Firedrake

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Cross-Element Vectorization in Firedrake

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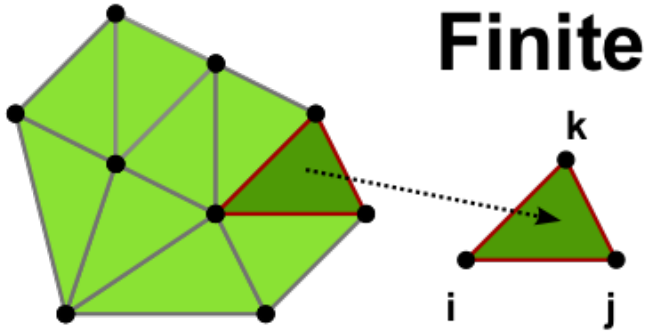
What is Firedrake

- Automated system for the portable solution of partial differential equations (PDEs) using the finite element method (FEM)
- DSL embedded in Python
- Optimizations at suitable (highest possible) abstraction layers
- Generate low level (e.g. C) code for performance
- Unstructured mesh → indirect data access
- Computationally, finite element ≈ assembly ≈ numerical integration

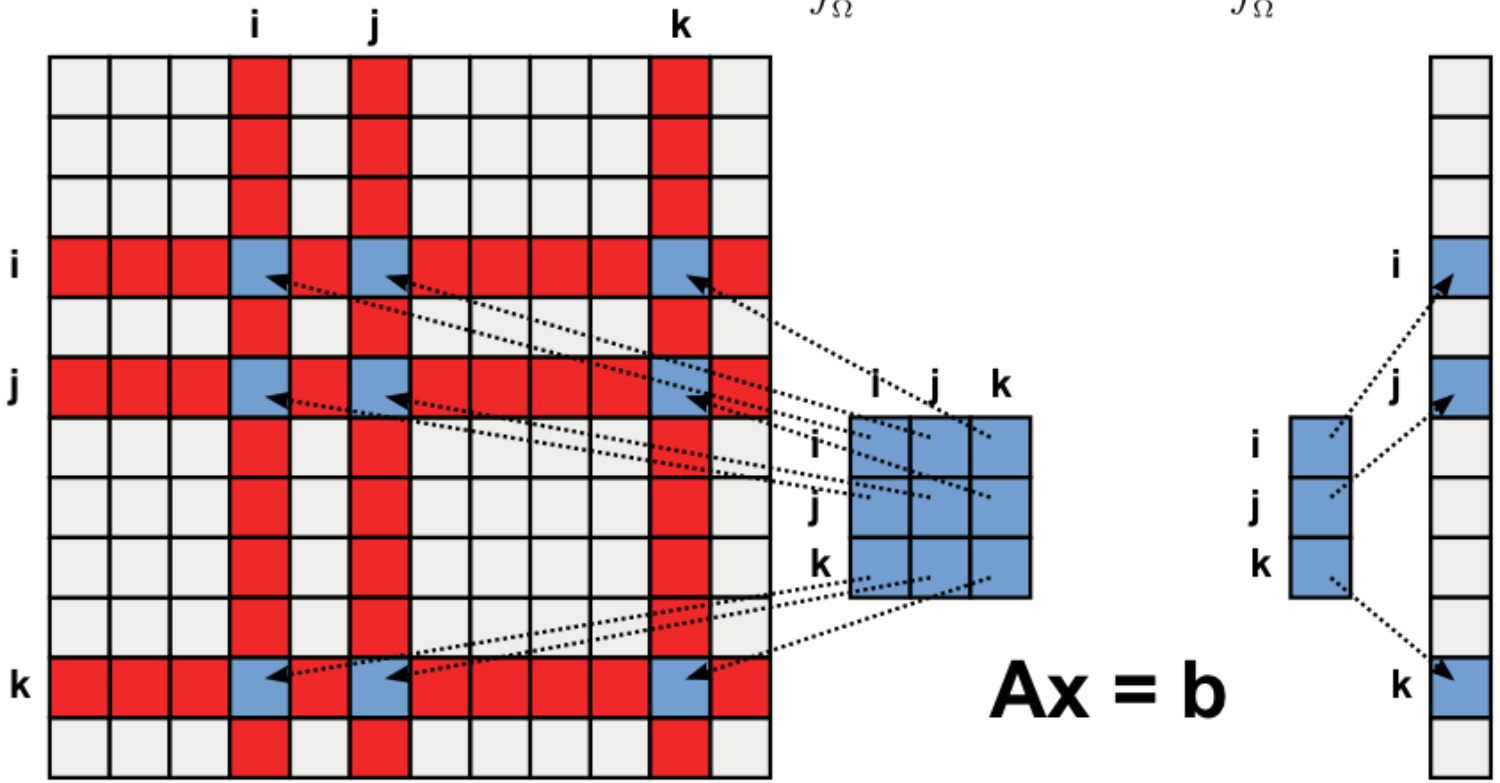
Firedrake

Finite-element assembly



The weak form of the Helmholtz equation:

$$\int_\Omega \nabla v \cdot \nabla u - \lambda v u \, \mathrm{d}V = \int_\Omega v f \, \mathrm{d}V$$

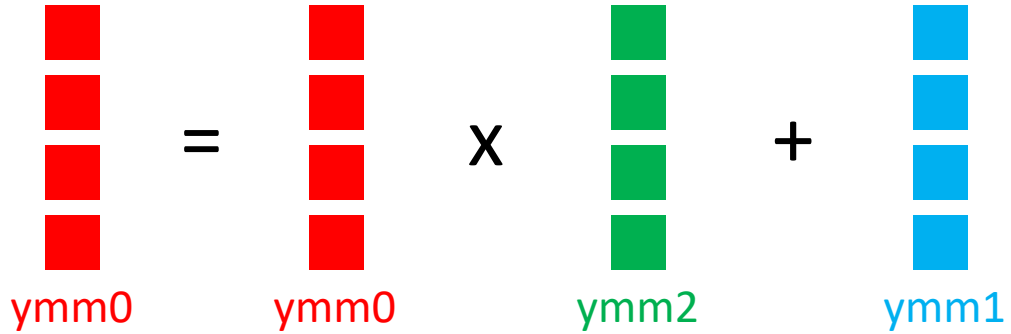


Ax = b

Source: Rathgeber

What is vectorization

- SIMD (single instruction multiple data) programming model
- e.g. VFMADD213PD ymm0, ymm1, ymm2 (in AVX2 instruction set)



- 8 double precision operations with 1 instruction
- Need to issue 2 FMA instructions in 1 cycle to get advertised performance
- SIMD width doubles every 4 years
  - AVX512 (2017) can do 8 doubles
- Naïve code usually achieves <10% peak performance
- This work is about generating vectorized code for finite element assembly

```
3  static inline void kernel(double *__restrict__ A, double const *__restrict__ coords, double const
4  {
5
6    /* ... */
7
8    for (int ip = 0; ip <= 5; ++ip)
9    {
10
11     /* ... */
12
13     for (int j0 = 0; j0 <= 9; ++j0)
14     {
15       A[2 * j0] = A[2 * j0] + t18[10 * ip + j0] * t36 + t19[10 * ip + j0] * t35;
16       A[1 + 2 * j0] = A[1 + 2 * j0] + t18[10 * ip + j0] * t32 + t19[10 * ip + j0] * t31;
17     }
18   }
19 }
20
21 void wrapper(int const start, int const end, double *__restrict__ dat0, double const *__restrict__
22 {
23   /* ... */
24   for (int n = start; n <= -1 + end; ++n)
25   {
26     for (int i7 = 0; i7 <= 2; ++i7)
27       for (int i8 = 0; i8 <= 1; ++i8)
28         t3[2 * i7 + i8] = dat1[2 * map1[3 * n + i7] + i8];
29     for (int i13 = 0; i13 <= 9; ++i13)
30       for (int i14 = 0; i14 <= 1; ++i14)
31         t4[2 * i13 + i14] = dat2[2 * map0[10 * n + i13] + i14];
32     for (int i1 = 0; i1 <= 9; ++i1)
33       for (int i2 = 0; i2 <= 1; ++i2)
34         t2[2 * i1 + i2] = 0.0;
35
36     form0_cell_integral_otherwise(t2, t3, t4);
37
38     for (int i20 = 0; i20 <= 1; ++i20)
39       for (int i19 = 0; i19 <= 9; ++i19)
40         dat0[2 * map0[10 * n + i19] + i20] = dat0[2 * map0[10 * n + i19] + i20] + t2[2 * i19 + i20
41   }
42 }
43
```

Action of linear elasticity operator on triangle mesh, Lagrange element of degree 3

```
3  static inline void kernel(double *__restrict__ A, double const *__restrict__ coords, double const
4  {
5
6    /* ... */
7
8    for (int ip = 0; ip <= 5; ++ip)
9    {
10
11     /* ... */
12
13     for (int j0 = 0; j0 <= 9; ++j0)
14     {
15       A[2 * j0] = A[2 * j0] + t18[10 * ip + j0] * t36 + t19[10 * ip + j0] * t35;
16       A[1 + 2 * j0] = A[1 + 2 * j0] + t18[10 * ip + j0] * t32 + t19[10 * ip + j0] * t31;
17     }
18   }
19 }
20
21 void wrapper(int const start, int const end, double *__restrict__ dat0, double const *__restrict__
22 {
23   /* ... */
24   for (int n = start; n <= -1 + end; ++n)
25   {
26     for (int i7 = 0; i7 <= 2; ++i7)
27       for (int i8 = 0; i8 <= 1; ++i8)
28         t3[2 * i7 + i8] = dat1[2 * map1[3 * n + i7] + i8];
29     for (int i13 = 0; i13 <= 9; ++i13)
30       for (int i14 = 0; i14 <= 1; ++i14)
31         t4[2 * i13 + i14] = dat2[2 * map0[10 * n + i13] + i14];
32     for (int i1 = 0; i1 <= 9; ++i1)
33       for (int i2 = 0; i2 <= 1; ++i2)
34         t2[2 * i1 + i2] = 0.0;
35
36     form0_cell_integral_otherwise(t2, t3, t4);
37
38     for (int i20 = 0; i20 <= 1; ++i20)
39       for (int i19 = 0; i19 <= 9; ++i19)
40         dat0[2 * map0[10 * n + i19] + i20] = dat0[2 * map0[10 * n + i19] + i20] + t2[2 * i19 + i20
41   }
42 }
43
```

Kernel

Wrapper

Action of linear elasticity operator on triangle mesh, Lagrange element of degree 3

```
static inline void kernel(double *__restrict__ A, double const *__restrict__ coords, double const
{

  /* ... */

  for (int ip = 0; ip <= 5; ++ip)
  {

    /* ... */

    for (int j0 = 0; j0 <= 9; ++j0)
    {
      A[2 * j0] =                                                    ;
      A[1 + 2 * j0                                                   0] * t31;
    }
  }
}

void wrapper(int const start, int const end, double *__restrict__ dat0, double const *__restrict__
{
  /* ... */
  for (int n = start; n <= -1 + end; ++n)
  {
    for (int i7 = 0; i7 <= 2; ++i7)
      for (int i8 = 0; i8 <= 1; ++i8)
        t3[2 * i7 + i8] = dat1[2 * map1[3 * n + i7] + i8];
    for (int i13 = 0; i13 <= 9; ++i13)
      for (int i14 = 0; i14 <= 1; ++i14)
        t4[2 * i13 + i14] = dat2[2 * map0[10 * n + i13] + i14];
    for (int i1 = 0; i1 <= 9; ++i1)
      for (int i2 = 0; i2 <= 1; ++i2)
        t2[2 * i1 + i2] = 0.0;

    form0_cell_integral_otherwise(t2, t3, t4);

    for (int i20 = 0; i20 <= 1; ++i20)
      for (int i19 = 0; i19 <= 9; ++i19)
        dat0[2 * map0[10 * n + i19] + i20] = dat0[2 * map0[10 * n + i19] + i20] + t2[2 * i19 + i20
  }
}
```

Outer loop over all elements in the mesh

```
for (int n = start; n <= -1 + end; ++n)
```

Wrapper

Action of linear elasticity operator on triangle mesh, Lagrange element of degree 3

Indirect gathering of input data for kernel

```
for (int i7 = 0; i7 <= 2; ++i7)
  for (int i8 = 0; i8 <= 1; ++i8)
    t3[2 * i7 + i8] = dat1[2 * map1[3 * n + i7] + i8];
for (int i13 = 0; i13 <= 9; ++i13)
  for (int i14 = 0; i14 <= 1; ++i14)
    t4[2 * i13 + i14] = dat2[2 * map0[10 * n + i13] + i14];
for (int i1 = 0; i1 <= 9; ++i1)
  for (int i2 = 0; i2 <= 1; ++i2)
    t2[2 * i1 + i2] = 0.0;
```

```
static inline void kernel(double *__restrict__ A, double const *__restrict__ coords, double const
{
  /* ... */
  for (int ip = 0
  {
    /*
    for
    {
      A
      A
    }
  }
}
void wr
{
  /* ..
  for (in
  {
    for (int i7 = 0; i7 <= 2; ++i7)
      for (int i8 = 0; i8 <= 1; ++i8)
        t3[2 * i7 + i8] = dat1[2 * map1[3 * n + i7] + i8];
    for (int i13 = 0; i13 <= 9; ++i13)
      for (int i14 = 0; i14 <= 1; ++i14)
        t4[2 * i13 + i14] = dat2[2 * map0[10 * n + i13] + i14];
    for (int i1 = 0; i1 <= 9; ++i1)
      for (int i2 = 0; i2 <= 1; ++i2)
        t2[2 * i1 + i2] = 0.0;

    form0_cell_integral_otherwise(t2, t3, t4);

    for (int i20 = 0; i20 <= 1; ++i20)
      for (int i19 = 0; i19 <= 9; ++i19)
        dat0[2 * map0[10 * n + i19] + i20] = dat0[2 * map0[10 * n + i19] + i20] + t2[2 * i19 + i20
  }
}
```

Action of linear elasticity operator on triangle mesh, Lagrange element of degree 3

```
3  static inline void kernel(double *__restrict__ A, double const *__restrict__ coords, double const
4  {
5
6    /* ... */
7
8    for (int ip = 0; ip <= 5; ++ip)
9    {
10
11     /* ... */
12
13     for (int j0 = 0; j0 <= 9; ++j0)
14     {
15       A[2 * j0] = A[2 * j0] + t18[10 * ip + j0] * t36 + t19[10 * ip + j0] * t35;
16       A[1 + 2 * j0] = A[1 + 2 * j0] + t18[10 * ip + j0] * t32 + t19[10 * ip + j0] * t31;
17     }
18   }
19 }
20
21 void wrapper(int const start, int const end, double *__restrict__ dat0, double const *__restrict__
22 {
23   /* ... */
24   for (int n = start; n <= -1 + en
25   {
26     for (int i7 = 0; i7 <= 2; ++i7
27       for (int i8 = 0; i8 <= 1; ++i8)
28         t3[2 * i7 + i8] = dat1[2 *
29     for (int i13 = 0; i13 <= 9; ++
30       for (int i14 = 0; i14 <= 1;
31         t4[2 * i13 + i14] = dat2[2
32     for (int i1 = 0; i1 <= 9; ++i1)
33       for (int i2 = 0; i2 <= 1; ++i2)
34         t2[2 * i1 + i2] = 0.0;
35
36     form0_cell_integral_otherwise(t2, t3, t4);
37
38     for (int i20 = 0; i20 <= 1; ++i20)
39       for (int i19 = 0; i19 <= 9; ++i19)
40         dat0[2 * map0[10 * n + i19] + i20] = dat0[2 * map0[10 * n + i19] + i20] + t2[2 * i19 + i20
41   }
42 }
43
```

Kernel “call”, actually it is inlined

```
kernel(t2, t3, t4);
```

Action of linear elasticity operator on triangle mesh, Lagrange element of degree 3

```
3  static inline void kernel(double *__restrict__ A, double const *__restrict__ coords, double const
4  {
5
6    /* ... */
7
8    for (int ip = 0; ip <= 5; ++ip)
9    {
10
11     /* ... */
12
13     for (int j0 = 0; j0 <= 9; ++j0)
14     {
15       A[2 * j0] = A[2 * j0] + t18[10 * ip + j0] * t36 + t19[10 * ip + j0] * t35;
16       A[1 + 2 * j0] = A[1 + 2 * j0] + t18[10 * ip + j0] * t32 + t19[10 * ip + j0] * t31;
17     }
18   }
19 }
20
21 void wrapper(int const start, int const end, double *__restrict__ dat0, double const *__restrict__
22 {
23   /* ... */
24   for (int n = start; n <= -1 + end; ++n)
25   {
26     for (int i
27       for (int
28         t3[2 *
29 for (int i20 = 0; i20 <= 1; ++i20)
30   for (int i19 = 0; i19 <= 9; ++i19)
31     dat0[2 * map0[10 * n + i19] + i20] += t2[2 * i19 + i20];
32
33
34
35
36     form0_cell_integral_otherwise(t2, t3, t4);
37
38     for (int i20 = 0; i20 <= 1; ++i20)
39       for (int i19 = 0; i19 <= 9; ++i19)
40         dat0[2 * map0[10 * n + i19] + i20] = dat0[2 * map0[10 * n + i19] + i20] + t2[2 * i19 + i20
41   }
42 }
43
```

Indirect scattering of local tensor to global tensor

Action of linear elasticity operator on triangle mesh, Lagrange element of degree 3

```
3  static inline void kernel(double *__restrict__ A, double const *__restrict__ coords, double const
4  {
5
6    /* ... */
7
8    for (int ip = 0; ip <= 5; ++ip)
9    {
10
11     /* ... */
12
13     for (int j0 = 0; j0 <= 9; ++j0)
14     {
15       A[2 * j0] = A[2 * j0] + t18[10 * ip + j0] * t36 + t19[10 * ip + j0] * t35;
16       A[1 + 2 * j0] = A[1 + 2 * j0] + t18[10 * ip + j0] * t32 + t19[10 * ip + j0] * t31;
17     }
18   }
19 }
20
21 void wrapper                                                                    * restrict
```

Kernel

Outer loop: contraction over quadrature points

```
for (int ip = 0; ip <= 5; ++ip)
{

  /* ...
```

inner loop over degrees of freedom

```
  for (int j0 = 0; j0 <= 9; ++j0)
  {
    A[2 * j0] += t18[10 * ip + j0] * t36 + t19[10 * ip + j0] * t35;
    A[1 + 2 * j0] += t18[10 * ip + j0] * t32 + t19[10 * ip + j0] * t31;
  }
}
```

```
38     for (int i20 = 0; i20 <= 1; ++i20)
39       for (int i19 = 0; i19 <= 9; ++i19)
40         dat0[2 * map0[10 * n + i19] + i20] = dat0[2 * map0[10 * n + i19] + i20] + t2[2 * i19 + i20
41   }
42 }
43
```

Action of linear elasticity operator on triangle mesh, Lagrange element of degree 3

Vectorization strategy

- "Intra-kernel" can be tricky
  - Trip count can be small and/or not multiple of SIMD width
  - Alignment to cache boundary
  - Stride 1 access
  - Operations outside of innermost loop hard to vectorize
  - Loop structure varies with PDE, discretization, mesh
  - And we have done many of these in firedrake [1]
- "Inter-kernel" provides a generic solution
  - Vector-expand the kernel to act on N elements together, N=SIMD width
  - All operations can be vectorized
  - Can always do this systematically
  - Downside: increasing working size

[1] F. Luporini, et al. ACM TACO 2015

```
3  static inline void kernel(double *__restrict__ A, double const *__restrict__ coords, double const *__restrict_
4  {
5
6    /* ... */
7
8    for (int ip = 0; ip <= 5; ++ip)
9    {
10     /* ... */
11
12     for (int j0 = 0; j0 <= 9; ++j0)
13       for (int elem = 0; elem <= 3; ++elem)
14       {
15         A[elem + 4 * 2 * j0] = A[elem + 4 * 2 * j0] + t18[10 * ip + j0] * t36[elem] + t19[10 * ip + j0] * t35[
16         A[elem + 4 * (1 + 2 * j0)] = A[elem + 4 * (1 + 2 * j0)] + t18[10 * ip + j0] * t32[elem] + t19[10 * ip
17       }
18   }
19 }
20
21 void wrap_form0_cell_integral_otherwise(int const start, int const end, double *__restrict__ dat0, double cons
22 {
23
24   /* ... */
25
26   for (int n_outer = (start / 4); n_outer <= ((-4 + end) / 4); ++n_outer)
27   {
28     for (int i7 = 0; i7 <= 2; ++i7)
29       for (int i8 = 0; i8 <= 1; ++i8)
30         for (int n_inner = 0; n_inner <= 3; ++n_inner)
31           t3[8 * i7 + 4 * i8 + n_inner] = dat1[2 * map1[3 * (4 * n_outer + n_inner) + i7] + i8];
32     for (int i13 = 0; i13 <= 9; ++i13)
33       for (int i14 = 0; i14 <= 1; ++i14)
34         for (int n_inner = 0; n_inner <= 3; ++n_inner)
35           t4[8 * i13 + 4 * i14 + n_inner] = dat2[2 * map0[10 * (4 * n_outer + n_inner) + i13] + i14];
36     for (int i1 = 0; i1 <= 9; ++i1)
37       for (int i2 = 0; i2 <= 1; ++i2)
38         for (int n_inner = 0; n_inner <= 3; ++n_inner)
39           t2[8 * i1 + 4 * i2 + n_inner] = 0.0;
40
41     kernel(t2, t3, t4);
42
43     for (int i20 = 0; i20 <= 1; ++i20)
44       for (int i19 = 0; i19 <= 9; ++i19)
45         for (int n_inner = 0; n_inner <= 3; ++n_inner)
46           dat0[2 * map0[10 * (4 * n_outer + n_inner) + i19] + i20] = dat0[2 * map0[10 * (4 * n_outer + n_inner
47   }
48 }
```

Action of linear elasticity operator on triangle mesh, batched by 4

```
3  static inline void kernel(double *__restrict__ A, double const *__restrict__ coords, double const *__restrict__
4  {
5
6    /* ... */
7
8    for (int ip = 0; ip <= 5; ++ip)
9    {
10     /* ... */
11
12     for (int j0 =
13       for (int ele
14       {
15         A[elem + 4
16         A[elem + 4
17       }
18
19   }
20
21 void wrap_form0_cell_integral_otherwise(int const start, int const end, double *__restrict__ dat0, double cons
22 {
23
24   /* ... */
25
26   for (int n_outer = (start / 4); n_outer <= ((-4 + end) / 4); ++n_outer)
27   {
28     for (int i7 = 0; i7 <= 2; ++i7)
29       for (int i8 = 0; i8 <= 1; ++i8)
30         for (int n_inner = 0; n_inner <= 3; ++n_inner)
31           t3[8 * i7 + 4 * i8 + n_inner] = dat1[2 * map1[3 * (4 * n_outer + n_inner)
32     for (int i13 = 0; i13 <= 9; ++i13)
33       for (int i14 = 0; i14 <= 1; ++i14)
34         for (int n_inner = 0; n_inner <= 3; ++n_inner)
35           t4[8 * i13 + 4 * i14 + n_inner] = dat2[2 * map0[10 * (4 * n_outer + n_inner) + i13] + i14];
36     for (int i1 = 0; i1 <= 9; ++i1)
37       for (int i2 = 0; i2 <= 1; ++i2)
38         for (int n_inner = 0; n_inner <= 3; ++n_inner)
39           t2[8 * i1 + 4 * i2 + n_inner] = 0.0;
40
41     kernel(t2, t3, t4);
42
43     for (int i20 = 0; i20 <= 1; ++i20)
44       for (int i19 = 0; i19 <= 9; ++i19)
45         for (int n_inner = 0; n_inner <= 3; ++n_inner)
46           dat0[2 * map0[10 *
47   }
48 }
```

Split n into n_outer and n_inner
Outer loop stride 4

36[elem] + t19[10 * ip + j0] * t35[
ip + j0] * t32[elem] + t19[10 * ip

```
for (int n_outer = (start / 4); n_outer <= ((-4 + end) / 4); ++n_outer)
```

Wrapper

Action of linear elasticity operator on triangle mesh, batched by 4

```
3  static inline void kernel(double *__restrict__ A, double const *__restrict__ coords, double const *__restrict__
4  {
5
6    /* ... */
7
8    for (int ip = 0; ip
9    {
10     /* ... */
11
12 for (int i7 = 0; i7 <= 2; ++i7)
13   for (int i8 = 0; i8 <= 1; ++i8)
14     for (int n_inner = 0; n_inner <= 3; ++n_inner)
15       t3[8 * i7 + 4 * i8 + n_inner] = dat1[2 * map1[3 * (4 * n_outer + n_inner) + i7] + i8];
16 for (int i13 = 0; i13 <= 9; ++i13)
17   for (int i14 = 0; i14 <= 1; ++i14)
18     for (int n_inner = 0; n_inner <= 3; ++n_inner)
19       t4[8 * i13 + 4 * i14 + n_inner] = dat2[2 * map0[10 * (4 * n_outer + n_inner) + i13] + i14];
20 for (int i1 = 0; i1 <= 9; ++i1)
21   for (int i2 = 0; i2 <= 1; ++i2)
22     for (int n_inner = 0; n_inner <= 3; ++n_inner)
23       t2[8 * i1 + 4 * i2 + n_inner] = 0.0;
24
25
26   for (int n_outer = (start /
27   {
28     for (int i7 = 0; i7 <= 2;
29       for (int i8 = 0; i8 <=
30         for (int n_inner = 0;
31           t3[8 * i7 + 4 * i8
32     for (int i13 = 0; i13 <= 9; ++i13)
33       for (int i14 = 0; i14 <= 1; ++i14)
34         for (int n_inner = 0; n_inner <= 3; ++n_inner)
35           t4[8 * i13 + 4 * i14 + n_inner] = dat2[2 * map0[10 * (4 * n_outer + n_inner) + i13] + i14];
36     for (int i1 = 0; i1 <= 9; ++i1)
37       for (int i2 = 0; i2 <= 1; ++i2)
38         for (int n_inner = 0; n_inner <= 3; ++n_inner)
39           t2[8 * i1 + 4 * i2 + n_inner] = 0.0;
40
41     kernel(t2, t3, t4);
42
43     for (int i20 = 0; i20 <= 1; ++i20)
44       for (int i19 = 0; i19 <= 9; ++i19)
45         for (int n_inner = 0; n_inner <= 3; ++n_inner)
46           dat0[2 * map0[10 *
47   }
48 }
```

Gathering input data for 4 elements
Arrays are vector-expanded

Data for different elements packed to inner most dimension

Action of linear elasticity operator on triangle mesh, batched by 4

```
3  static inline void kernel(double *__restrict__ A, double const *__restrict__ coords, double const *__restrict__
4  {
5
6    /* ... */
7
8    for (int ip = 0; ip <= 5; ++ip)
9    {
10     /* ... */
11
12     for (int j0 = 0; j0 <= 9; ++j0)
13       for (int elem = 0; elem <= 3; ++elem)
14       {
15         A[elem + 4 * 2 * j0] = A[elem + 4 * 2 * j0] + t18[10 * ip + j0] * t36[elem] + t19[10 * ip + j0] * t35[
16         A[elem + 4 * (1 + 2 * j0)] = A[elem + 4 * (1 + 2 * j0)] + t18[10 * ip + j0] * t32[elem] + t19[10 * ip
17       }
18   }
19 }
20
21 void wrap_form0_cell_integral_otherwise(int const start, int const end, double *__restrict__ dat0, double cons
22 {
23
24   /* ... */
25
26   for (int n_outer = (start / 4); n_outer <= ((-4 + end) / 4); ++n_outer)
27   {
28     for (int i7 = 0; i7 <= 2; ++i7)
29       for (int i8 = 0; i8 <= 1;
30         for (int n_inner = 0; n
31           t3[8 * i7 + 4 * i8 +                                  outer + n_inner) + i7] + i8];
32     for (int i13 = 0; i13 <= 9; ++i13)
33       for (int i14 = 0;
34         for (int n_inner
35           t4[8 * i13 +                                          nner) + i13] + i14];
36     for (int i1 = 0; i1
37       for (int i2 = 0; i2 <= 1; ++i2)
38         for (int n_inner = 0; n_inner <= 3; ++n_inner)
39           t2[8 * i1 + 4 * i2 + n_inner] = 0.0;
40
41     kernel(t2, t3, t4);
42
43     for (int i20 = 0; i20 <= 1; ++i20)
44       for (int i19 = 0; i19 <= 9; ++i19)
45         for (int n_inner = 0; n_inner <= 3; ++n_inner)
46           dat0[2 * map0[10 * (4 * n_outer + n_inner) + i19] + i20] = dat0[2 * map0[10 * (4 * n_outer + n_inner
47   }
48 }
```

Kernel call

```
kernel(t2, t3, t4);
```

Action of linear elasticity operator on triangle mesh, batched by 4

```
3  static inline void kernel(double *__restrict__ A, double const *__restrict__ coords, double const *__restrict
4  {
5
6    /* ... */
7
8    for (int ip = 0; ip <= 5; ++ip)
9    {
10     /* ... */
11
12     for (int j0 = 0; j0 <= 9; ++j0)
13       for (int elem = 0; elem <= 3; ++elem)
14       {
15         A[elem + 4 * 2 * j0] = A[elem + 4 * 2 * j0] + t18[10 * ip + j0] * t36[elem] + t19[10 * ip + j0] * t35[
16         A[elem + 4 * (1 + 2 * j0)] = A[elem + 4 * (1 + 2 * j0)] + t18[10 * ip + j0] * t32[elem] + t19[10 * ip
17       }
18   }
19 }
20
21 void wrap_form0_cell_integral_otherwise(int const start, int const end, double *__restrict__ dat0, double cons
22 {
23
24   /* ... */
25
26   for (int n_outer = (start / 4); n_outer <= ((-4 + end) / 4); ++n_outer)
27   {
28     for (int i7 = 0; i7 <= 2; ++i7)
29       for (int i8 = 0; i8
30         for (int n_inner
31           t3[8 * i7 + 4 *                                                    7] + i8];
```

Scattering might have race condition

```
for (int i20 = 0; i20 <= 1; ++i20)
  for (int i19 = 0; i19 <= 9; ++i19)
    for (int n_inner = 0; n_inner <= 3; ++n_inner)
      dat0[2 * map0[10 * (4 * n_outer + n_inner) + i19] + i20] += t2[8 * i19 + 4 * i20 + n_inner];
```

```
38         for (int n_inner = 0; n_inner <= 3; ++n_inner)
39           t2[8 * i1 + 4 * i2 + n_inner] = 0.0;
40
41     kernel(t2, t3, t4);
42
43     for (int i20 = 0; i20 <= 1; ++i20)
44       for (int i19 = 0; i19 <= 9; ++i19)
45         for (int n_inner = 0; n_inner <= 3; ++n_inner)
46           dat0[2 * map0[10 *
47   }
48 }
```

Action of linear elasticity operator on triangle mesh, batched by 4

```
3  static inline void kernel(double *__restrict__ A, double const *__restrict__ coords, double const *__restrict__
4  {
5
6    /* ... */
7
8    for (int ip = 0; ip <= 5; ++ip)
9    {
10     /* ... */
11
12     for (int j0 = 0; j0 <= 9; ++j0)
13       for (int elem = 0; elem <= 3; ++elem)
14       {
15         A[elem + 4 * 2 * j0] = A[elem + 4 * 2 * j0] + t18[10 * ip + j0] * t36[elem] + t19[10 * ip + j0] * t35[
16         A[elem + 4 * (1 + 2 * j0)] = A[elem + 4 * (1 + 2 * j0)] + t18[10 * ip + j0] * t32[elem] + t19[10 * ip
17       }
18   }
19 }
20
```

Kernel

```
for (int ip = 0; ip <= 5; ++ip)
{
  /* ... */

  for (int j0 = 0; j0 <= 9; ++j0)
    for (int elem = 0; elem <= 3; ++elem)
    {
      A[elem + 4 * 2 * j0] += t18[10 * ip + j0] * t36[elem] + t19[10 * ip + j0] * t35[elem];
      A[elem + 4 * (1 + 2 * j0)] += t18[10 * ip + j0] * t32[elem] + t19[10 * ip + j0] * t31[elem];
    }
}
```

"element" loop pushed to innermost

Trip count 4, stride 1, aligned, independent

```
35           t4[8 * i13 + 4 * i14 + n_inner] = dat2[2 * map0[10 * (4 * n_outer + n_inner) + i13] + i14];
36       for (int i1 = 0; i1 <= 9; ++i1)
37         for (int i2 = 0; i2 <= 1; ++i2)
38           for (int n_inner = 0; n_inner <= 3; ++n_inner)
39             t2[8 * i1 + 4 * i2 + n_inner] = 0.0;
40
41       kernel(t2, t3, t4);
42
43       for (int i20 = 0; i20 <= 1; ++i20)
44         for (int i19 = 0; i19 <= 9; ++i19)
45           for (int n_inner = 0; n_inner <= 3; ++n_inner)
46             dat0[2 * map0[10 *
47     }
48   }
```

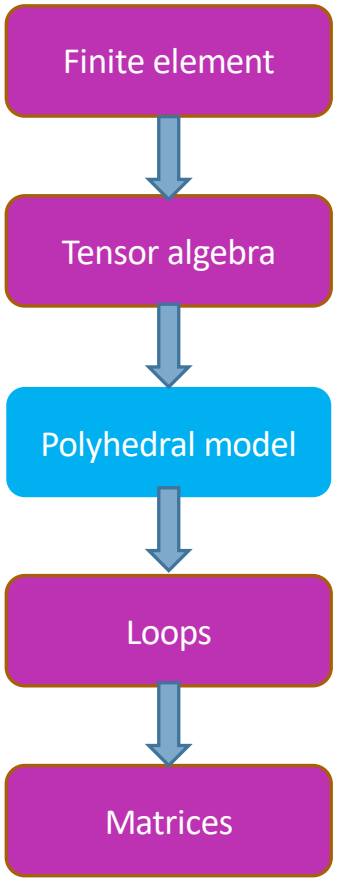
Action of linear elasticity operator on triangle mesh, batched by 4

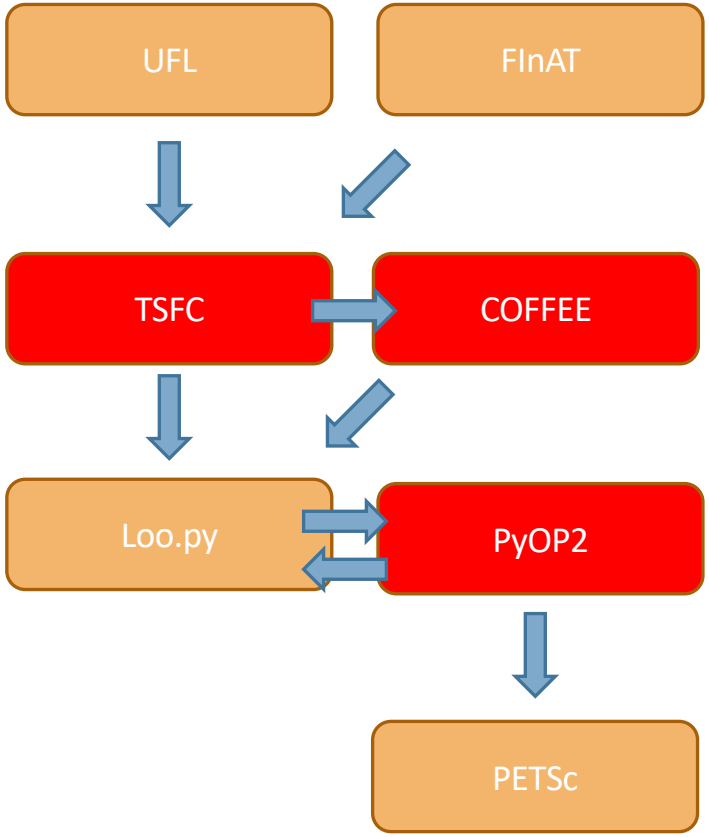
Implementation

Introducing loo.py

- Andreas Klöckner et al. (UIUC)
- ≈ isl model of loops + transformations
- Not a blackbox
  - But handy if you tell it exactly what to do
- Support multiple backends
  - CPU
  - ISPC
  - OpenCL, PyOpenCL
  - Cuda

Abstraction layers





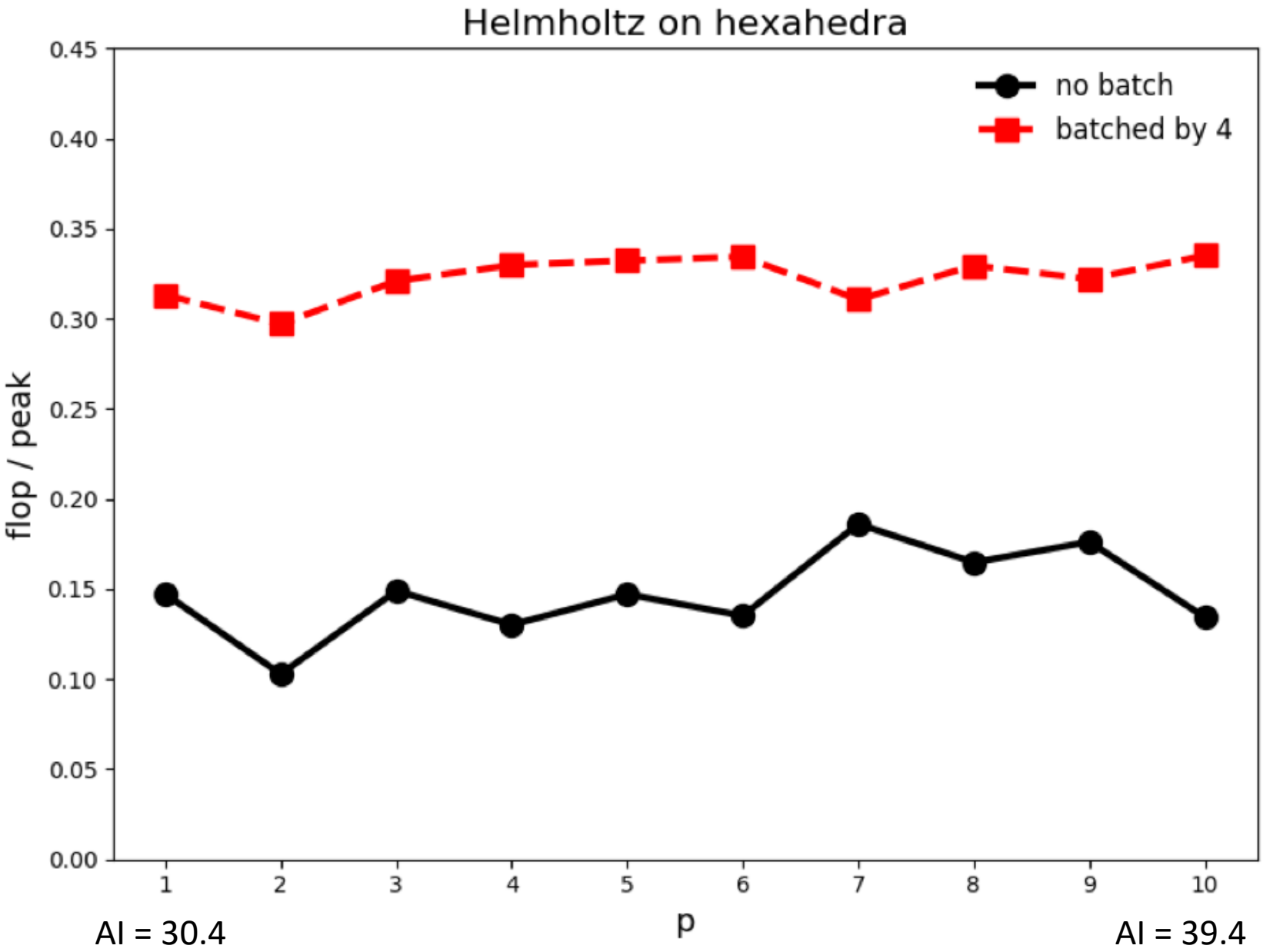
Firedrake

Experimental setup

- Hardware: Haswell i7-4790 (single core measurement)
  - SIMD width = 4 (avx2)
  - Peak flop = 3.6 GHz x 4 (avx2) x 2 (fma) x 2 (issue) = 57.6 Gflops
  - Running Intel LINPACK binary: 51.0 Gflops
  - STREAM triad bandwidth: 10.4 GB / s
  - Roofline AI “regime switching point” = 5.54 flops / byte
- Mesh: hexahedra (3D tensor product element)
  - TSFC automates sum factorisation [2]

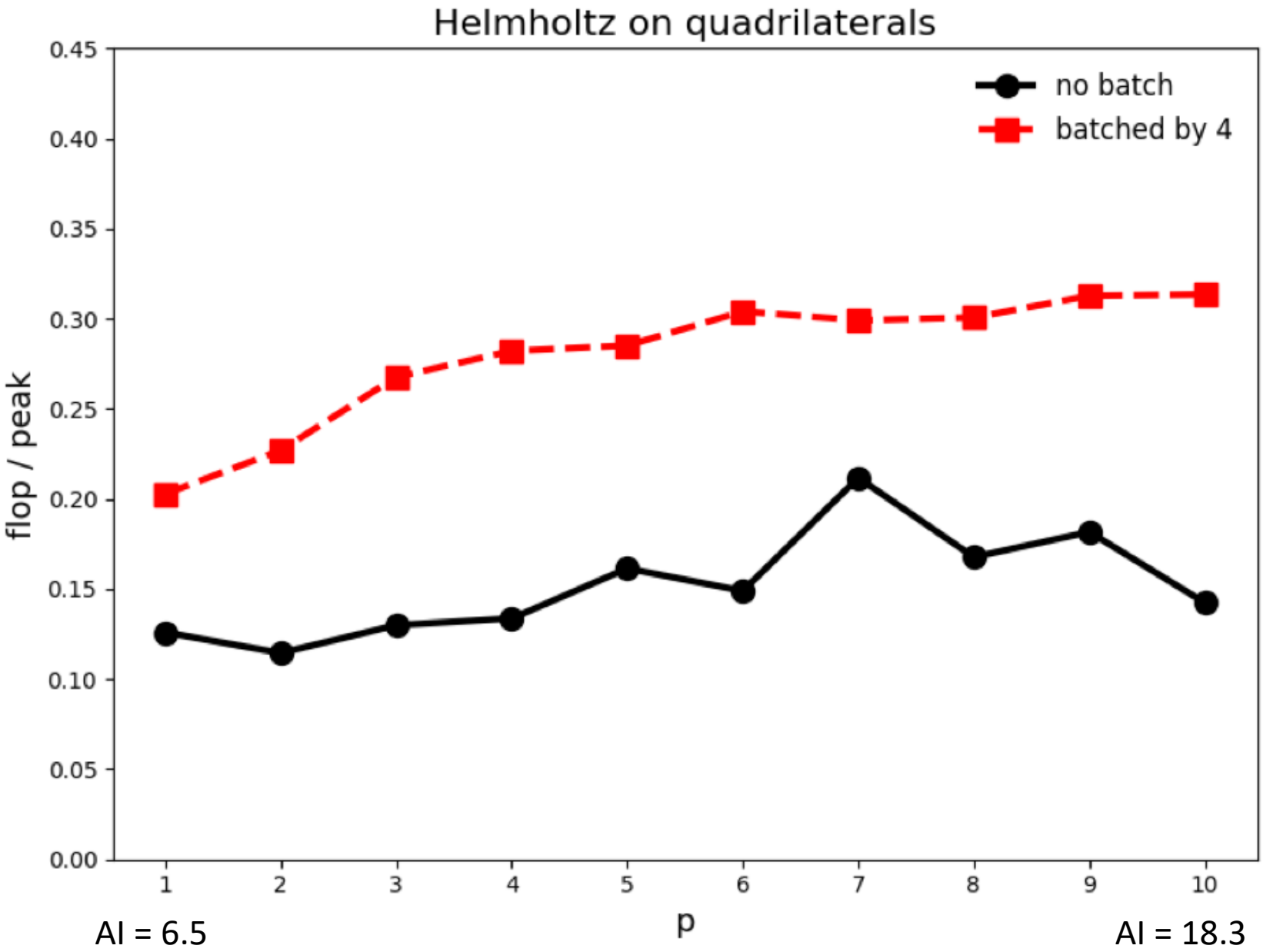
    -> Innermost loop trip count = polynomial degree + 1
- Action of Helmholtz operator
  - Arithmetic Intensity (perfect cache) 30.4 to 33.6 → compute bound
- We present achieved flops / 57.6 Gflops

[2] M. Homolya, et al. arXiv:1711.02473 (2017).


Helmholtz on hexahedra
no batch
batched by 4
flop / peak
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
1
2
3
4
5
6
7
8
9
10
p

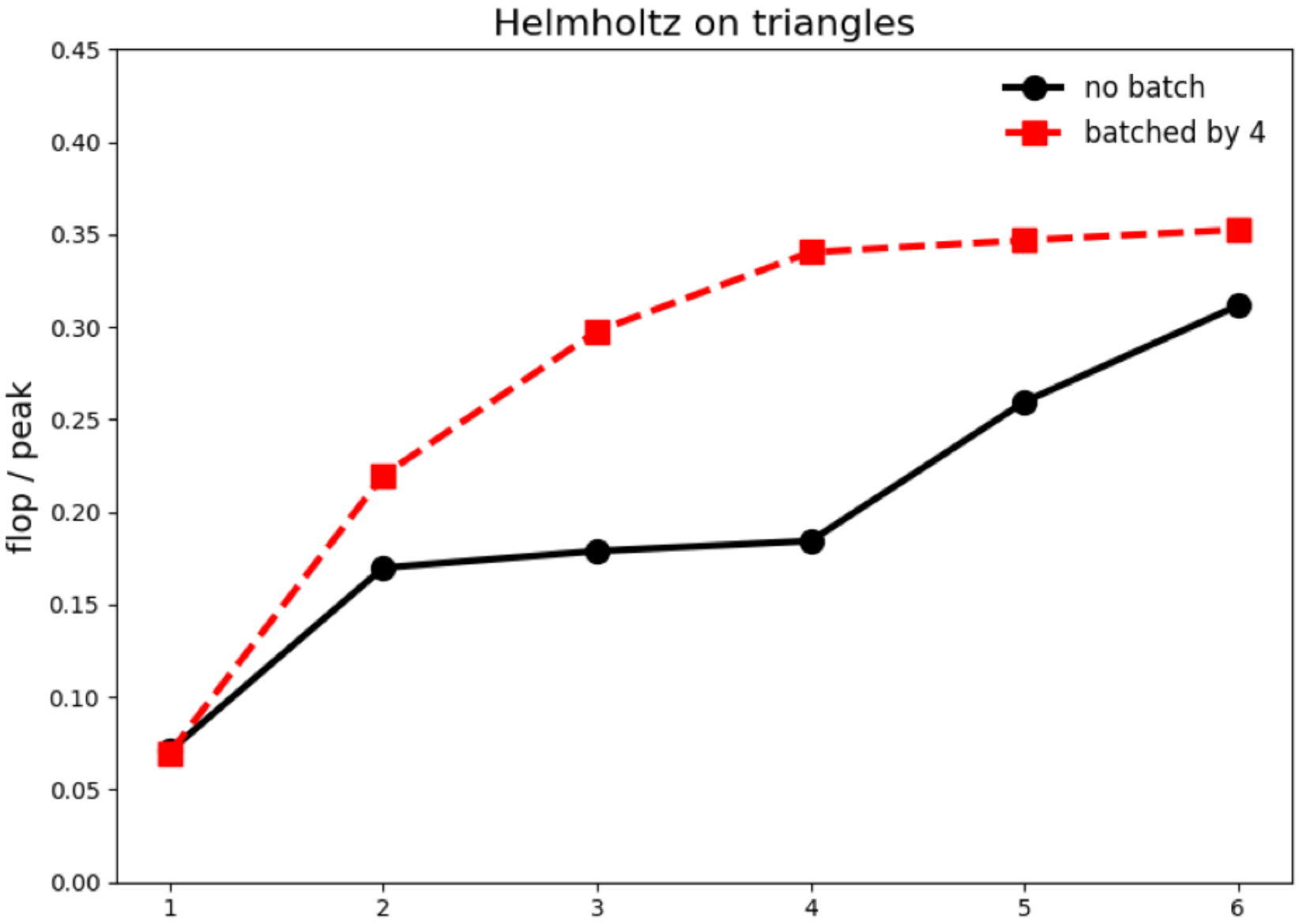
AI = 30.4

AI = 39.4


Helmholtz on quadrilaterals
no batch
batched by 4
flop / peak
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
1
2
3
4
5
6
7
8
9
10
p

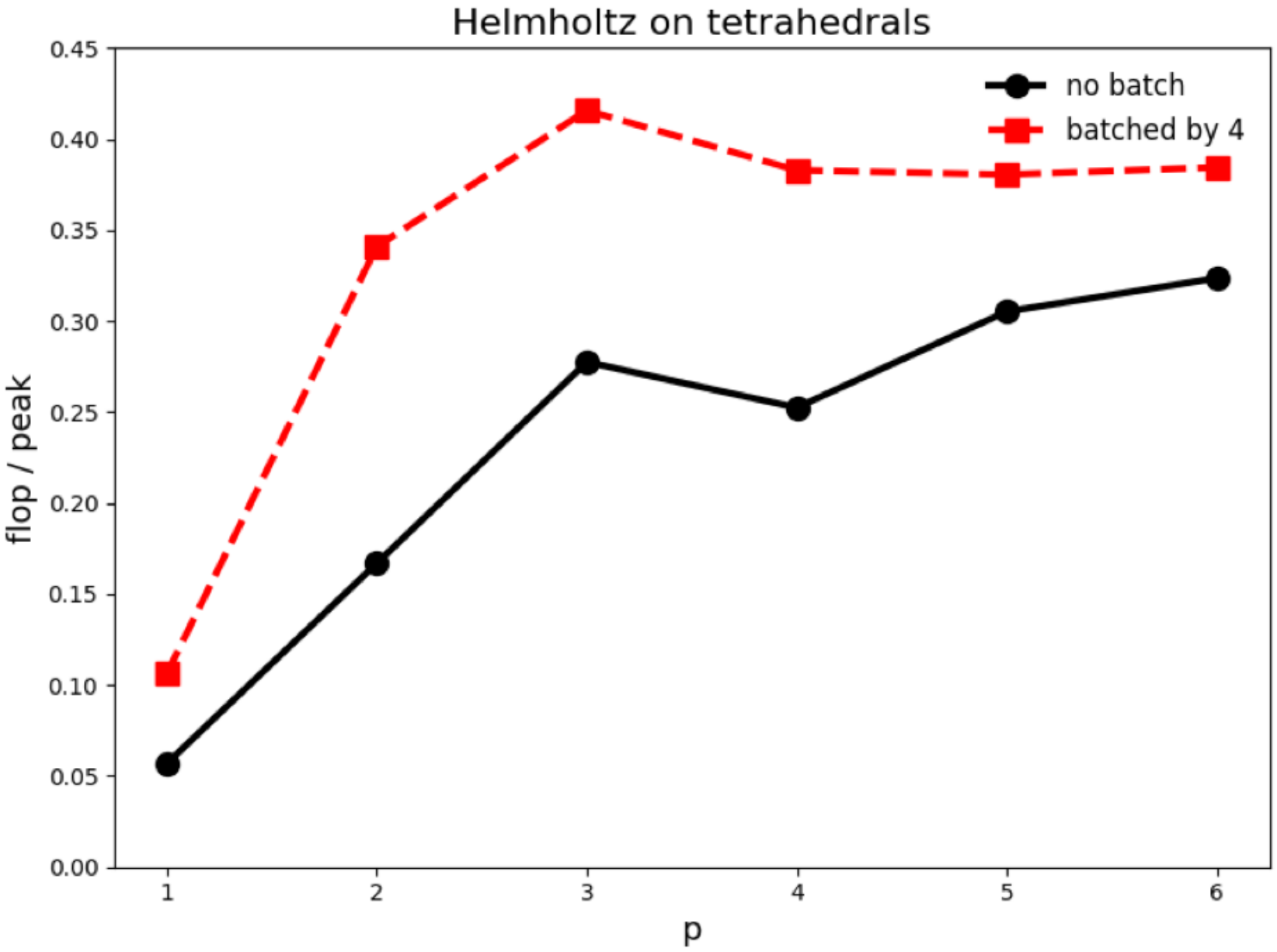
AI = 6.5

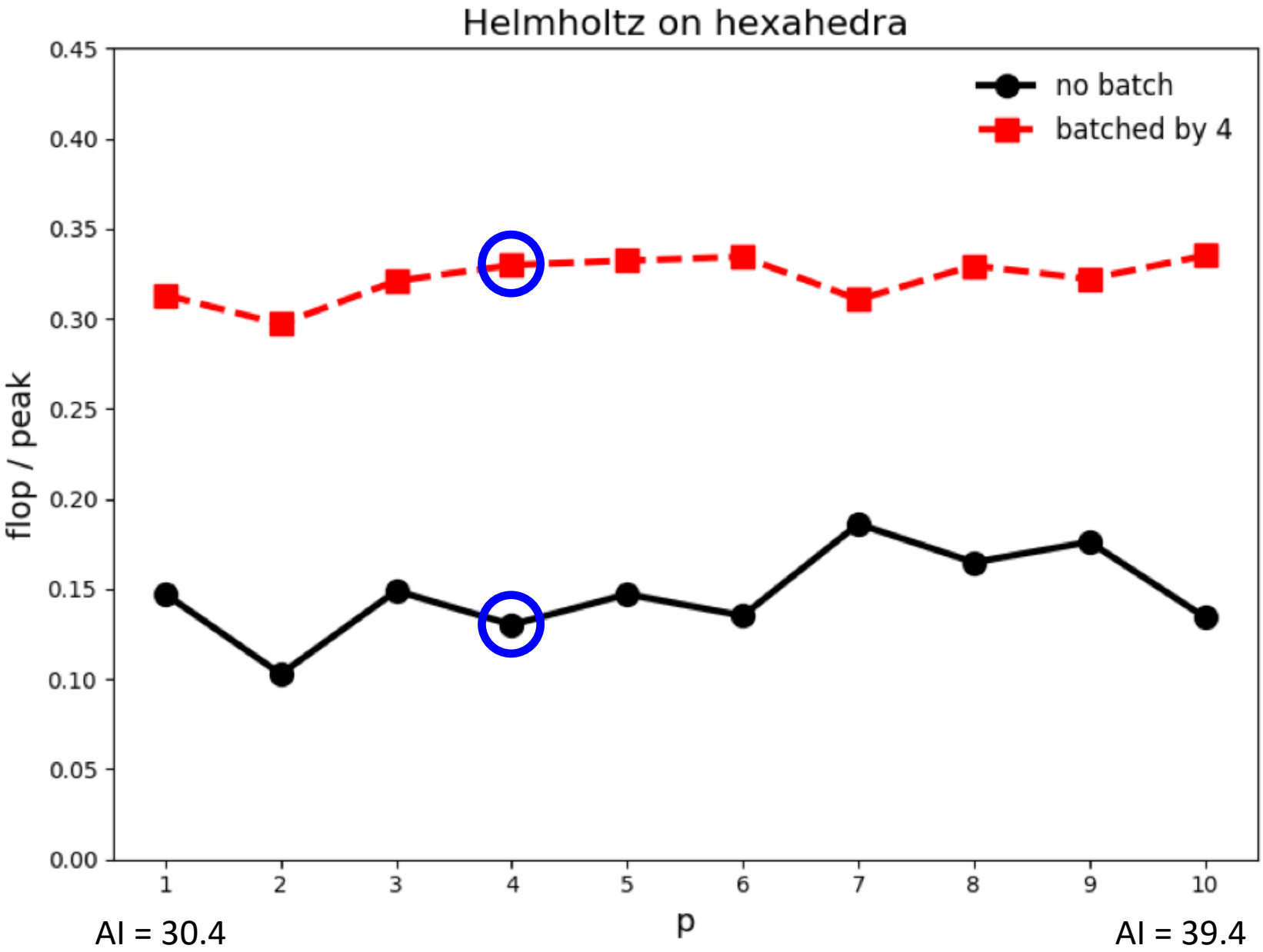
AI = 18.3


Helmholtz on triangles
no batch
batched by 4
flop / peak
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
1
2
3
4
5
6
p

AI = 2.8

AI = 56.7

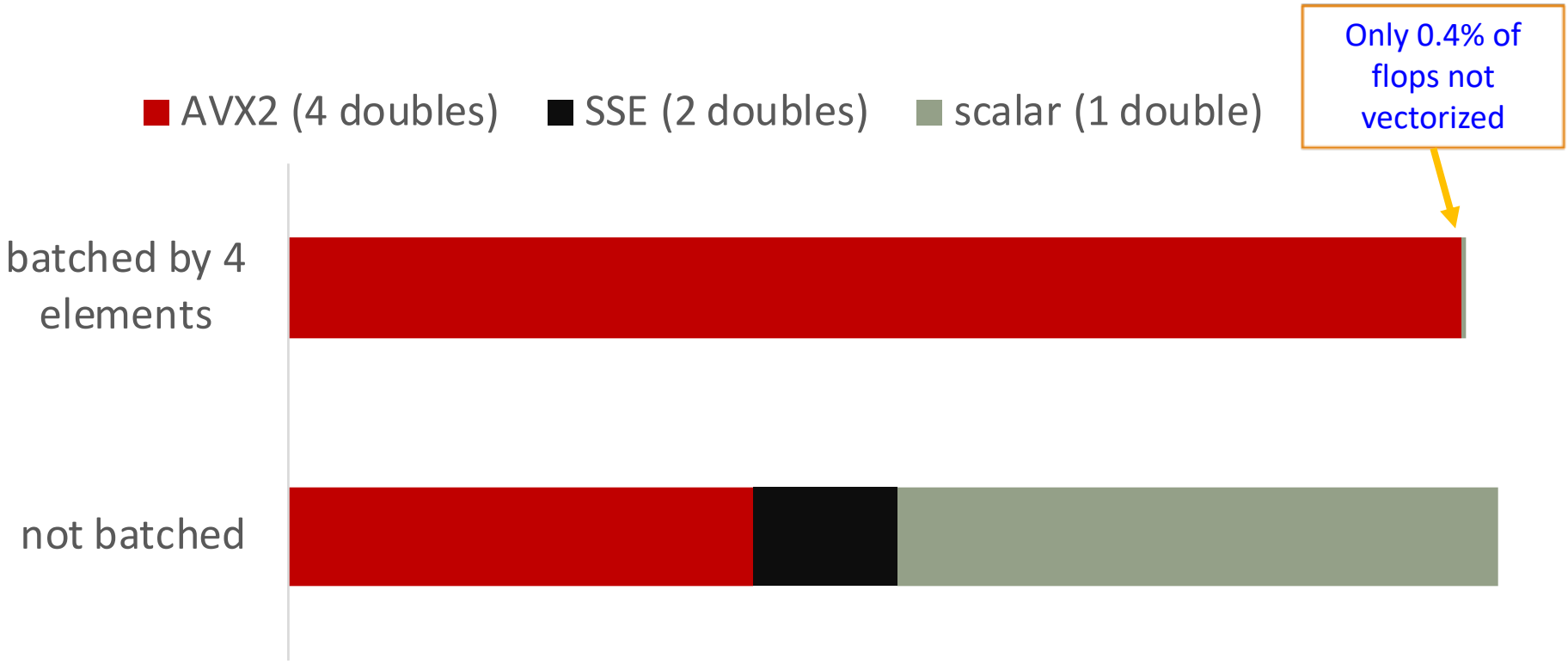

Helmholtz on tetrahedrals
no batch
batched by 4
flop / peak
p
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
1
2
3
4
5
6


Helmholtz on hexahedra
no batch
batched by 4
flop / peak
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
1
2
3
4
5
6
7
8
9
10
p

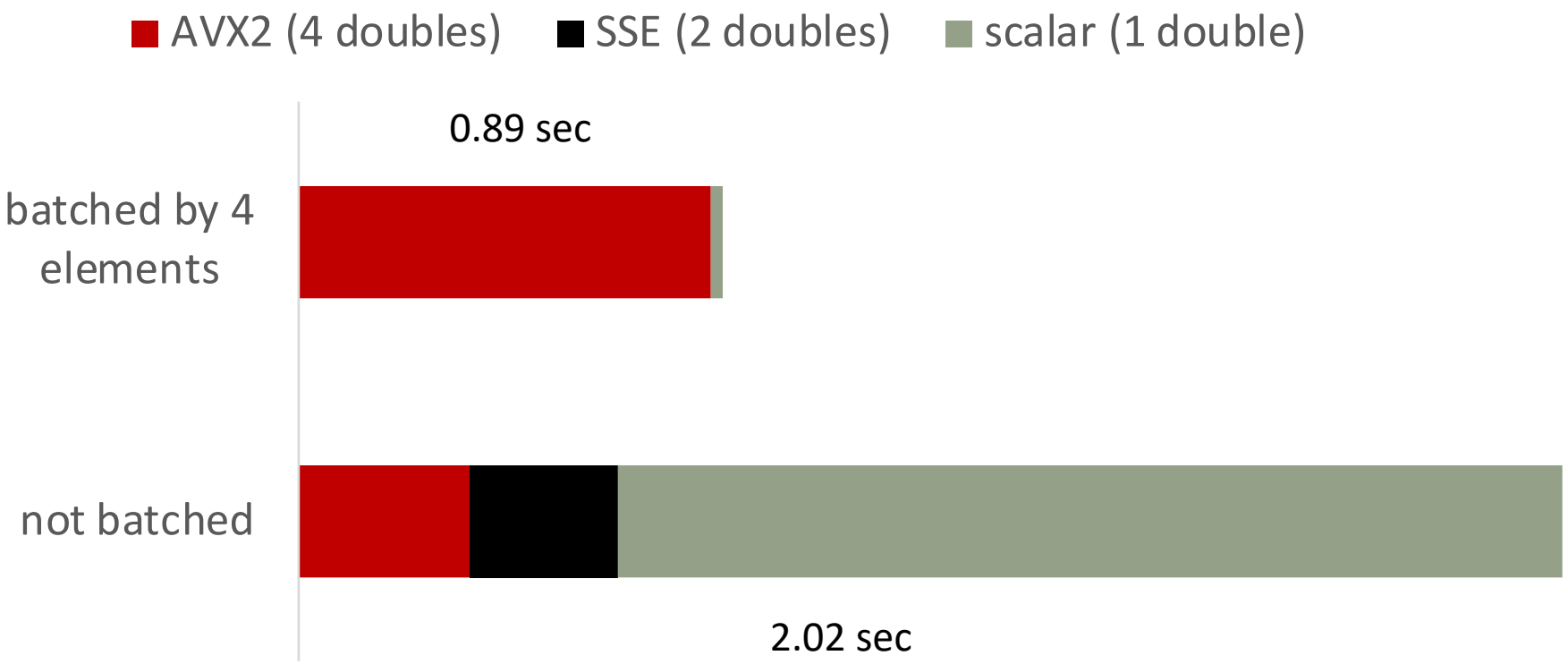
AI = 30.4

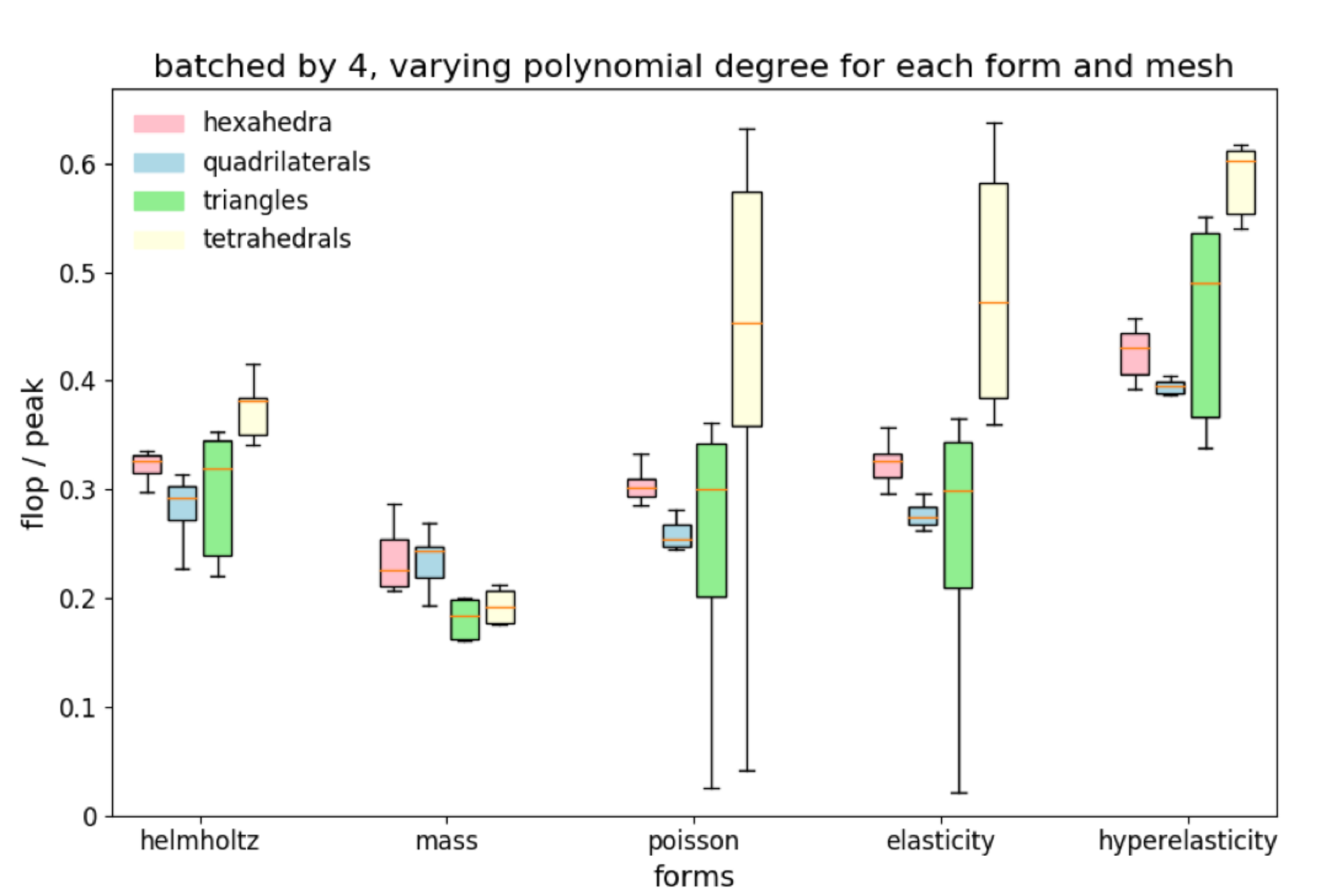
AI = 39.4

Flop contributions by instruction types



Instruction counts by instruction types




batched by 4, varying polynomial degree for each form and mesh
hexahedra
quadrilaterals
triangles
tetrahedrals
flop / peak
0
0.1
0.2
0.3
0.4
0.5
0.6
helmholtz
mass
poisson
elasticity
hyperelasticity
forms

To be continued…

- We are building an abstraction layer of loops (via loo.py)
- Pathway to GPUs
  - …which requires a better performance model
- Try it out:
  - firedrake branch `tsfc2loopy`
  - tsfc branch `tsfc2loopy`
  - PyOP2 branch `tsfc_loopying`
- Get in touch:
  - firedrakeproject.org
  - Email: firedrake@imperial.ac.uk
  - Slack channel: firedrakeproject

Firedrake

Aside…

- Local assembly in continuous form

$$\int \nabla u \cdot \nabla v \, \mathrm{d}x = \int J^{-T} \hat{\nabla} \tilde{u} \cdot J^{-T} \hat{\nabla} \tilde{v} |\det J| \, \mathrm{d}\hat{x}$$

- Discretization → "just tensor contraction"

$$\mathbf{A}_{ij} = \sum_{q\alpha\beta\gamma} w_q K_{q\beta\alpha} \frac{\partial \Psi_i}{\partial \hat{x}_\beta}(\xi_q) K_{q\gamma\alpha} \frac{\partial \Psi_j}{\partial \hat{x}_\gamma}(\xi_q) |\det J_q|$$

- Many open questions
  - Should we always apply flop-minimising optimizations?
  - Pre-evaluation of (some) compiler time constant tensors?
  - Best combination of tensor representation and polyhedral model?
  - Parameter search & autotuning
- … for now, I'm focusing on building the infrastructure