G \urcorner$. By Lemma 5.2, $r \not\models -\ulcorner G \urcorner$ iff $-r \not\models \ulcorner G \urcorner$, so we are done by the induction hypothesis.

Case $F = F_1 \otimes F_2$. We require to show:

$$\begin{array}{rcl} r \models F_1 \otimes F_2 & \Leftrightarrow & r \models \ulcorner \sim(\sim F_1 * \sim F_2) \urcorner \\ & \Leftrightarrow & r \models -((-\ulcorner F_1 \urcorner \rightarrow \bot) * (-\ulcorner F_2 \urcorner \rightarrow \bot)) \rightarrow \bot \\ & \Leftrightarrow & -r \models (-\ulcorner F_1 \urcorner \rightarrow \bot) * (-\ulcorner F_2 \urcorner \rightarrow \bot) \\ & \Leftrightarrow & \neg \exists r_1, r_2.\ -r = r_1 \circ r_2 \text{ and } -r_1 \not\models \ulcorner F_1 \urcorner \text{ and } -r_2 \not\models \ulcorner F_2 \urcorner \\ & \Leftrightarrow & \forall r_1, r_2.\ -r = r_1 \circ r_2 \text{ implies } -r_1 \models \ulcorner F_1 \urcorner \text{ or } -r_2 \models \ulcorner F_2 \urcorner \\ & \Leftrightarrow & r \models \ulcorner F_1 \urcorner \otimes \ulcorner F_2 \urcorner \end{array}$$

Note that we use Lemma 5.2 in some of the equivalences above. The required equivalence thus follows from the induction hypothesis. This completes the proof for part (i).

For part (ii), let $\Gamma \vdash \Delta$ be a CBI-sequent. The embedded sequent $\ulcorner \Gamma \vdash \Delta \urcorner = \ulcorner \Gamma \urcorner \vdash \ulcorner \Psi_\Delta \urcorner$ is true in M if for all $r \in R$ we have $r \models \Phi_{\ulcorner \Gamma \urcorner}$ implies $r \models \ulcorner \Psi_\Delta \urcorner$, where $\Phi_{\ulcorner \Gamma \urcorner}$ and Ψ_Δ are the formulas constructed from $\ulcorner \Gamma \urcorner$ and Δ in Defn. 4.5. Now one can prove easily by structural induction on Γ that $\Phi_{\ulcorner \Gamma \urcorner} = \ulcorner \Phi_\Gamma \urcorner$, so:

$$\begin{array}{rcll} \ulcorner \Gamma \vdash \Delta \urcorner \text{ true in } M & \Leftrightarrow & \ulcorner \Gamma \urcorner \vdash \ulcorner \Psi_\Delta \urcorner \text{ true in } M & \\ & \Leftrightarrow & \forall r \in R.\ r \models \ulcorner \Phi_\Gamma \urcorner \text{ implies } r \models \ulcorner \Psi_\Delta \urcorner & \\ & \Leftrightarrow & \forall r \in R.\ r \models \Phi_\Gamma \text{ implies } r \models \Psi_\Delta & \text{(by part (i))} \\ & \Leftrightarrow & \Gamma \vdash \Delta \text{ true in } M & \end{array}$$

Thus $\Gamma \vdash \Delta$ is valid if and only if $\ulcorner \Gamma \vdash \Delta \urcorner$ is. □

Lemma 5.8. *For any bunch Γ with sub-bunch Γ', we have $\ulcorner \Gamma(\Gamma') \urcorner = \ulcorner \Gamma \urcorner (\ulcorner \Gamma' \urcorner)$.*

Proof. If $\Gamma = \Gamma'$ then this is trivial. Otherwise, proceed by a straightforward induction on the structure of Γ. □

Lemma 5.9. *The proof rules of* CBI *are admissible in* BI$^+$ *under the embedding* $\ulcorner - \urcorner$*. That is, for any* CBI *proof rule, say:*

$$\frac{\{\Gamma_i \vdash \Delta_i \mid i \in \{1,2\}\}}{\Gamma \vdash \Delta}\,(R)$$

if $\ulcorner \Gamma_i \vdash \Delta_i \urcorner$ *is derivable in* BI$^+$ *for each* $i \in \{1,2\}$*, then so is* $\ulcorner \Gamma \vdash \Delta \urcorner$*.*

Proof. We distinguish a case for each proof rule of CBI. All derivations shown are BI^+ derivations, except that we use the rule symbol $(=)$ to denote rewriting a sequent according to Defns. 4.4 and 5.6 and Lemma 5.8. Because of space limitations, we show only certain representative cases.

Cases (Id), (WkL), (CtrL), ($\bot$L), ($\vee$L), ($\wedge$L), ($\rightarrow$L), ($*$L), ($-\!\!*$L). Under the embedding $\ulcorner - \urcorner$, these rules become instances of the corresponding BI^+ rule. For example, in the case of ($-\!\!*$L) we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\ulcorner \Gamma' \vdash F_1 \urcorner}}{\ulcorner \Gamma' \urcorner \vdash \ulcorner F_1 \urcorner}\,(=) \qquad \dfrac{\dfrac{\vdots}{\ulcorner \Gamma(F_2) \vdash \Delta \urcorner}}{\ulcorner \Gamma \urcorner(\ulcorner F_2 \urcorner) \vdash \ulcorner \Psi_\Delta \urcorner}\,(=)}{\ulcorner \Gamma \urcorner(\ulcorner \Gamma' \urcorner, \ulcorner F_1 \urcorner -\!\!* \ulcorner F_2 \urcorner) \vdash \ulcorner \Psi_\Delta \urcorner}\,(-\!\!*\mathrm{L})}{\ulcorner \Gamma(\Gamma', F_1 -\!\!* F_2) \vdash \Delta \urcorner}\,(=)$$

Cases ($\vee$R), ($\otimes$R). The premise and conclusion of these rules are identified under $\ulcorner - \urcorner$, so we are trivially done. For example, in the case of ($\otimes$R), a straightforward structural induction on Δ shows that $\Psi_{\Delta(F_1,F_2)} = \Psi_{\Delta(F_1 \otimes F_2)}$. Thus using Defn. 5.6 we have:

$$\ulcorner \Gamma \vdash \Delta(F_1 \otimes F_2) \urcorner = \ulcorner \Gamma \urcorner \vdash \ulcorner \Psi_{\Delta(F_1 \otimes F_2)} \urcorner = \ulcorner \Gamma \urcorner \vdash \ulcorner \Psi_{\Delta(F_1,F_2)} \urcorner = \ulcorner \Gamma \vdash \Delta(F_1, F_2) \urcorner$$

Cases ($\rightarrow$R), ($\neg$R), ($\neg$L), ($\sim$L), ($\sim$R), ($-\!\!*$R), ($*$R), ($\otimes$L), (ACut), (MCut). These rules are derivable under $\ulcorner - \urcorner$ by using (Cut) to introduce the needed premises and using one or more of the derivable sequents and equivalences given in Lemma 5.5 to discharge the resulting proof burden. For example, in the case of ($\sim$R) we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\ulcorner \Gamma, F \vdash \Delta \urcorner}}{\ulcorner \Gamma \urcorner, \ulcorner F \urcorner \vdash \ulcorner \Psi_\Delta \urcorner}\,(=)}{\ulcorner \Gamma \urcorner \vdash \ulcorner F \urcorner -\!\!* \ulcorner \Psi_\Delta \urcorner}\,(-\!\!*\mathrm{R}) \qquad \dfrac{\text{Lemma 5.5, part 4}}{\dfrac{\vdots}{\ulcorner F \urcorner -\!\!* \ulcorner \Psi_\Delta \urcorner \vdash \neg{-}(\ulcorner F \urcorner * \neg{-}\ulcorner \Psi_\Delta \urcorner)}}}{\ulcorner \Gamma \urcorner \vdash \neg{-}(\ulcorner F \urcorner * \neg{-}\ulcorner \Psi_\Delta \urcorner)}\,(\mathrm{Cut})}{\ulcorner \Gamma \urcorner \vdash \neg{-}(\neg{-}\neg{-}\ulcorner F \urcorner * \neg{-}\ulcorner \Psi_\Delta \urcorner)}\,(\dashv\vdash\mathrm{R})}{\ulcorner \Gamma \vdash {\sim}F, \Delta \urcorner}\,(=)$$

where the application of the derived rule ($\dashv\vdash$R) (cf. Lemma 5.4) rewrites the formula on the right of the sequent according to the derivable equivalence $\neg{-}\neg{-}F \dashv\vdash F$ given by part 3 of Lemma 5.5.

Cases ($\top$R), (WkR), (CtrR), ($\wedge$R). These rules operate inside bunches on the right hand side of sequents. As well as the techniques for the rules in the cases above, we additionally require structural induction on these bunches. For example, in the case of (WkR) we proceed as follows:

$$
\dfrac{\dfrac{\dfrac{\vdots}{\ulcorner \Gamma \vdash \Delta(\Delta') \urcorner}}{\ulcorner \Gamma \urcorner \vdash \ulcorner \Psi_{\Delta(\Delta')} \urcorner}\,(=) \qquad \ulcorner \Psi_{\Delta(\Delta')} \urcorner \vdash \ulcorner \Psi_{\Delta(\Delta';\Delta'')} \urcorner}{\dfrac{\ulcorner \Gamma \urcorner \vdash \ulcorner \Psi_{\Delta(\Delta';\Delta'')} \urcorner}{\ulcorner \Gamma \vdash \Delta(\Delta';\Delta'') \urcorner}\,(=)}\,(\mathrm{Cut})
$$

It thus remains to show that $\ulcorner \Psi_{\Delta(\Delta')} \urcorner \vdash \ulcorner \Psi_{\Delta(\Delta';\Delta'')} \urcorner$ is derivable, which we do by structural induction on $\Delta(-)$, observing that the latter can be defined inductively, up to coherent equivalence ($\equiv$), as follows:

$$
\Delta(-) ::= - \mid \Delta(-);\Delta \mid \Delta(-),\Delta
$$

where Δ ranges over (complete) bunches.

Subcase $\Delta(-) = -$. We have $\Delta(\Delta') = \Delta'$ and $\Delta(\Delta';\Delta'') = \Delta';\Delta''$ and proceed as follows:

$$
\dfrac{\dfrac{\dfrac{}{\ulcorner \Psi_{\Delta'} \urcorner \vdash \ulcorner \Psi_{\Delta'} \urcorner}\,(\mathrm{Id})}{\ulcorner \Psi_{\Delta'} \urcorner \vdash \ulcorner \Psi_{\Delta'} \urcorner \vee \ulcorner \Psi_{\Delta''} \urcorner}\,(\vee R_1)}{\ulcorner \Psi_{\Delta'} \urcorner \vdash \ulcorner \Psi_{\Delta';\Delta''} \urcorner}\,(=)
$$

Subcase $\Delta(-) = \Delta_1(-);\Delta_2$. We proceed as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\text{(I.H.)}}{\vdots}}{\ulcorner \Psi_{\Delta_1(\Delta')} \urcorner \vdash \ulcorner \Psi_{\Delta_1(\Delta';\Delta'')} \urcorner}}{\ulcorner \Psi_{\Delta_1(\Delta')} \urcorner \vdash \ulcorner \Psi_{\Delta_1(\Delta';\Delta'')} \urcorner \vee \ulcorner \Psi_{\Delta_2} \urcorner}\,(\vee R_1) \quad \dfrac{\dfrac{}{\ulcorner \Psi_{\Delta_2} \urcorner \vdash \ulcorner \Psi_{\Delta_2} \urcorner}\,(\mathrm{Id})}{\ulcorner \Psi_{\Delta_2} \urcorner \vdash \ulcorner \Psi_{\Delta_1(\Delta';\Delta'')} \urcorner \vee \ulcorner \Psi_{\Delta_2} \urcorner}\,(\vee R_2)}{\dfrac{\ulcorner \Psi_{\Delta_1(\Delta')} \urcorner \vee \ulcorner \Psi_{\Delta_2} \urcorner \vdash \ulcorner \Psi_{\Delta_1(\Delta';\Delta'')} \urcorner \vee \ulcorner \Psi_{\Delta_2} \urcorner}{\ulcorner \Psi_{\Delta_1(\Delta');\Delta_2} \urcorner \vdash \ulcorner \Psi_{\Delta_1(\Delta';\Delta'');\Delta 2} \urcorner}\,(=)}\,(\vee \mathrm{L})
$$

Subcase $\Delta(-) = \Delta_1(-),\Delta_2$. We proceed as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\!-\ulcorner \Psi_{\Delta_2} \urcorner \vdash \neg\!-\ulcorner \Psi_{\Delta_2} \urcorner}\,(\mathrm{Id}) \quad \dfrac{\dfrac{\text{(I.H.)}}{\vdots}}{\ulcorner \Psi_{\Delta_1(\Delta')} \urcorner \vdash \ulcorner \Psi_{\Delta_1(\Delta';\Delta'')} \urcorner}}{\neg\!-\ulcorner \Psi_{\Delta_2} \urcorner \mathrel{-\!\!*} \ulcorner \Psi_{\Delta_1(\Delta')} \urcorner, \neg\!-\ulcorner \Psi_{\Delta_2} \urcorner \vdash \ulcorner \Psi_{\Delta_1(\Delta';\Delta'')} \urcorner}\,(\mathrel{-\!\!*}\mathrm{L})}{\neg\!-\ulcorner \Psi_{\Delta_2} \urcorner \mathrel{-\!\!*} \ulcorner \Psi_{\Delta_1(\Delta')} \urcorner \vdash \neg\!-\ulcorner \Psi_{\Delta_2} \urcorner \mathrel{-\!\!*} \ulcorner \Psi_{\Delta_1(\Delta';\Delta'')} \urcorner}\,(\mathrel{-\!\!*}\mathrm{R})}{\neg\!-(\neg\!-\ulcorner \Psi_{\Delta_1(\Delta')} \urcorner * \neg\!-\ulcorner \Psi_{\Delta_2} \urcorner) \vdash \neg\!-\ulcorner \Psi_{\Delta_2} \urcorner \mathrel{-\!\!*} \ulcorner \Psi_{\Delta_1(\Delta';\Delta'')} \urcorner}\,(\dashv\vdash\mathrm{L})}{\dfrac{\neg\!-(\neg\!-\ulcorner \Psi_{\Delta_1(\Delta')} \urcorner * \neg\!-\ulcorner \Psi_{\Delta_2} \urcorner) \vdash \neg\!-(\neg\!-\ulcorner \Psi_{\Delta_1(\Delta';\Delta'')} \urcorner * \neg\!-\ulcorner \Psi_{\Delta_2} \urcorner)}{\ulcorner \Psi_{\Delta_1(\Delta'),\Delta_2} \urcorner \vdash \ulcorner \Psi_{\Delta_1(\Delta';\Delta''),\Delta 2} \urcorner}\,(=)}\,(\dashv\vdash\mathrm{R})
$$

Note that in the applications of ($\dashv\vdash$R) and ($\dashv\vdash$L) we use the derivable equivalence $\neg -(F * \neg -G) \dashv\vdash F \multimap\!\!* G$ given by Lemma 5.5, part 4. This completes the induction and thus the case.

Case (Equiv). It is easy to show that if $\Gamma \equiv_L \Gamma'$ then $\ulcorner\Gamma\urcorner \equiv_L \ulcorner\Gamma'\urcorner$. Since $\equiv_L$ is the same relation as $\equiv$ we can proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\ulcorner\Gamma' \vdash \Delta'\urcorner}}{\ulcorner\Gamma'\urcorner \vdash \ulcorner\Psi_{\Delta'}\urcorner}(=) \quad \ulcorner\Psi_{\Delta'}\urcorner \vdash \ulcorner\Psi_{\Delta}\urcorner}{\ulcorner\Gamma'\urcorner \vdash \ulcorner\Psi_{\Delta}\urcorner}(\text{Cut}) \quad \ulcorner\Gamma\urcorner \equiv \ulcorner\Gamma'\urcorner}{\dfrac{\ulcorner\Gamma\urcorner \vdash \ulcorner\Psi_{\Delta}\urcorner}{\ulcorner\Gamma \vdash \Delta\urcorner}(=)}(\text{Equiv})$$

It thus suffices to show that if $\Delta' \equiv_R \Delta$ then $\ulcorner\Psi_{\Delta'}\urcorner \vdash \ulcorner\Psi_{\Delta}\urcorner$ is derivable. This follows by induction on the conditions defining $\Delta' \equiv_R \Delta$. Most of the cases are straightforward (making appropriate use of Lemmas 5.4 and 5.5). For congruence, we need to show that if $\ulcorner\Psi_{\Delta'}\urcorner \vdash \ulcorner\Psi_{\Delta}\urcorner$ is derivable then so is $\ulcorner\Psi_{\Delta''(\Delta')}\urcorner \vdash \ulcorner\Psi_{\Delta''(\Delta)}\urcorner$. This follows by a structural induction on $\Delta''(-)$. □

Theorem 5.10. CBI *is sound with respect to validity in classical* BI*-models.*

Proof. Since the proof rules of CBI are admissible in BI$^+$ under the embedding $\ulcorner-\urcorner$ by Lemma 5.9, it follows that if $\Gamma \vdash \Delta$ is derivable in CBI then $\ulcorner\Gamma \vdash \Delta\urcorner$ is derivable in BI$^+$, and hence $\ulcorner\Gamma \vdash \Delta\urcorner$ is valid by the soundness of BI$^+$ (Proposition 5.3). Since $\ulcorner-\urcorner$ preserves validity by Lemma 5.7, $\Gamma \vdash \Delta$ is also valid. □

6 Conclusions and future work

Our starting point in investigating the issues considered here was to observe that (boolean) BI has additive connectives with no multiplicative equivalent, and to ask whether any computationally significant models would be admitted by a classical version of BI or, for that matter, any non-trivial models (i.e., models in which the connectives do not collapse). Our main conceptual contribution in the present paper is to make the connection between classical BI and our class of classical BI-models, and to show how to interpret the new multiplicative connectives in these models. In particular, similar approaches to our multiplicative negation have been proposed before, both by Pym [7] and by the relevant logic community. We just recently learned that our definition of classical BI-model extends the models of relevant logic [4] based on the Routley star (our $-$ operator) with the ∞ element, and its relation to $-$. Further investigations will be the subject of future work.

Our main technical contribution here is to show that the CBI sequent calculus is a sound basis for reasoning about our models. This is done via a translation from CBI proofs into proofs in an extension of boolean BI with one extra formula and two extra axioms which directly express properties of our models. In fact, we have recently shown that, if one considers a generalisation of our models to relational models (i.e. models in which the monoidal operation is a relation rather than a partial function) then this extension is complete [3]. This completeness result should straightforwardly transfer to CBI.

A concern must be the failure of cut-elimination for the CBI sequent calculus proof rules as we have formulated them. The reason for this failure appears to be the tension between the "deep" rules for the implications and the additive units on the one hand and the "shallow" rules for the negations and other connectives on the other, despite provable equivalences linking the connectives such as $F \mathrel{-\!\!*} G \dashv\vdash {\sim}F \otimes G$. While one might expect suitable deep formulations of the shallow rules to restore cut-elimination, finding such formulations appears a difficult problem. On the other hand, we have recently formulated a cut-eliminating proof system for classical BI based on Belnap's *display logic* [1], which is also sound and complete for the relational version of our models [3].

At the present time it remains to be seen whether our models, with their notion of "resource involution", will turn out to have widespread application in computer science, as is the case for BI-models as used e.g. in separation logic. We hope that the present paper, together with the more recent developments reported in [3], will serve to stimulate interest in our models, and in potential applications.

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